Real Analysis Preliminary Exam January 2009

Your grade will be based on your answers to the first three questions and 4 of the last five questions. In question 1, each correct answer is worth 3 points, in question 2 a fully correct answer is worth 4 points and in question 3 a correct answer is worth 4 points, an incorrect answer deducts 3 points and no answer is 0 points. The remaining questions require complete proofs and are 12 points each.

Question 1: (3 points each)

(a): Let X be a non-empty set. Define a $\sigma-$ *algebra* $\mathcal{M}$ *of subsets of* X *and a measure* on $\mathcal{M}$.

(b): Suppose $\{a_m : m \geq 1\}$ is a sequence of real numbers. Define $\limsup_{m\to\infty} a_m$.

(c): Let H be a Hilbert space, and $P : H \to H$ be a continuous linear transformation. What are the properties of P that ensure that P is a self-adjoint projection on H?

(d): Suppose $f \in L^1(\mathbb{R})$. Give the formula for the Fourier transformation of f.

Question 2: (4 points each)

State carefully the following results; making sure that all important terms are defined. Here $(X, \mathcal{M}, \mu)$ is a $\sigma-$finite measure space, and "e.r.v" means takes values in $[-\infty, \infty]$.

(a): The dominated convergence theorem for e.r.v. functions on X.

(b): Holder's inequality for e.r.v. functions on X.

(c): State the Fourier inversion theorem for complex-valued functions in $L^1(\mathbb{R})$.

(d): The Jordan decomposition theorem for a signed measure on $\mathcal{M}$.

Question 3: (4 points for a correct answer, -3 points for an incorrect answer and 0 points for no answer). Answer T (true) or F (false) for each of the following statements. λ will denote Lebesgue measure and all functions are assumed to be Lebesgue measurable on their domains. S_1 is the unit circle in the plane.

(a) If E is a Borel measurable subset of $\mathbb{R}$ with $\lambda(E) > 0$, then E contains a (non-empty) open interval.

(b) If $\{A_k : k \geq 1\}$ is a sequence of Lebesgue measurable subsets of $\mathbb{R}^n$ and $\lambda(A)$ is the Lebesgue measure of A, then $\lambda(\limsup_{k\to\infty} A_k) = \limsup_{k\to\infty} \lambda(A_k)$

(c) If $\{f_m\}$ converges to f in $L^1[0,1]$ then $\{f_m\}$ converges to f $\lambda\, a.e.$ on [0,1].

(d) When $f \in L^2(S_1)$ and $s_m(f)$ is the m-th partial sum of the Fourier series of f, then $s_m(f)$ converges to f in $L^2(S_1)$.

(e) Every real Hilbert space has a countable orthonormal basis.

(f) The space of all bounded continuous functions on $\mathbb{R}$ is a dense subset of $L^1(\mathbb{R})$.

For the following questions give reasons for your claims and detailed proofs. You may use theorems proved in class or in the textbook. Your grade will be based on your answers to 4 out of the five problems. $\mathbb{N}$ is the set of natural numbers, $\mathbb{Z}$ is the set of positive and negative integers.

Question 4: Define the function $g : \mathbb{R}\backslash\{0\} \to \mathbb{R}$ by $g(x) := x - (1/x)$.

(a) Is g one-to-one and onto?

(b) Suppose $I := [a, b]$ is a finite closed interval in $(0, \infty)$, show that

$$\int_{\mathbb{R}} f(g(x))\, dx \;=\; \int_{\mathbb{R}} f(x)\, dx$$

whenever $f = \chi_I$ is the characteristic function of the interval I.

(c) Does the preceding proof extend to an arbitrary finite closed interval I of $\mathbb{R}$?

Question 5: Suppose that S_1 is the unit disc and $f, g : S_1 \to \mathbb{R}$ are Lebesgue integrable real valued functions on $S_1 \simeq (-\pi, \pi]$. Define the k-th complex Fourier coefficient of f by

$$c_k(f) \;:=\; \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)\, e^{-ik\theta}\, d\theta \qquad k \in \mathbb{Z}$$

and the convolution of the functions f, g by

$$(f * g)(\theta) \;:=\; \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta - \psi)\, g(\psi)\, d\psi.$$

(a) Show that $c_k(f)$ is finite and give a bound on its absolute value.

(b) Show that if f is odd on S_1, that is $f(-\theta) = f(\theta)$ for all θ, then $c_k(f)$ is pure imaginary and find a formula for the imaginary component.

(c) Prove that $c_k(f * g) = c_k(f)\, c_k(g)$.

Question 6: Let I be a closed interval in $\mathbb{R}$ and $f : I \to \overline{\mathbb{R}}$ be a Lebesgue measurable function.

(a) Show that if $|I| < \infty$ and $f \in L^p(I)$ for some $p > 1$, then $f \in L^q(I)$ for all $1 \leq q \leq p$.

(b) Show that if $f \in L^p(I) \cap L^q(I)$ for $1 \leq p < q \leq \infty$, and $r \in (p, q)$ then there is an $\theta \in (0, 1)$ such that

$$\int_I |f|^r \leq \left(\int_I |f|^p\right)^{(1-\theta)p} \left(\int_I |f|^q\right)^{\theta q}$$

What is the value of theta here?

Question 7: Suppose $f(x) := e^{-ax^2}$ for $x \in \mathbb{R}, a > 0$.

(a) Find the Fourier transform of f.

(b) Hence find the Fourier transform $g(x) := x\, e^{-x^2}$.

Question 8: Let $H = L^2(-1, 1)$be the usual real Hilbert space of Lebesgue measurable functions on $(-1, 1)$ with the inner product.

$$\langle f, g \rangle := \int_{-1}^{1} f(x)g(x)\, dx$$

A function $f \in H$ is said to be even (odd) provided $f(-x) = f(x)$ $(f(-x) = -f(x))$ respectively a.e. on $(-1, 1)$

(a) Prove that the class V_e of all even functions in H is a closed subspace of H.

(b) Show that if g is an odd function then $\langle f, g \rangle = 0$ for all $f \in V_e$.

(c) Prove that if $f \in H$, then there are unique (equivalence classes of) functions $f_e, f_o \in H$ with $f_e \in V_e$ and f_o an odd function on $(-1, 1)$ such that $f(x) = f_e(x) + f_o(x) a.e.$.

(d) Show that your mapping of f to f_e defines a continuous self-adjoint linear projection of H to itself.