UNICITY RESULTS FOR GAUSS MAPS OF MNINIMAL SURFACES IMMERSED IN $\mathbb{R}^m$

A Dissertation

Presented to

the Faculty of the Department of Mathematics

University of Houston

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

By

Jungim Park, student

July 2016

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Abstract

The purpose of this thesis is to discuss the theory of holomorphic curves in order to study value distributions of the (generalized) Gauss map of complete minimal surfaces immersed in $\mathbb{R}^m$. The study was initiated by S.S. Chern and R. Osserman [4] in 1960s. Since then, it has been developed by F. Xavier [27], H. Fujimoto [7], M. Ru [22], etc. In this thesis, we prove a unicity theorem for two conformally diffeomorphic complete minimal surfaces immersed in $\mathbb{R}^m$ whose generalized Gauss maps f and g agree on the pre-image $\cup_{j=1}^{q} f^{-1}(H_j)$ for given hyperplanes $H_j, 1 \leq j \leq q$, in $\mathbb{P}^{m-1}(\mathbb{C})$, located in general position, under the assumption that $\bigcap_{j=1}^{k+1} f^{-1}(H_{i_j}) = \varnothing$. In the case when $k = m-1$, the result obtained gives an improvement of the earlier result of Fujimoto [10].

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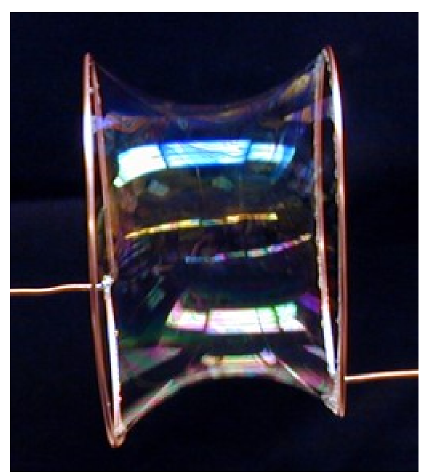
Introduction

Figure 1.1: Catenoid Soap Film



http://faraday.physics.uiowa.edu

Minimal surfaces are surfaces which have minimal areas for all small perturbations. Minimal surfaces have been studied not only in mathematics but also in many other fields such as physics, biology, architecture, art, and so on. Physicians have studied the minimal surface using soap films. The picture on the right is the catenoid created from a soap bubble by a physician. After Alan Schoen discovered a minimal surface called a "gyroid", many biologists have found gyroid structures in the membrane of butterflies' wings and in certain surfactant or lipid mesophases. In recent years, new types of architecture and art using minimal surfaces have been created and developed. In short, interest in the minimal surface has been growing in many fields over the past several decades.

Minimal surface theories was studied for the first time by Joseph-Louis Lagrange who in 1762 considered the variational problem of finding the surface of least area stretched across a given closed contour. Since Lagrange's initial investigation, there have been multiple discoveries on minimal surface theories by many great mathematicians. In 1967, the value distribution theory of Gauss map of complete minimal surfaces started to be studied by S.S. chern and R. Osserman [4]. Since then, it have been studied by many mathematicians such as F. Xavier, H. Fujimoto, M. Ru, S.J. Kao, H.P. Hoang, etc. From 1992 onward, the unicity theorem for Gauss maps of complete minimal surfaces also have been studied by H. Fujimoto, but there have been no particular improvements. That is, until now. In this thesis, we shall give an improvement of Fujimoto's unicity theorem of Gauss maps of complete minimal surfaces.

For a minimal surface $x := (x_1, \cdots, x_m) : M \to \mathbb{R}^m$ immersed in $\mathbb{R}^m$ with $m \geq 3$, its generalized Gauss map G is defined as the map which maps each $p \in M$ to the point in $Q_{m-2}(\mathbb{C}) := \{(w_1 : \cdots : w_m) \mid w_1^2 + \cdots + w_m^2 = 0\}$ corresponding to the oriented tangent plane of M at p. We may regard M as a Riemann surface with a conformal metric and G as a holomorphic map of M into $\mathbb{P}^{m-1}(\mathbb{C})$. With this setting, many value-distribution-theoretic properties of holomorphic curves in the complex projective space can be carried into the study of the Gauss maps of complete minimal surfaces in $\mathbb{R}^m$. This thesis specifically focuses on the corresponding unicity results for two conformally diffeomorphic complete minimal surfaces immersed in $\mathbb{R}^m$.

We begin with by recalling H. Fujimoto's result [10]. Let $x : M \to \mathbb{R}^m$ and $\tilde{x} : \tilde{M} \to \mathbb{R}^m$ be two oriented non-flat complete minimal surfaces immersed in $\mathbb{R}^m$ and let $\mathsf{G} : M \to \mathbb{P}^{m-1}(\mathbb{C})$ and $\tilde{\mathsf{G}} : \tilde{M} \to \mathbb{P}^{m-1}(\mathbb{C})$ be their generalized Gauss maps. Assume that there is a conformal diffeomorphism Φ of M onto $\tilde{M}$. Then the Gauss map of the minimal surface $\tilde{x} \circ \Phi : M \to \mathbb{R}^m$ is given by $\tilde{\mathsf{G}} \circ \Phi$. Fujimoto obtained the following result.

Fujimoto's Theorem (H. Fujimoto, [10]). *Under the notations above, consider the holomorphic maps $f = \mathsf{G} : M \to \mathbb{P}^{m-1}(\mathbb{C})$ and $g = \tilde{\mathsf{G}} \circ \Phi : M \to \mathbb{P}^{m-1}(\mathbb{C})$. Assume that there exist hyperplanes $H_1, \cdots, H_q$ in $\mathbb{P}^{m-1}(\mathbb{C})$ located in general position such that*

(i) $f^{-1}(H_j) = g^{-1}(H_j)$ for every $j = 1, \cdots, q$,

(ii) $f = g$ on $\bigcup_{j=1}^{q} f^{-1}(H_j)$.

If $q > m^2 + m(m-1)/2$, then $f \equiv g$.

Main Theorem. *Assume that both $f = \mathsf{G} : M \to \mathbb{P}^{m-1}(\mathbb{C})$ and $g = \tilde{\mathsf{G}} \circ \Phi : M \to \mathbb{P}^{m-1}(\mathbb{C})$ are linearly non-degenerate (i.e. the images of f and g are not contained in any linear subspaces of $\mathbb{P}^{m-1}(\mathbb{C})$) and that there exist hyperplanes $H_1, \cdots, H_q$ in $\mathbb{P}^{m-1}(\mathbb{C})$ located in general position and a positive integer $k > 0$ such that*

(i) *$f^{-1}(H_j) = g^{-1}(H_j)$ for every $j = 1, \cdots, q$,*

(ii) *$\bigcap_{j=1}^{k+1} f^{-1}(H_{i_j}) = \varnothing$ for any $\{i_1, \cdots, i_{k+1}\} \subset \{1, \cdots, q\}$,*

(iii) $f = g$ *on* $\bigcup_{j=1}^{q} f^{-1}(H_j)$.

$$\textit{If } q > \frac{(m^2+m+4k) + \sqrt{(m^2+m+4k)^2 + 16k(m-2)m(m+1)}}{4}, \tag{1.1}$$

then $f \equiv g$.

When $k = 1$, the condition (1.1) becomes

$$q > \frac{(m^2+m+4) + \sqrt{(m^2+m+4)^2 + 16(m-2)m(m+1)}}{4}.$$

In this case, notice that

$$3m - 2 + \frac{m(m-1)}{2} > \frac{(m^2+m+4) + \sqrt{(m^2+m+4)^2 + 16(m-2)m(m+1)}}{4}$$

for all $m \geq 3$, then the Main Theorem gives the following corollary.

Corollary 1. *Under the notations above, consider the holomorphic maps* $f = \mathsf{G} : M \to \mathbb{P}^{m-1}(\mathbb{C})$, $g = \tilde{\mathsf{G}} \circ \Phi : M \to \mathbb{P}^{m-1}(\mathbb{C})$. *Assume that f and g are linearly non-degenerate, and that there exist hyperplanes $H_1, \cdots, H_q$ in $\mathbb{P}^{m-1}(\mathbb{C})$ located in general position such that*

(i) $f^{-1}(H_j) = g^{-1}(H_j)$ *for every* $j = 1, \cdots, q$,

(ii) $f^{-1}(H_i \bigcap H_j) = \varnothing$ *for* $i \neq j$,

(iii) $f = g$ *on* $\bigcup_{j=1}^{q} f^{-1}(H_j)$.

If $q \geq 3m - 2 + \frac{m(m-1)}{2}$, *then* $f \equiv g$.

When $k = m-1$, since the condition (ii) in the Main Assumption automatically holds under the assumption that $H_1, \cdots, H_q$ are in general position, we can omit it. Then, the Main Theorem gives the following Corollary.

Corollary 2. *Under the notations above, consider the holomorphic maps* $f = \mathsf{G} : M \to \mathbb{P}^{m-1}(\mathbb{C})$, $g = \tilde{\mathsf{G}} \circ \Phi : M \to \mathbb{P}^{m-1}(\mathbb{C})$. *Assume that* f *and* g *are linearly non-degenerate, and that there exist hyperplanes* $H_1, \cdots, H_q$ *in* $\mathbb{P}^{m-1}(\mathbb{C})$ *located in general position such that*

(i)$f^{-1}(H_j) = g^{-1}(H_j)$ *for every* $j = 1, \cdots, q$,

(ii)$f = g$ *on* $\bigcup_{j=1}^{q} f^{-1}(H_j)$.

If $q > \frac{(m^2+5m-4)+\sqrt{(m^2+5m-4)^2+16(m-2)(m-1)m(m+1)}}{4}$, *then* $f \equiv g$.

Furthermore, if $m = 3$ and $k = m-1 = 2$ in the Main Theorem, we obtain $q > 6$, which matches the result in Fujimoto's paper [8] for the case of the complete minimal surface immersed in $\mathbb{R}^3$ exactly.

Now we shall compare the number of hyperplanes, q, in order to show that the result in this thesis gives an improvement of Fujimoto's theorem mentioned before. Let $q_1 = m^2 + \frac{m(m-1)}{2}$ as in Fujimoto's result and

$$q_2 = \frac{m^2 + 5m - 4 + \sqrt{(m^2 + 5m - 4)^2 + 16(m-2)(m-1)m(m+1)}}{4}$$

as in our Corollary 2, we verify that $q_1 \geq q_2$ for $m \geq 3$. Indeed, for $m \geq 3$,

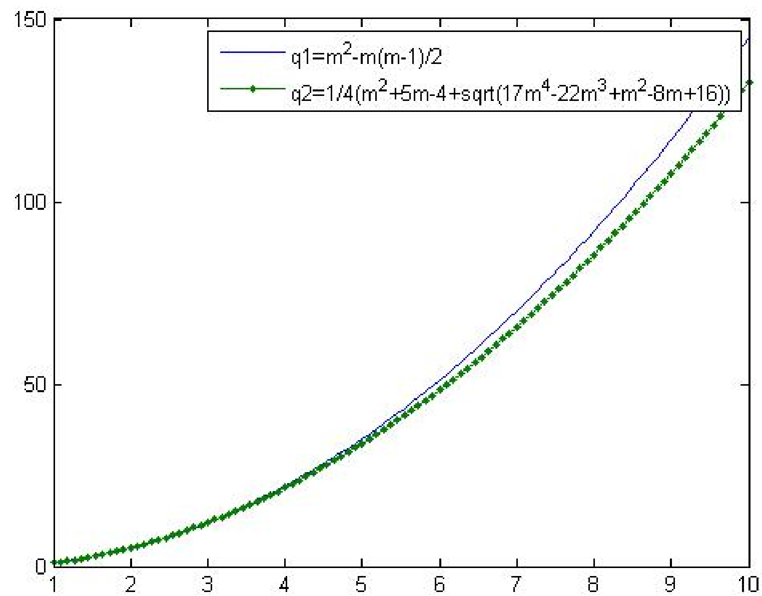
$$(5m^2 - 7m + 4)^2 - (\sqrt{17m^4 - 22m^3 + m^2 - 8m + 16})^2$$

$$= 8m^4 - 48m^3 + 88m^2 - 48m = 8m(m-1)(m-2)(m-3) \geq 0.$$

This implies that $5m^2 - 7m + 4 \geq \sqrt{17m^4 - 22m^3 + m^2 - 8m + 16}$. Hence,

$$q_1 - q_2 = \frac{5m^2 - 7m + 4 - \sqrt{17m^4 - 22m^3 + m^2 - 8m + 16}}{4} \geq 0$$

for $m \geq 3$. The same argument show that $q_1 > q_2$ for $m > 3$. Thus our Corollary 2 gives an improvement of Fujimoto's result mentioned above. The graph below gives how q_1 and q_2 grows as m becomes larger.



We outline here the strategies of proving our Main Theorem: (a) Instead of using Fujimoto's approach in [10], we use the method of Ru [19] which gave an explicit construction of the negative curvature on the unit disc; (b) Instead of the auxiliary function χ constructed in Fujimoto's paper [10], we use the new auxiliary function constructed in Chen-Yan [14] (see also [15] or [16]) which allows us to improve Fujimoto's result, as well as working for general k (while Fujimoto's case is only for $k = m - 1$).

Chapter 2

Basic Facts about Complete Minimal Surfaces in $\mathbb{R}^m$

2.1 Minimal Surfaces in $\mathbb{R}^3$

In this section, we define the classical Gauss map and study the relation between the generalized Gauss map ($\mathsf{G} = [\phi_1 : \phi_2 : \phi_3] : M \to \mathbb{P}^2(\mathbb{C})$) and the classical Gauss map ($g : M \to \overline{\mathbb{C}}$) of a surface in $\mathbb{R}^3$. Via the Weierstass-Enneper Representations, the Gauss map of a minimal surface is considered a meromorphic function on the corresponding Riemann surface. From this, we can see a remarkable analogy between the value distribution theory and the minimal surface theory. That analogy applies to their respective unicity theorems as well, so in this section we prove the Enneper-Weierstrass Representation theorem and see some examples.

In $\mathbb{R}^3$, each oriented plane $\mathcal{P} \in G_{2,\mathbb{R}^m}$ is uniquely determined by the unit vector N such that it is perpendicular to $\mathcal{P}$ and the system $\{X, Y, N\}$ is a positive orthonormal basis of $\mathbb{R}^3$ for arbitrarily chosen positively oriented orthonormal basis $\{X, Y\}$ of $\mathcal{P}$. For an oriented surface in $\mathbb{R}^3$ the tangent plane is uniquely determined by the positively oriented unit normal vector. On the other hand, the unit sphere S^2 of all unit vectors in $\mathbb{R}^3$ is bijectively mapped onto the extended complex plane $\bar{\mathbb{C}} = \{\mathbb{C} \cup \infty\}$ by the stereographic projection $\bar{\omega}$.

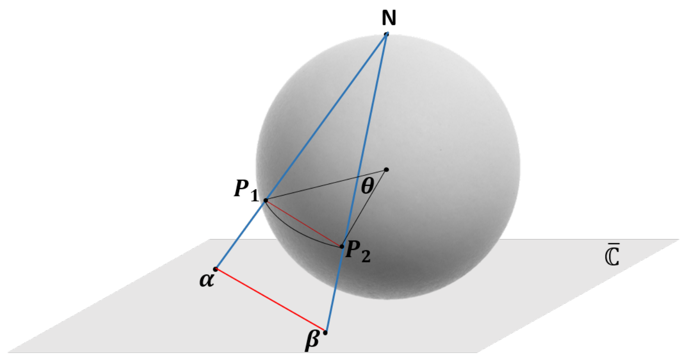
Definition 2.1.1. For a minimal surface M immersed in $\mathbb{R}^3$ the *classical Gauss map* $g : M \to \bar{\mathbb{C}}$ of M is defined as the map which maps each point $p \in M$ to the point $\bar{\omega}(N_p) \in \bar{\mathbb{C}}$, where N_p is the positively oriented unit normal vector N_p of M at p.

We begin by studying the stereographic projection $\bar{\omega}$. For an arbitrary point $(\xi, \eta, \zeta) \in S^2$ set $z = x + \sqrt{-1}y := \bar{\omega}(\xi, \eta, \zeta) \in \bar{\mathbb{C}}$, which means that, if $P \neq (0, 0, 1)$, then the points $\mathrm{N}(0, 0, 1), P(\xi, \eta, \zeta)$, and $P'(x, y, 0)$ are collinear and, otherwise, $z = \infty$. Using $\gamma(t) = \mathrm{N}+t(P-\mathrm{N}) := P'$, we obtain $x = \frac{\xi}{1-\zeta},\ y = \frac{\eta}{1-\zeta},\ \xi^2 + \eta^2 + \zeta^2 = 1$. Then by elementary calculation, we see

$$\xi = \frac{z + \bar{z}}{|z|^2 + 1},\ \eta = \sqrt{-1}\frac{\bar{z} - z}{|z|^2 + 1},\ \zeta = \frac{|z|^2 - 1}{|z|^2 + 1}. \tag{2.1}$$

For two points (ξ_1, η_1, ζ_1) and (ξ_2, η_2, ζ_2) in S^2 we denote by θ $(0 \leq \theta \leq \pi)$ the angle between two vectors $P_1(\xi_1, \eta_1, \zeta_1)$, $P_2(\xi_2, \eta_2, \zeta_2)$ and $\alpha = x_1 + \sqrt{-1}y_1 := \bar{\omega}(\xi_1, \eta_1, \zeta_1)$, $\beta = x_2 + \sqrt{-1}y_2 := \bar{\omega}(\xi_2, \eta_2, \zeta_2)$.

Figure 2.1: Chordal Distance



Define

$$|\alpha,\ \beta| = \sin\frac{\theta}{2} \quad (\leq 1).$$

Appling Law of Cosine, by definition we have

$$P_1P_2 = \sqrt{2 - 2\cos\theta} = 2\sin\frac{\theta}{2} = 2|\alpha,\ \beta|$$

Thus, geometrically $2|\alpha,\ \beta|$ is the chordal distance between P_1 and P_2. If $\alpha \neq \infty$ and $\beta \neq \infty$, by (2.1)

$$\begin{aligned}
&|\alpha,\ \beta| \\
&= \frac{1}{2}P_1P_2 = \frac{1}{2}\sqrt{(\xi_1-\xi_2)^2 + (\eta_1-\eta_2)^2 + (\zeta_1-\zeta_2)^2} \\
&= \frac{1}{2}\sqrt{2(1-\xi_1\xi_2-\eta_1\eta_2-\zeta_1\zeta_2)} \\
&= \frac{\sqrt{2}}{2}\sqrt{1-\left(\frac{\alpha+\overline{\alpha}}{|\alpha|^2+1}\right)\left(\frac{\beta+\overline{\beta}}{|\beta|^2+1}\right)+\left(\frac{\overline{\alpha}-\alpha}{|\alpha|^2+1}\right)\left(\frac{\overline{\beta}-\beta}{|\beta|^2+1}\right)-\left(\frac{|\alpha|^2-1}{|\alpha|^2+1}\right)\left(\frac{|\beta|^2-1}{|\beta|^2+1}\right)} \\
&= \frac{|\alpha-\beta|}{\sqrt{|\alpha|^2+1}\sqrt{|\beta|^2+1}}\ .
\end{aligned}$$

If $\beta = \infty$, then $(\xi_2, \eta_2, \zeta_2) = (0, 0, 1)$, so

$$\begin{aligned}
&|\alpha,\ \beta| \\
&= \frac{1}{2} P_1 P_2 = \frac{1}{2}\sqrt{(\xi_1)^2 + (\eta_1)^2 + (\zeta_1 - 1)^2} \\
&= \frac{1}{2}\sqrt{2(1-\zeta_1)} \\
&= \frac{\sqrt{2}}{2}\sqrt{1 - \frac{|\alpha|^2 - 1}{|\alpha|^2 + 1}} \\
&= \frac{1}{\sqrt{|\alpha|^2 + 1}}\ .
\end{aligned}$$

We define the chordal distance between α and β by $|\alpha,\ \beta|$. By definition, we see $0 \leq |\alpha,\ \beta| \leq 1$.

Now we take an arbitrary point $[w_1 : w_2 : w_3] \in \mathbb{Q}^1(\mathbb{C})$. Write $w_i = x_i - \sqrt{-1} y_i$ $(1 \leq i \leq 3)$ with $x_i,\ y_i \in \mathbb{R}$, and set

$$W := (w_1, w_2, w_3), \quad X := (x_1, x_2, x_3), \quad Y := (y_1, y_2, y_3).$$

Since $W \in \mathbb{Q}^1(\mathbb{C})$, we have

$$\begin{aligned}
w_1^2 + w_2^2 + w_3^2 = 0 &\Longleftrightarrow (x_1 - \sqrt{-1}y_1)^2 + (x_2 - \sqrt{-1}y_2)^2 + (x_3 - \sqrt{-1}y_3)^2 = 0 \\
&\Longleftrightarrow x_1^2 + x_2^2 + x_3^2 = y_1^2 + y_2^2 + y_3^2 \ \text{ and } \ x_1 y_1 + x_2 y_2 + x_3 y_3 = 0 \\
&\Longleftrightarrow |X| = |Y| \text{ and } \langle X, Y \rangle = 0.
\end{aligned}$$

Without loss of generality, we may assume that $|X| = |Y| = 1$. Then, the unit normal vector of the plane which has a positive basis $\{X, Y\}$ is given by

$$N := X \times Y = (x_2 y_3 - x_3 y_2,\ x_3 y_1 - x_1 y_3,\ x_1 y_2 - x_2 y_1) = \mathbf{Im}\{(w_2 \overline{w_3},\ w_3 \overline{w_1},\ w_1 \overline{w_2})\}$$

For the case where $w_1 \neq \sqrt{-1}w_2$, we assign to W the point

$$z = \frac{w_3}{w_1 - \sqrt{-1}w_2} \tag{2.2}$$

and, otherwise, the point $z = \infty$. This correspondence is continuous inclusively at ∞. To see this, we rewrite (2.2) as

$$z = \frac{w_3(w_1 + \sqrt{-1}w_2)}{w_1^2 + w_2^2} = -\frac{w_1 + \sqrt{-1}w_2}{w_3}.$$

If W tends to the point with $w_1 = \sqrt{-1}w_2$, z tends to ∞ because

$$w_1^2 + w_2^2 + w_3^2 = 0 \Leftrightarrow (\sqrt{-1}w_2)^2 + w_2^2 + w_3^2 = 0 \Leftrightarrow w_3 = 0.$$

If $w_1 \neq \sqrt{-1}w_2$, from

$$z = -\frac{w_1 + \sqrt{-1}w_2}{w_3} \text{ and } \frac{1}{z} = \frac{w_1 - \sqrt{-1}w_2}{w_3}$$

we have

$$\frac{w_1}{w_3} = \frac{1}{2}(\frac{1}{z} - z) \text{ and } \frac{w_2}{w_3} = \frac{\sqrt{-1}}{2}(\frac{1}{z} + z). \tag{2.3}$$

Since $|w_1|^2 + |w_2|^2 + |w_3|^2 = x_1^2 + y_1^2 + x_2^2 + y_2^2 + x_3^2 + y_3^2 = |X|^2 + |Y|^2 = 2$, we get

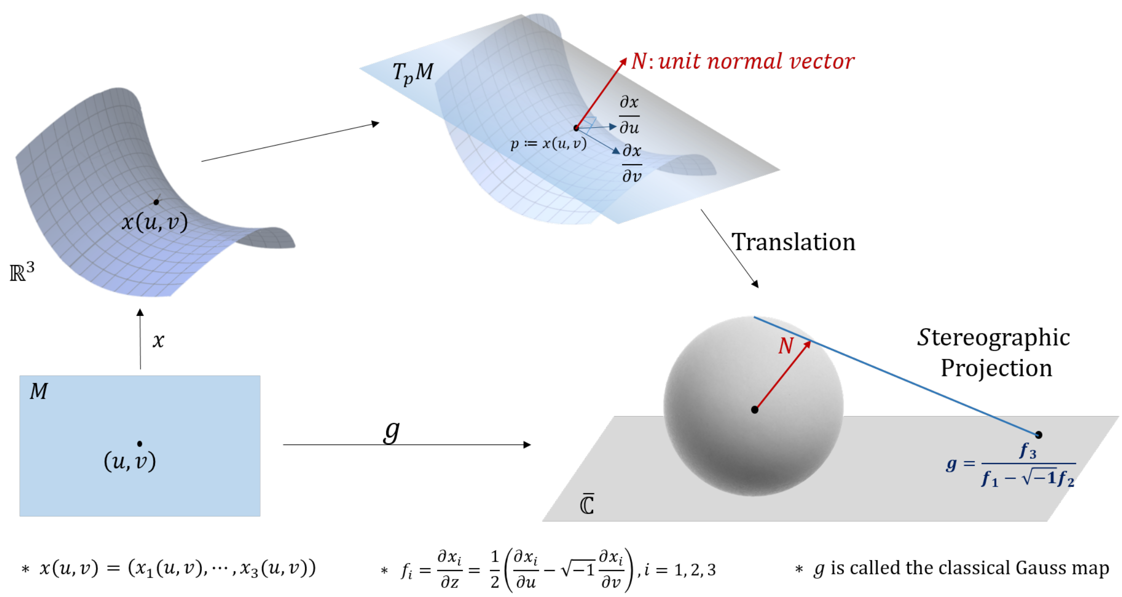
$$|w_3|^2 = \frac{2}{|\frac{w_1}{w_3}|^2 + |\frac{w_2}{w_3}|^2 + 1} = \frac{4|z|^2}{(|z|^2 + 1)^2}. \tag{2.4}$$

Then, (2.3) and (2.4) yield that

$$\begin{aligned} N &= \mathbf{Im}\{(w_2\overline{w_3},\ w_3\overline{w_1},\ w_1\overline{w_2})\} \\ &= |w_3|^2\,\mathbf{Im}\left\{\left(\frac{w_2}{w_3},\ \overline{\left(\frac{w_1}{w_3}\right)},\ \frac{w_1}{w_3}\overline{\left(\frac{w_2}{w_3}\right)}\right)\right\} \\ &= \left(\frac{2\,\mathbf{Re}\,z}{|z|^2+1},\ \frac{2\,\mathbf{Im}\,z}{|z|^2+1},\ \frac{|z|^2-1}{|z|^2+1}\right). \end{aligned}$$

By (2.1) this shows that the point in S^2 corresponding to $[w_1 : w_2 : w_3] \in \mathbb{Q}^1(\mathbb{C})$ is mapped to the point $z = w_3 \in \overline{\mathbb{C}}$ by the stereographic projection.

Figure 2.2: The Classical Gauss Map



Now we go back to the study of surfaces in $\mathbb{R}^3$. Let $x = (x_1, x_2, x_3) : M \to \mathbb{R}^3$ be a non-flat surface immersed in $\mathbb{R}^3$. Then its generalized Gauss map G is not a constant, and M may be considered as a Riemann surface (we shall discuss this more in the section 2.2) with a conformal metric ds^2. For a holomorphic local coordinate $z = u + \sqrt{-1}v$, G is represented as $G = [\phi_1 : \phi_2 : \phi_3] = [f_1 : f_2 : f_3]$, where

$$\phi_i = f_i dz = \frac{\partial x_i}{\partial z} dz. \tag{2.5}$$

Set

$$hdz = \phi_1 - \sqrt{-1}\phi_2 \ (\not\equiv 0) \text{ and } g := \frac{f_3}{f_1 - \sqrt{-1}f_2} \tag{2.6}$$

By the above discussion, the function g is the classical Gauss map of M.

Since the above correspondences are all biholomorphic, we obtain the following proposition which is the special case of Proposition 2.3.2. Refer to Proposition 2.3.2 for the proof of the following proposition.

Proposition 2.1.2. *For a surface M immersed in $\mathbb{R}^3$, M is a minimal surface if and only if the classical Gauss map is meromorphic on M.*

Now we explain the following Enneper-Weierstrass representaion theorem for minimal surfaces.

Theorem 2.1.3. (H. Fujimoto, [9]) *Let $x = (x_1, x_2, x_3) : M \to \mathbb{R}^3$ be a non-flat minimal surface immersed in $\mathbb{R}^3$. Consider the holomorphic forms $\phi_1, \phi_2, \phi_3, hdz$ and the meromorphic function g which is defined by (2.5) and (2.6) respectively. Then,*

(i) it holds that

$$\phi_1 = \frac{1}{2}(1-g^2)hdz,\ \phi_2 = \frac{\sqrt{-1}}{2}(1+g^2)hdz,\ \text{and } \phi_3 = ghdz \tag{2.7}$$

and

$$(x_1, x_2, x_3) = \left(2\,\mathbf{Re}\int_{z_0}^{z}\phi_1 + x_1(z_0),\ 2\,\mathbf{Re}\int_{z_0}^{z}\phi_2 + x_2(z_0),\ 2\,\mathbf{Re}\int_{z_0}^{z}\phi_3 + x_3(z_0)\right), \tag{2.8}$$

(ii) the metric induced from the standard metric on $\mathbb{R}^3$ is given by

$$ds^2 = (1+|g|^2)^2|h|^2|dz|^2, \tag{2.9}$$

(iii) the holomorphic form h has a zero of order $2k$ when and only when g has a pole of order k.

Proof. Consider the function f_i and h for a holomorphic local coordinate z. Obviously, $ghdz = \phi_3$. Since $f_1^2 + f_2^2 + f_3^2 = 0$, we have

$$\begin{aligned}
\frac{1}{2}(1-g^2)hdz &= \frac{1}{2}\left(1 - \left(\frac{f_3}{f_1 - \sqrt{-1}f_2}\right)^2\right)\left(f_1 - \sqrt{-1}f_2\right) dz \\
&= \frac{f_1^2 - f_2^2 - 2\sqrt{-1}f_1f_2 - f_3^2}{2(f_1 - \sqrt{-1}f_2)}dz \\
&= f_1 dz \\
&= \phi_1
\end{aligned}$$

Similarly,

$$\frac{\sqrt{-1}}{2}(1+g^2)hdz = \frac{\sqrt{-1}}{2}\frac{f_1^2 - f_2^2 - 2\sqrt{-1}f_1f_2 + f_3^2}{2(f_1 - \sqrt{-1}f_2)}dz = f_2dz = \phi_2$$

On the other hand, by (2.2), for $i = 1, 2, 3$, we have obtained

$$dx_i = 2\,\mathbf{Re}(\phi_i).$$

This implies $x_i(z) - x_i(z_0) = 2\mathbf{Re}\int_{z_0}^{z}\phi_i$, so the assertion (i) holds.

The assertion (ii) is shown by the direct calculations

$$\begin{aligned}
ds^2 &= 2(|f_1|^2 + |f_2|^2 + |f_3|^2)|dz|^2 \\
&= \frac{1}{2}\left(|1-g^2|^2 + |1+g^2|^2 + 4|g|^2\right)|h|^2|dz|^2 \\
&= \frac{1}{2}\left((1-g^2)(1-\overline{g}^2) + (1+g^2)(1+\overline{g}^2) + 4|g|^2\right)|h|^2|dz|^2 \\
&= \left(1+|g|^2\right)^2|h|^2|dz|^2
\end{aligned}$$

If h has a zero at a point p where g is holomorphic, then f_1, f_2, f_3 have a common zero at p, which contradicts the definition $[\phi_1 : \phi_2 : \phi_3] = [f_1 : f_2 : f_3]$. On the other hand, if g has a pole of order k at a point p, then h has a zero of exact order $2k$ at p. It is because otherwise some f_i has a pole or f_i's have a common zero, which contradicts the assumption $\phi_i = f_i dz$ $(i = 1, 2, 3)$ are holomorphic forms and the definition $[\phi_1 : \phi_2 : \phi_3] = [f_1 : f_2 : f_3]$. Thus, the assertion (iii) holds. Furthermore, from (iii), we see that the metric ds^2 in (ii) is continuous. Q.E.D.

Theorem 2.1.4. (H. Fujimoto, [9])*Let M be an open Riemann surface, hdz be a nonzero holomorphic form and g a nonconstant meromorphic function on M. Assume that hdz has a zero of order $2k$ when and only when g has a pole of order k and that the holomorphic forms ϕ_1, ϕ_2, ϕ_3 defined by (2.7) have no real periods. Then, for the functions x_1, x_2, x_3 defined by (2.8), the surface*

$$x = (x_1, x_2, x_3) : M \to \mathbb{R}^3$$

is a minimal surface immersed in $\mathbb{R}^3$ whose classical Gauss map is the map g and whose induced metric is given by (2.9).

Proof. We shall prove the general case ($\mathbb{R}^m$) of this theorem in Theorem 2.3.3.

Example 2.1.5. We regard a helicoid as Riemann surface $\mathbb{C}$, and by (2.7) it may be obtained from $g = e^z$ and $h = \frac{\sqrt{-1}}{2e^z}$ which give us

$$(\phi_1, \phi_2, \phi_3) = (\frac{-\sqrt{-1}}{2} \sinh z \, dz, \ \frac{-1}{2} \cosh z \, dz, \ \frac{\sqrt{-1}}{2} dz).$$

Now using (2.8) we compute $x = (x_1,\ x_2,\ x_3)$ in the following way.

$$
\begin{aligned}
x_1 &= 2\,\mathbf{Re}\int_{z_0}^{z} \frac{-\sqrt{-1}}{2}\sinh z\,dz + x_1(z_0) = \mathbf{Re}\int_{z_0}^{z} \frac{-\sqrt{-1}}{2}(e^z - e^{-z})dz + x_1(z_0)\\
&= \frac{1}{2}\,\mathbf{Re}(-\sqrt{-1}e^z - \sqrt{-1}e^{-z})\\
&= \frac{1}{2}\,\mathbf{Re}[-\sqrt{-1}e^u(\cos v + \sqrt{-1}\sin v) - \sqrt{-1}e^{-u}(\cos v - \sqrt{-1}\sin v)]\\
&= \frac{1}{2}\,(e^u - e^{-u})\sin v\\
&= \sinh u \cdot \sin v
\end{aligned}
$$

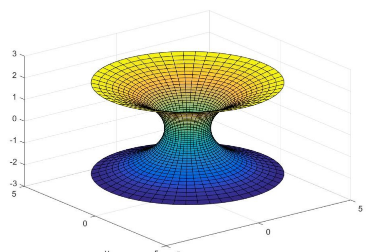
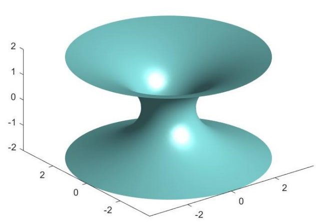
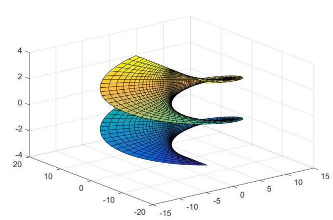
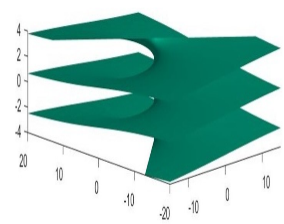
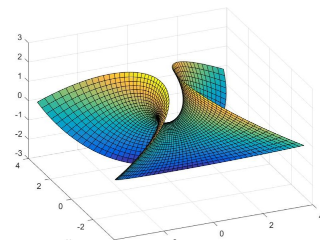
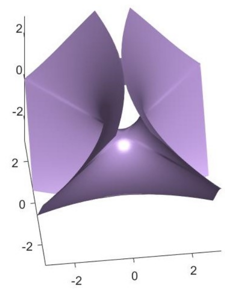
$$
\begin{aligned}
x_2 &= 2\,\mathbf{Re}\int_{z_0}^{z} \frac{-1}{2}\cosh z\,dz + x_2(z_0) = \mathbf{Re}\int_{z_0}^{z} \frac{-1}{2}(e^z + e^{-z})dz + x_2(z_0)\\
&= \frac{1}{2}\,\mathbf{Re}(e^{-z} - e^z) = \frac{1}{2}\,\mathbf{Re}[e^{-u}(\cos v - \sqrt{-1}\sin v) - e^u(\cos v + \sqrt{-1}\sin v)]\\
&= \frac{1}{2}\,(e^{-u} - e^u)\cos v\\
&= -\sinh u \cdot \cos v
\end{aligned}
$$

$$
\begin{aligned}
x_3 &= 2\,\mathbf{Re}\int_{z_0}^{z} \frac{\sqrt{-1}}{2}dz\\
&= \mathbf{Re}(\sqrt{-1}z)\\
&= \mathbf{Re}(\sqrt{-1}u - v)\\
&= -v
\end{aligned}
$$

Thus, we have obtained the helicoid $x(u,v) = (\sinh u \cdot \sin v,\ -\sinh u \cdot \cos v,\ -v)$.

The following picture shows the graph of the helicoid and more examples.

Table 2.1: Non-flat complete minimal surfaces immersed in $\mathbb{R}^3$

Catenoid	**Helicoid**	**Enneper's Surface**
 	 	 
$M = \mathbb{C} \setminus \{\infty\}$ $g = e^{\sqrt{-1}z},\ h = \frac{-\sqrt{-1}}{2e^{\sqrt{-1}z}}$ $\phi_1 = \frac{-1}{2} \sin z\, dz$ $\phi_2 = \frac{1}{2} \cos z\, dz$ $\phi_3 = -\frac{\sqrt{-1}}{2} dz$	$M = \mathbb{C}$ $g = e^z,\ h = \frac{\sqrt{-1}}{2e^z}$ $\phi_1 = \frac{-\sqrt{-1}}{2} \sinh z\, dz$ $\phi_2 = \frac{-1}{2} \cosh z\, dz$ $\phi_3 = \frac{\sqrt{-1}}{2} dz$	$M = \mathbb{C}$ $g = z,\ h = 1$ $\phi_1 = \frac{1}{2}(1 - z^2)dz$ $\phi_2 = \frac{\sqrt{-1}}{2}(1 + z^2)dz$ $\phi_3 = zdz$
$x_1 = \cos u \cosh v$ $x_2 = \sin u \cosh v$ $x_3 = v$	$x_1 = \sinh u \cdot \sin v$ $x_2 = -\sinh u \cdot \cos v$ $x_3 = -v$	$x_1 = u - \frac{u^3}{3} + uv^2$ $x_2 = -v + \frac{v^3}{3} - u^2 v$ $x_3 = u^2 - v^2$

2.2 Minimal Surfaces in $\mathbb{R}^m$

Let M be an oriented real 2-dimensional differentiable manifold immersed in $\mathbb{R}^m$ and $x = (x_1, \cdots, x_m) : M \to \mathbb{R}^m$ be an immersion. For a point $p \in M$, take a local coordinate system (u_1, u_2) around p which is positively oriented. The tangent plane of M at p is given by

$$T_p(M) = \left\{ \lambda \frac{\partial x}{\partial u_1}\Big|_p + \mu \frac{\partial x}{\partial u_2}\Big|_p \;:\; \lambda,\ \mu \in \mathbb{R} \right\}$$

and the normal space of M at p is given by

$$N_p(M) = \left\{ N \;:\; \left\langle N,\ \frac{\partial x}{\partial u_1}\Big|_p \right\rangle \;=\; \left\langle N,\ \frac{\partial x}{\partial u_2}\Big|_p \right\rangle = 0 \right\}$$

where $\langle X, Y \rangle$ denotes the inner product of vectors X and Y. The metric ds^2 on M induced from the standard metric on $\mathbb{R}^m$ is called the first fundamental form on M and given by

$$\begin{aligned} ds^2 &= |dx|^2 \\ &= \langle dx, dx \rangle \\ &= \left\langle \frac{\partial x}{\partial u_1} du_1 + \frac{\partial x}{\partial u_2} du_2,\ \frac{\partial x}{\partial u_1} du_1 + \frac{\partial x}{\partial u_2} du_2 \right\rangle \\ &= g_{11} du_1^2 + 2 g_{12} du_1 du_2 + g_{22} du_2^2 \end{aligned}$$

where $g_{ij} = \left\langle \frac{\partial x}{\partial u_i}, \frac{\partial x}{\partial u_j} \right\rangle$ for $1 \leq i,\ j \leq 2$, and the second fundamental form of M with respect to a unit normal vector N is given by

$$d\sigma^2 = b_{11}(N) du_1^2 + 2 b_{12}(N) du_1 du_2 + b_{22}(N) du_2^2$$

where $b_{ij}(N) = \left\langle \frac{\partial^2 x}{\partial u_i \partial u_j}, N \right\rangle$ for $1 \leq i,\ j \leq 2$.

Then the mean curvature of M for the normal direction N at p is defined by

$$H_p(N) = \frac{g_{11}b_{22}(N) + g_{22}b_{11}(N) - 2g_{12}b_{12}(N)}{2(g_{11}g_{22} - g_{12}^2)}.$$

Definition 2.2.1. A surface M is called a *minimal surface* in $\mathbb{R}^m$ if $H_p(N) = 0$ for all $p \in M$ and $N \in N_p(M)$.

Definition 2.2.2. A local coordinate system (u_1, u_2) on an open set U in M is called *isothermal* on U if ds^2 can be represented as

$$ds^2 = \lambda^2(du_1^2 + du_2^2)$$

for a positive smooth function λ on U. This means that $\lambda^2 := g_{11} = g_{22}$ and $g_{12} = 0$.

Theorem 2.2.3. (S. S. Chern, [3]) *For every surface M, there is a system of isothermal local coordinates whose domains cover the whole M.*

Propositon 2.2.4. (H. Fujimoto, [9]) *For an oriented surface M with a metric ds^2, if we take two positively oriented isothermal local coordinate (x, y) and (u, v), then $w = u + \sqrt{-1}v$ is a holomorphic function in $z = x + \sqrt{-1}y$ on the common domain.*

Proof. By assumption, there exists a positive differentiable function λ such that $du^2 + dv^2 = \lambda^2(dx^2 + dy^2)$. Then, we have

$$\begin{aligned}
&du^2 + dv^2 = \lambda^2(dx^2 + dy^2) \\
\Longleftrightarrow\ &(u_x dx + u_y dy)^2 + (v_x dx + v_y dy)^2 = \lambda^2(dx^2 + dy^2) \\
\Longleftrightarrow\ &(u_x^2 + v_x^2)dx^2 + 2(u_x u_y + v_x v_y)dxdy + (u_y^2 + v_y^2)dy^2 = \lambda^2(dx^2 + dy^2) \\
\Longleftrightarrow\ &A := u_x^2 + v_x^2 = u_y^2 + v_y^2,\ u_x u_y + v_x v_y = 0
\end{aligned}$$

This means that the Jacobi matrix

$$\mathbf{J} := \begin{pmatrix} u_x & v_x \\ u_y & v_y \end{pmatrix}$$

satisfies the identity

$$\mathbf{J}\ {}^t\mathbf{J} = \begin{pmatrix} u_x^2 + v_x^2 & u_x u_y + v_x v_y \\ u_x u_y + v_x v_y & u_y^2 + v_y^2 \end{pmatrix} = \begin{pmatrix} u_x^2 + v_x^2 & 0 \\ 0 & u_x^2 + v_x^2 \end{pmatrix} = A\,\mathbf{I}_2$$

where $\mathbf{I}_2$ is the unit matrix of degree 2. We then have $\mathbf{J}^{-1} = \frac{1}{A}\,{}^t\mathbf{J}$ and

$$\mathbf{J}^{-1} = \frac{1}{det(\mathbf{J})} \begin{pmatrix} v_y & -v_x \\ -u_y & u_x \end{pmatrix} = \frac{1}{A} \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}.$$

On the other hand, since $[det(\mathbf{J})]^2 = det(\mathbf{J}\,{}^t\mathbf{J}) = det(A\,\mathbf{I}_2) = A^2$ and $det(\mathbf{J}) > 0$, we have $det(\mathbf{J}) = A$. These imply that $u_x = v_y$ and $u_y = -v_x$. Therefore, the function $w = u + \sqrt{-1}v$ is holomorphic in z. Q.E.D.

Let $x = (x_1, \cdots, x_m) : M \to \mathbb{R}^m$ be an oriented surface with a Riemannian metric ds^2. With each isothermal local coordinate (u, v), we associate the complex function $z = u + \sqrt{-1}v$. Let $\bar{z} = u - \sqrt{-1}v$. Notice that

$$u = \frac{z + \bar{z}}{2} \text{ and } v = \frac{z - \bar{z}}{2\sqrt{-1}}.$$

Then, $x(u, v) = (x_1(u, v), \cdots, x_m(u, v))$ may be written as

$$x(z, \bar{z}) = (x_1(z, \bar{z}), x_2(z, \bar{z}), \cdots, x_m(z, \bar{z})).$$

By Proposition 2.2.4, we can see that the surface M has a complex structure, and these complex valued functions define holomorphic local coordinates on M. Thus, we may regard M as a Riemann surface. By the use of complex differentiations

$$\frac{\partial x_i}{\partial z} = \frac{1}{2}\left(\frac{\partial x_i}{\partial u} - \sqrt{-1}\frac{\partial x_i}{\partial v}\right) \text{ and } \frac{\partial x_i}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial x_i}{\partial u} + \sqrt{-1}\frac{\partial x_i}{\partial v}\right)$$

we have

$$\left|\frac{\partial x_i}{\partial z}\right|^2 = \frac{1}{4}\left|\frac{\partial x_i}{\partial u} - \sqrt{-1}\frac{\partial x_i}{\partial v}\right|^2 = \frac{1}{4}\left[\left(\frac{\partial x_i}{\partial u}\right)^2 + \left(\frac{\partial x_i}{\partial v}\right)^2\right].$$

From this property, we obtain that

$$\begin{aligned}\left|\frac{\partial x_1}{\partial z}\right|^2 + \cdots + \left|\frac{\partial x_m}{\partial z}\right|^2 &= \frac{1}{4}\sum_{i=1}^{m}\left[\left(\frac{\partial x_i}{\partial u}\right)^2 + \left(\frac{\partial x_i}{\partial v}\right)^2\right] \\ &= \frac{1}{4}\left[\left\langle\frac{\partial x}{\partial u}, \frac{\partial x}{\partial u}\right\rangle + \left\langle\frac{\partial x}{\partial v}, \frac{\partial x}{\partial v}\right\rangle\right] \\ &= \frac{1}{2}\left\langle\frac{\partial x}{\partial u}, \frac{\partial x}{\partial u}\right\rangle\end{aligned} \tag{2.10}$$

and

$$|dz|^2 = dz \cdot d\bar{z} = (du + \sqrt{-1}dv)(du - \sqrt{-1}dv) = du^2 + dv^2.$$

Then, by (2.10) we may rewrite

$$\begin{aligned}ds^2 &= \lambda^2(du^2 + dv^2) \\ &= \left\langle\frac{\partial x}{\partial u}, \frac{\partial x}{\partial u}\right\rangle(du^2 + dv^2) \\ &= 2\left(\left|\frac{\partial x_1}{\partial z}\right|^2 + \left|\frac{\partial x_2}{\partial z}\right|^2 + \cdots + \left|\frac{\partial x_m}{\partial z}\right|^2\right)|dz|^2.\end{aligned}$$

Let $\lambda_z^2 = 2\left(\left|\frac{\partial x_1}{\partial z}\right|^2 + \left|\frac{\partial x_2}{\partial z}\right|^2 + \cdots + \left|\frac{\partial x_m}{\partial z}\right|^2\right)$. Then M may be considered as a Riemann surface with a metric $ds^2 = \lambda_z^2|dz|^2$ where λ_z is a positive C^∞ function in terms of a holomorphic local coordinate z.

We denote by $\Delta = \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}$, the Laplacian in terms of the holomorphic local coordinate $z = u + \sqrt{-1}v$.

Proposition 2.2.5. (H. Fujimoto, [9]) *For each isothermal local coordinate* (u, v), *we have the following property:*

$$(i)\, \langle \Delta x, X\rangle = 0 \text{ for each } X \in T_p(M)$$

$$(ii)\, \langle \Delta x, N\rangle = 2\lambda^2 H(N) \text{ for each } N \in N_p(M)$$

Proof. By the assumption, we have

$$\lambda^2 = \left\langle \frac{\partial x}{\partial u}, \frac{\partial x}{\partial u} \right\rangle = \left\langle \frac{\partial x}{\partial v}, \frac{\partial x}{\partial v} \right\rangle, \quad \left\langle \frac{\partial x}{\partial u}, \frac{\partial x}{\partial v} \right\rangle = 0$$

Defferentiating these identities, we have

$$\left\langle \frac{\partial^2 x}{\partial u^2}, \frac{\partial x}{\partial u} \right\rangle = \left\langle \frac{\partial^2 x}{\partial u \partial v}, \frac{\partial x}{\partial v} \right\rangle, \quad \left\langle \frac{\partial^2 x}{\partial v \partial u}, \frac{\partial x}{\partial v} \right\rangle + \left\langle \frac{\partial x}{\partial u}, \frac{\partial^2 x}{\partial v^2} \right\rangle = 0$$

These imply

$$\left\langle \Delta x, \frac{\partial x}{\partial u} \right\rangle = \left\langle \frac{\partial^2 x}{\partial u^2}, \frac{\partial x}{\partial u} \right\rangle + \left\langle \frac{\partial^2 x}{\partial v^2}, \frac{\partial x}{\partial u} \right\rangle = \left\langle \frac{\partial^2 x}{\partial v \partial u}, \frac{\partial x}{\partial v} \right\rangle + \left\langle \frac{\partial x}{\partial u}, \frac{\partial^2 x}{\partial v^2} \right\rangle = 0$$

By similar method, we have

$$\left\langle \Delta x, \frac{\partial x}{\partial v} \right\rangle = 0$$

Since $\frac{\partial x}{\partial u}$ and $\frac{\partial x}{\partial v}$ generate the tangent plane, we write $T_p(M) = \{a\frac{\partial x}{\partial u} + b\frac{\partial x}{\partial v} |\ a, b \in \mathbb{R}\}$. For each $X = a\frac{\partial x}{\partial u} + b\frac{\partial x}{\partial v} \in T_p(M)$ we have

$$\langle \Delta x, X\rangle = \left\langle \Delta x, a\frac{\partial x}{\partial u} + b\frac{\partial x}{\partial v} \right\rangle = a\left\langle \Delta x, \frac{\partial x}{\partial u} \right\rangle + b\left\langle \Delta x, \frac{\partial x}{\partial v} \right\rangle = 0$$

as desired in (i). On the other hand, for every normal vector N to M it holds that

$$H(N) = \frac{b_{11}(N) + b_{22}(N)}{2\lambda^2} = \frac{\left\langle \frac{\partial^2 x}{\partial u^2}, N \right\rangle + \left\langle \frac{\partial^2 x}{\partial v^2}, N \right\rangle}{2\lambda^2} = \frac{\langle \Delta x, N\rangle}{2\lambda^2}$$

because $\lambda^2 = g_{11} = g_{22}$ and $g_{12} = 0$. This shows (ii). Q.E.D

Theorem 2.2.6. (H. Fujimoto, [9]) *Let $x = (x_1, \cdots, x_m) : M \to \mathbb{R}^m$ be a surface immersed in $\mathbb{R}^m$, which is considered as a Riemann surface as above. Then, M is minimal if and only if each x_i is a harmonic function on M, namely,*

$$\Delta x_i = \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) x_i = 0$$

for every holomorphic local coordinate $z = u + \sqrt{-1}v$.

Proof. $\Rightarrow$) Assume that M is a minimal surface. Then, by definition $H(N) = 0$. Now we apply proposition 2.2.5 to get

$$\langle \Delta x, N \rangle = 2\lambda^2 H(N) = 0.$$

Since each $N \in N_p(M)$ is not equal to the zero vector, we have $\Delta x_i = 0$. Thus, each x_i is a harmonic function on M.

$\Leftarrow$) Assume that each x_i is a harmonic function on M. Then, by definition, each $\Delta x_i = 0$. By Proposition 2.2.5, we have $0 = \Delta x = 2\lambda^2 H(N)$. Since $\lambda^2 \neq 0$, we have $H(N) = 0$. Therefore, M is a minimal surface. Q.E.D.

Corollary 2.2.7. (H. Fujimoto, [9]) *There exists no compact minimal surface without boundary in $\mathbb{R}^m$.*

Proof. For a minimal surface $x = (x_1, \cdots, x_m) : M \to \mathbb{R}^m$ immersed in $\mathbb{R}^m$, if M is compact, then each x_i takes the maximum value at a point in M. By the maximum principle of harmonic functions, x_i is a constant. This is impossible because x is an immersion. Q.E.D.

By the uniformization theorem, a simply connected Riemann surface is conformally equivalent to either the sphere S^2, the complex plane $\mathbb{C}$, or the unit disk $\Delta = \{z : |z| < 1\}$. Because of Corollary 2.2.7. the first case is excluded, and we obtain the following corollary.

Corollary 2.2.8. *For a minimal surface $x = (x_1, \cdots, x_m) : M \to \mathbb{R}^m$ immersed in $\mathbb{R}^m$, if M is simply connected, then M is conformally equivalent to the complex plane or the unit disk.*

2.3 The Generalized Gauss Map of Minimal Surfaces in $\mathbb{R}^m$

First, we consider the set of all oriented 2-dimensional planes in $\mathbb{R}^m$ which contain the origin and denote it by $G_{2,\mathbb{R}^m}$. To clarify the set $G_{2,\mathbb{R}^m}$, we regard it as a subspace of the $(m-1)$-dimensional complex projective space $\mathbb{P}^{m-1}(\mathbb{C})$ as following. To each $P \in G_{2,\mathbb{R}^m}$, taking a positively oriented basis $\{X, Y\}$ of P such that $|X| = |Y|$, $\langle X, Y \rangle = 0$, we assign the point $\Phi(P) = \pi(X - \sqrt{-1}Y)$ where π denotes the cannonical projection of $\mathbb{C}^m \setminus \{0\}$ onto $\mathbb{P}^{m-1}(\mathbb{C})$, namely, the map which maps each $P = (w_1, \cdots, w_m) \neq (0, \cdots, 0)$ to the equivalence class

$$[w_1 : \cdots : w_m] := \{(cw_1, \cdots, cw_m)|\ c \in \mathbb{C} \setminus \{0\}\}.$$

For another positive basis $\{\tilde{X}, \tilde{Y}\}$ of P satisfying $|\tilde{X}| = |\tilde{Y}|$ and $\left\langle \tilde{X}, \tilde{Y} \right\rangle = 0$, we can find a real number θ such that

$$\tilde{X} = r(X \cos\theta + Y \sin\theta)$$
$$\tilde{Y} = r(-X \sin\theta + Y \cos\theta)$$

where $r := |\tilde{X}|/|X| = |\tilde{Y}|/|Y|$. Therefore, we can write

$$\tilde{X} - \sqrt{-1}\tilde{Y} = re^{i\theta}(X - \sqrt{-1}Y).$$

This shows that the value $\Phi(P)$ does not depend on the choice of a positive basis of P but only on P. Since

$$\left\langle X - \sqrt{-1}Y,\ X - \sqrt{-1}Y \right\rangle = |X|^2 - 2\sqrt{-1}\left\langle X,\ Y \right\rangle - |Y|^2 = 0,$$

we have $w_1^2 + \cdots + w_m^2 = 0$ via Φ. Thus, $\Phi(P)$ is contained in the quadric

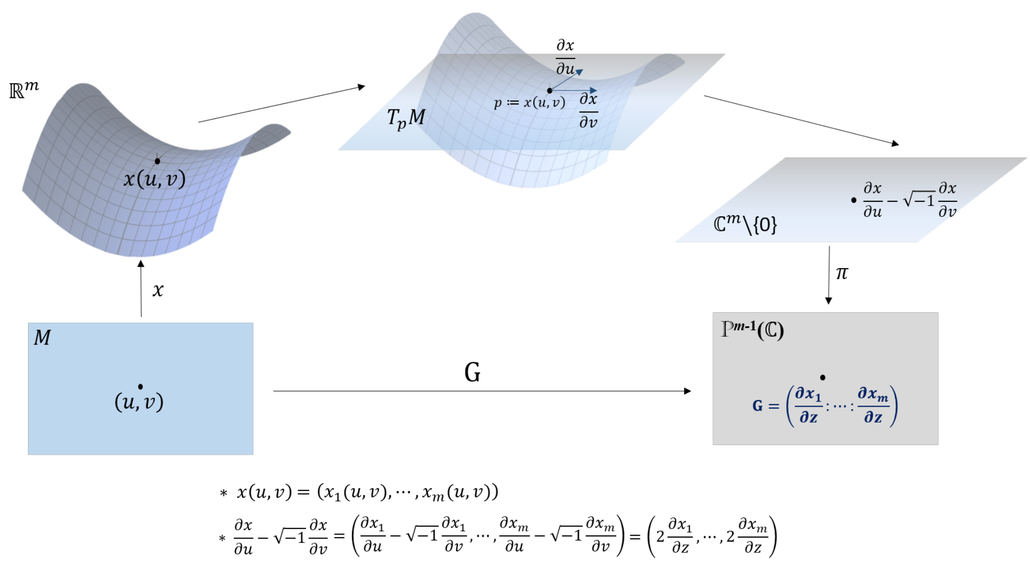
$$\mathbb{Q}^{n-2}(\mathbb{C}) := \{[w_1 : \cdots : w_m] |\ w_1^2 + \cdots + w_m^2 = 0\} \subset \mathbb{P}^{m-1}(\mathbb{C}).$$

Conversely, take an arbitrary point $Q \in \mathbb{Q}^{n-2}(\mathbb{C})$. If we choose some $W \in \mathbb{C}^m \setminus \{0\}$ with $\pi(W) = Q$ and write $W = X - \sqrt{-1}Y$ with real vectors X and Y, then X and Y satisfy the conditions $|X| = |Y|$, $\langle X,\ Y \rangle = 0$. It is easily seen that there is a unique oriented 2-dimensional plane W such that $\Phi(W) = Q$. This shows that Φ is bijective. Therefore, in the following sections, there will be no confusion if we identify the set of all oriented 2-dimensional planes in $\mathbb{R}^m$, $G_{2,\mathbb{R}^m}$, with $\mathbb{Q}^{n-2}(\mathbb{C})$.

Now, we consider a surface $x = (x_1, \cdots, x_m) : M \to \mathbb{R}^m$ immersed in $\mathbb{R}^m$. For each point $p \in M$, the oriented tangent plane $T_p(M)$ is canonically identified with an element of $G_{2,\mathbb{R}^m}$ after the parallel translation which maps p to the origin.

Definition 2.3.1. The *(generalized) Gauss map* of a surface M is identified as the map of M into $\mathbb{Q}^{n-2}(\mathbb{C})$ which maps each point $p \in M$ to $\Phi(T_p(M))$.

Figure 2.3: The Generalized Gauss Map



For a positively oriented isothermal local coordinate (u, v), the vectors

$$X = \frac{\partial x}{\partial u},\ Y = \frac{\partial x}{\partial v}$$

give a positive basis of $T_p(M)$ satisfying the conditions $|X| = |Y|$, $\langle X,\ Y\rangle = 0$. Therefore, the Gauss map G is locally given by

$$\mathsf{G} = \pi(X - \sqrt{-1}Y) = \left[\frac{\partial x_1}{\partial z} : \cdots : \frac{\partial x_m}{\partial z}\right] \tag{2.11}$$

where $z = u + \sqrt{-1}v$. Take a reduced representation $G = \left(\frac{\partial x_1}{\partial z}, \cdots, \frac{\partial x_m}{\partial z}\right)$. Then we have

$$ds^2 = 2\left(\left|\frac{\partial x_1}{\partial z}\right|^2 + \left|\frac{\partial x_2}{\partial z}\right|^2 + \cdots + \left|\frac{\partial x_m}{\partial z}\right|^2\right)|dz|^2 = 2|G|^2|dz|^2$$

We may write $\mathsf{G} = [\phi_1 : \cdots : \phi_m]$ with globally defined holomorphic 1-forms $\phi_i = \frac{\partial x_i}{\partial z} dz,\ 1 \leq i \leq m$.

Proposition 2.3.2. (H. Fujimoto, [9]) *A surface $x = (x_1, \cdots, x_m) : M \to \mathbb{R}^m$ is minimal if and only if the Gauss map $\mathsf{G} : M \to \mathbb{P}^{m-1}(\mathbb{C})$ is holomorphic.*

Proof.$\Rightarrow$) Assume that M is minimal. Then, from $\frac{\partial x}{\partial z} = \frac{1}{2}\left(\frac{\partial x}{\partial u} - \sqrt{-1}\frac{\partial x}{\partial v}\right)$ and $\frac{\partial x}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial x}{\partial u} + \sqrt{-1}\frac{\partial x}{\partial v}\right)$ we have

$$\frac{\partial}{\partial \bar{z}}\left(\frac{\partial x}{\partial z}\right) = \frac{1}{2}\left[\frac{1}{2}\frac{\partial}{\partial u}\left(\frac{\partial x}{\partial u} - \sqrt{-1}\frac{\partial x}{\partial v}\right) + \frac{1}{2}\sqrt{-1}\frac{\partial}{\partial v}\left(\frac{\partial x}{\partial u} - \sqrt{-1}\frac{\partial x}{\partial v}\right)\right] = \frac{1}{4}\Delta x = 0$$

by Theorem 2.2.6. This shows that $\frac{\partial x}{\partial z}$ satisfies Cauchy-Riemann equation. Hence, the Gauss map G is holomorphic.

$\Leftarrow$) Assume that G is holomorphic. For a holomorphic local coordinate z we set $f_i = \frac{\partial x_i}{\partial z}$ $(1 \leq i \leq m)$. After a suitable change of indices, we may assume that f_m has no zero. Since $\frac{f_i}{f_m}$ are holomorphic, we have

$$\begin{aligned} \frac{1}{4}\Delta x_i &= \frac{\partial}{\partial \bar{z}}\left(\frac{\partial x_i}{\partial z}\right) = \frac{\partial}{\partial \bar{z}}\left(\frac{f_i}{f_m} \cdot f_m\right) = \left[\frac{\partial}{\partial \bar{z}}\left(\frac{f_i}{f_m}\right)\right] f_m + \frac{f_i}{f_m}\left[\frac{\partial f_m}{\partial \bar{z}}\right] \\ &= \frac{f_i}{f_m}\left[\frac{\partial f_m}{\partial \bar{z}}\right] = f_i \cdot \frac{1}{f_m}\left[\frac{\partial f_m}{\partial \bar{z}}\right] \end{aligned}$$

for $i = 1, 2, \cdots, m$. Write

$$\frac{1}{f_m}\left[\frac{\partial f_m}{\partial \bar{z}}\right] = h_1 + \sqrt{-1}h_2$$

with real-valued functions $h_1,\ h_2$. Then we have

$$\Delta x_i = 4 \cdot f_i \cdot \frac{1}{f_m}\left[\frac{\partial f_m}{\partial \bar{z}}\right] = 4 \cdot \frac{\partial x_i}{\partial z} \cdot (h_1 + \sqrt{-1}h_2) = 4 \cdot \frac{1}{2}\left(\frac{\partial x}{\partial u} - \sqrt{-1}\frac{\partial x}{\partial v}\right) \cdot (h_1 + \sqrt{-1}h_2)$$

Take the real parts of both sides of the above equation to see

$$\Delta x = 2\left(\frac{\partial x}{\partial u}h_1 + \frac{\partial x}{\partial v}h_2\right) \in T_p(M).$$

According to Proposition 2.2.5. (i), we obtain $\langle \Delta x,\ \Delta x \rangle = 0$ because $\Delta x \in T_p(M)$ by the above. Thus, we get $\Delta x = 0$. This implies that M is a minimal surface by Theorem 2.2.6. Q.E.D.

We say that a holomorphic form ϕ on a Riemann surface M has no real period if

$$\mathbf{Re} \int_\gamma \phi = 0$$

for every closed curve γ in M.

Since

$$\begin{aligned}
dx_i &= \frac{\partial x_i}{\partial z} dz + \frac{\partial x_i}{\partial \bar{z}} d\bar{z} \\
&= \frac{1}{2}\left(\frac{\partial x_i}{\partial u} - \sqrt{-1}\frac{\partial x_i}{\partial v}\right)(du + \sqrt{-1}dv) + \frac{1}{2}\left(\frac{\partial x_i}{\partial u} + \sqrt{-1}\frac{\partial x_i}{\partial v}\right)(du - \sqrt{-1}dv) \\
&= \frac{\partial x_i}{\partial u} du + \frac{\partial x_i}{\partial v} dv \\
&= 2\ \mathbf{Re}\left[\frac{1}{2}\left(\frac{\partial x_i}{\partial u} - \sqrt{-1}\frac{\partial x_i}{\partial v}\right)(du + \sqrt{-1}dv)\right] \\
&= 2\ \mathbf{Re}\left(\frac{\partial x_i}{\partial z} dz\right) \\
&= 2\ \mathbf{Re}(\phi_i),
\end{aligned} \tag{2.12}$$

we have

$$x_i(z) = 2\ \mathbf{Re} \int_{\gamma_{z_0}^z} \phi_i + x_i(z_0)$$

for a piecewise smooth curve $\gamma_{z_0}^z$ in M joining z_0 and z. If ϕ has no real period, then the quantity

$$x(z) = 2\ \mathbf{Re} \int_{\gamma_{z_0}^z} \phi + x(z_0)$$

depends only on z, and so x is a well-defined function of z on M, which we denote by

$$x(z) = 2\,\mathbf{Re}\int_{z_0}^{z}\phi + x(z_0)$$

from here on. Related to Proposition 2.3.2, we state here the following construction theorem of minimal surfaces.

Theorem 2.3.3. (H. Fujimoto, [9]) *Let M be an open Riemann surface and let $\phi_1, \phi_2, \cdots, \phi_m$ be holomorphic forms on M such that they have no common zero, no real periods and locally satisfy the identity*

$$\left(\frac{\partial x_1}{\partial z}\right)^2 + \left(\frac{\partial x_2}{\partial z}\right)^2 + \cdots + \left(\frac{\partial x_m}{\partial z}\right)^2 = 0 \tag{2.13}$$

for holomorphic functions $\frac{\partial x_i}{\partial z}$ with $\phi_i = \frac{\partial x_i}{\partial z}dz$. Set

$$x_i = 2\,\mathbf{Re}\int_{z_0}^{z}\phi_i + x_i(z_0)$$

for an arbitrarily fixed point z_0 of M. Then, the surface $x = (x_1, \cdots, x_m) : M \to \mathbb{R}^m$ is a minimal surface immersed in $\mathbb{R}^m$ such that the Gauss map is the map $\mathsf{G} = \left[\frac{\partial x_1}{\partial z} : \cdots : \frac{\partial x_m}{\partial z}\right] : M \to \mathbb{Q}^{m-2}(\mathbb{C})$ and the induced metric is given by

$$ds^2 = 2\left(\left|\frac{\partial x_1}{\partial z}\right|^2 + \left|\frac{\partial x_2}{\partial z}\right|^2 + \cdots + \left|\frac{\partial x_m}{\partial z}\right|^2\right)|dz|^2.$$

Proof. By the assumption, the x_i are well-defined single-valued functions on M.

Consider the map $x = (x_1, \cdots, x_m) : M \to \mathbb{R}^m$. By (2.13) we have

$$\begin{aligned}\left\langle \frac{\partial x}{\partial u}, \frac{\partial x}{\partial u} \right\rangle - 2\sqrt{-1}\left\langle \frac{\partial x}{\partial u}, \frac{\partial x}{\partial v} \right\rangle - \left\langle \frac{\partial x}{\partial v}, \frac{\partial x}{\partial v} \right\rangle &= \left\langle \frac{\partial x}{\partial u} - \sqrt{-1}\frac{\partial x}{\partial v}, \ \frac{\partial x}{\partial u} - \sqrt{-1}\frac{\partial x}{\partial v} \right\rangle \\ &= \left\langle 2\,\frac{\partial x}{\partial z}, 2\,\frac{\partial x}{\partial z} \right\rangle \\ &= 4\left[\left(\frac{\partial x_1}{\partial z}\right)^2 + \left(\frac{\partial x_2}{\partial z}\right)^2 + \cdots + \left(\frac{\partial x_m}{\partial z}\right)^2 \right] \\ &= 0\end{aligned}$$

for $z = u + \sqrt{-1}v$. This gives that

$$\left\langle \frac{\partial x}{\partial u}, \frac{\partial x}{\partial u} \right\rangle = \left\langle \frac{\partial x}{\partial v}, \frac{\partial x}{\partial v} \right\rangle, \ \left\langle \frac{\partial x}{\partial u}, \frac{\partial x}{\partial v} \right\rangle = 0$$

Moreover, By (2.10)

$$\begin{aligned}\sum_{i<j} \left| \frac{\partial(x_i, x_j)}{\partial(u, v)} \right|^2 &= \left\langle \frac{\partial x}{\partial u}, \frac{\partial x}{\partial u} \right\rangle \left\langle \frac{\partial x}{\partial v}, \frac{\partial x}{\partial v} \right\rangle - \left\langle \frac{\partial x}{\partial u}, \frac{\partial x}{\partial v} \right\rangle^2 = \left\langle \frac{\partial x}{\partial u}, \frac{\partial x}{\partial u} \right\rangle^2 \\ &= 4\left(\left|\frac{\partial x_1}{\partial z}\right|^2 + \left|\frac{\partial x_2}{\partial z}\right|^2 + \cdots + \left|\frac{\partial x_m}{\partial z}\right|^2 \right)^2 > 0\end{aligned}$$

Hence, x is an immersion. Then, the induced metric is given by

$$ds^2 = \left\langle \frac{\partial x}{\partial u}, \frac{\partial x}{\partial u} \right\rangle (du^2 + dv^2) = 2\left(\left|\frac{\partial x_1}{\partial z}\right|^2 + \left|\frac{\partial x_2}{\partial z}\right|^2 + \cdots + \left|\frac{\partial x_m}{\partial z}\right|^2 \right) |dz|^2$$

and (u, v) gives a system of isothermal coordinates for the induced metric ds^2. On the other hand, by (2.11) the Gauss map G of M is given by $\mathsf{G} = \left[\frac{\partial x_1}{\partial z} : \cdots : \frac{\partial x_m}{\partial z}\right]$ with holomorphic functions $\frac{\partial x_i}{\partial z}$, and so holomorphic. According to Proposition 2.3.2, the surface M is a minimal surface. Q.E.D.

Let M be a Riemann surface with a metric ds^2 which is conformal, namely represented as

$$ds^2 = \lambda_z^2 |dz|^2$$

with a positive C^∞ function λ_z in term of a holomorphic local coordinate z.

Definition 2.3.4. For each point $p \in M$ we define the *Gaussian Curvature* of M at p by

$$K_{ds^2} = -\frac{\Delta \ln \lambda_z}{\lambda_z^2}$$

For a minimal surface M immersed in $\mathbb{R}^m$ consider the system of holomorphic functions $G = (f_1, \cdots, f_m)$ and set $|G| = (|f_1|^2 + \cdots + |f_m|^2)^{1/2}$ for $f_i = \frac{\partial x_i}{\partial z}$. From the definition of Gaussian curvature, we get

$$K_{ds^2} = -\frac{2}{|G|^2}\frac{\partial^2}{\partial \bar{z}\partial z}\ln |G| = -\frac{1}{|G|^6}\left(\sum_{i<j}\left|f_i f_j' - f_j f_i'\right|^2\right) \tag{2.14}$$

This implies that the Gaussian curvature of a minimal surface is always nonpositive.

Definition 2.3.5. A surface with a metric $ds^2 = \lambda_z^2|dz|^2$ is called to be *flat* if the Gaussian curvature K_{ds^2} vanishes identically.

If a minimal surface is flat, then (2.14) implies that $(f_i/f_{i_0})' = 0,\ 1 \leq i \leq m$ for some i_0 with $f_{i_0} \not\equiv 0$, which means f_i/f_{i_0} is constant. Therefore, the Gauss map G is a constant.

2.4 Completeness of Minimal Surfaces

We will prove the Main Theorem of this paper using the Completeness of Minimal Surfaces, so we shall explain it in this section.

Definition 2.4.1. A *divergent curve on a Riemann manifold* M is a differentiable map $\gamma : [0,1) \to M$ such that for every compact subset $K \subset M$ there exist a $t_0 \in (0,1)$ with $\gamma(t) \notin K$ for all $t > t_0$. That is, γ leaves every compact subset of M.

Definition 2.4.2. Riemann manifold M is said to be *complete* if every divergent curve $\gamma : [0,1) \to M$ has unbounded length.

Definition 2.4.3. A *divergent curve on a minimal surface* $x : M \to \mathbb{R}^m$ is a continuous map $\Gamma : [0,1) \to \mathbb{R}^m$ of the form $\Gamma = x \circ \gamma$ where $\gamma : [0,1) \to M$ is a divergent curve on the Riemann maniflod M endowed with the metric of $\mathbb{R}^m$ via the mapping x.

Definition 2.4.4. A minimal surface $x : M \to \mathbb{R}^m$ immersed in $\mathbb{R}^m$ is said to be *complete* if every divergent curve $\Gamma : [0,1) \to \mathbb{R}^m$ on x has unbounded length.

Example 2.4.5. The helicoid in Example 2.3.4.

$$x(u,v) = (\sinh u \cdot \sin v,\ -\sinh u \cdot \cos v,\ -v)$$

obtained from $g = e^z$ and $h = \frac{\sqrt{-1}}{2e^z}$ is a non-flat complete minimal surface.

Proof. By 2.3.3. we have $ds^2 = \left(1 + |g|^2\right)^2 |h|^2 |dz|^2$. Then the length of a curve γ is defined as

$$d(\gamma) = \int_\gamma \left(1 + |g|^2\right) |h||dz|.$$

where $\gamma : [0,1) \to M,\ t \to \gamma(t)$ is a diffeomorphism. The only way a curve can be divergent here is when it tends to ∞. As the curve $\gamma(t)$ tends to ∞,

$$\int_\gamma \left(1 + |g|^2\right) |h||dz| = \int_\gamma (1 + |e^z|^2) \cdot \frac{1}{2|e^z|} |dz| = \frac{1}{2} \int_\gamma \left(\frac{1}{|e^z|} + |e^z|\right) |dz|$$

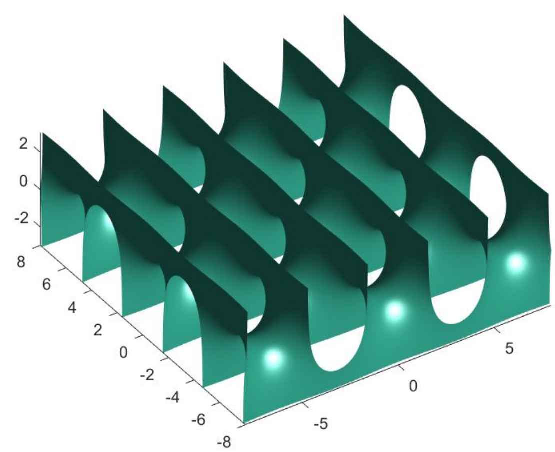
tends to ∞. Thus $d(\gamma)$ is unbounded. Therefore, we conclude the helicoid is complete. As a matter of fact, all helicoid are complete minimal surfaces. Q.E.D.

Now we consider a doubly periodic Scherk's surface in $\mathbb{R}^3$. Let $\tilde{M} = \mathbb{C} \setminus \{1, -1, \sqrt{-1}, -\sqrt{-1}\}$, $\tilde{g} = z$, and $\tilde{h} = \frac{4}{z^4-1}$. Then

$$(\tilde{\phi}_1,\ \tilde{\phi}_2,\ \tilde{\phi}_3) = \left(\frac{-2}{z^2+1}dz,\ \frac{2\sqrt{-1}}{z^2-1}dz,\ \frac{4z}{z^4-1}dz \right).$$

But then $\tilde{x} = 2\,\mathbf{Re}\int \tilde{\phi}$ is not well defined because the 1-form $\tilde{\phi}_i,\ i = 1, 2, 3$ have real periods on $\tilde{M}$ as we see the picture below. To solve this, we define M as the universal covering of $\tilde{M}$, and we take g and h as the lifts of $\tilde{g}$ and $\tilde{h}$ to M, respectively. Thus, $\tilde{x} = 2\,\mathbf{Re}\int \tilde{\phi}$ is well defined.

Figure 2.4: The graph of a doubly periodic Scherk's surface in $\mathbb{R}^3$



As we have seen in example of the doubly periodic Scherk's surface, sometimes we need to deal with surfaces $\tilde{x} : \tilde{M} \to \mathbb{R}^3$ which is defined on Riemann surfaces $\tilde{M}$, where a simpley connected manifold M of the same dimension as $\tilde{M}$ is regarded as the universal covering of $\tilde{M}$. Any minimal surfaces $\tilde{x} : \tilde{M} \to \mathbb{R}^3$ can be lifted from $\tilde{M}$ to M as a minimal surface $x : M \to \mathbb{R}^3$, and we shall see that x is complete if and only if $\tilde{x}$ is complete.

If $\tilde{M}$ is a Riemann surface with conformal structure $\tilde{c}$, then π^{-1} induces a conformal sturcture c on M such that the projection map $\pi : (M, c) \to (\tilde{M}, \tilde{c})$ becomes a holomorphic mapping of the Riemann surface (M, c) onto the Riemann surface $(\tilde{M}, \tilde{c})$. Consequently if $\tilde{x} : \tilde{M} \to \mathbb{R}^3$ is a minimal surface with $\tilde{M}$ as parameter domain, and if $\pi : M \to \tilde{M}$ is the universal map, then $x := \tilde{x} \circ \pi : M \to \mathbb{R}^3$ defines a minimal surface. We call this map *the universal covering of the minimal surface* $\tilde{x}$. Note that x is regular if and only if $\tilde{x}$ is regular, and the images of the Gauss maps G of M and the image of the Gauss maps $\tilde{\mathsf{G}}$ of $\tilde{M}$ coincide.

Proposition 2.4.6. (U.Dierkes, S.Hildebrandt, and F.Sauvigny, [17]) *A minimal surface* $\tilde{x} : \tilde{M} \to \mathbb{R}^3$ *is complete if and only if its universal covering* $x : M \to \mathbb{R}^3$ *is complete.*

Proof. $\Leftarrow$) Suppose that x is complete. We consider an arbitrary divergent curve $\tilde{\Gamma}$ on $\tilde{x}$. Lifting $\tilde{\Gamma}$ to the covering surface x, we obtain a divergent curve Γ on x which must have infinite length as x is complete. Since $\pi : M \to \tilde{M}$ is a local isometry, it follows that $\tilde{\Gamma}$ has infinite length, and we conclude that $\tilde{x}$ is complete.

$\Rightarrow$) Now Suppose that $\tilde{x}$ is complete. Consider an arbirary divergent curve Γ on x given by $\Gamma = x \circ \gamma,\ \gamma : [0, 1) \to M$. Let $\tilde{\Gamma} := \tilde{x} \circ \tilde{\gamma}$ be the curve on $\tilde{x}$ with $\tilde{\gamma} = \pi \circ \gamma$ on

$\tilde{M}$. Then, we have $\tilde{\Gamma} := \tilde{x} \circ \tilde{\gamma} = \tilde{x} \circ \pi \circ \gamma = x \circ \gamma = \Gamma$. We have to show that the length of Γ is infinite. If $\tilde{\gamma}$ is divergent on $\tilde{M}$, then by definition $\tilde{\Gamma}$ is divergent on $\tilde{x}$, so the completeness of $\tilde{x}$ implies that $\tilde{\Gamma}$ has infinite lengh, and hence Γ has infinite length since π is locally an isometry. On the other hand, if $\tilde{\gamma}$ is not divergent, then there is a compact subset K of $\tilde{M}$ and a sequence of parameter values t_n in $[0, 1)$ converging to 1 such that $\tilde{\gamma}(t_n)$ belongs to K for all n. We may assume that the points $\tilde{\gamma}(t_n)$ converges to a point $\tilde{p} \in \tilde{M}$. Then we choose a chart $\varphi : U \to \mathbb{R}^2$ around $\tilde{p}$ such that $\varphi(\tilde{p}) = 0$, and that $\pi^{-1}(U)$ is the disjoint union of open sheets V_i. Since the branch points are isolated, there is an $\varepsilon > 0$ such that $\Omega_\varepsilon := B_\varepsilon(0) \setminus \overline{B}_{\varepsilon/2}(0)$ is contained in $\varphi(U)$ and that the metric of M is positive definite on $\varphi^{-1}(\overline{\Omega}_\varepsilon)$. Since the points $\tilde{\gamma}(t_n)$ converge to $\tilde{p}$, almost all of them belong to the compact set $\varphi^{-1}\left(\overline{B}_{\varepsilon/2}(0)\right)$. Since γ is divergent on the universal covering M, the points $\gamma(t_n)$ are distributed over infinitely many sheets V_i. From this fact we infer that the path $\varphi \circ \tilde{\gamma}$ has to cross Ω_ε an infinite number of times. This implys that the length of $\tilde{\gamma}$ is infinite. Hence the length of γ is infinite via π, so is Γ.

Chapter 3

Theory of Holomorphic Curves

3.1 Holomorphic Curves

Let f be a holomorphic curve in $\mathbb{P}^n(\mathbb{C})$ defined on an open Rieman surface M, which means a nonconstant holomorphic map of M into $\mathbb{P}^n(\mathbb{C})$. For a fixed system of homogeneous coordinates $[w_0 : \cdots : w_n]$ we set

$$V_i := \{[w_0 : \cdots : w_n] \ : \ w_i \neq 0\}, \quad 1 \leq i \leq q.$$

Then, every $z_0 \in M$ has a neighborhood U of z_0 such that $f(U) \subset V_i$ for some i and f has a representation

$$f = [f_0 : \cdots : f_{i-1} : 1 : f_{i+1} : \cdots : f_n]$$

on U with holomorphic functions $f_0, \cdots, f_{i-1}, f_{i+1}, \cdots, f_n$.

Definition 3.1.1. For an open subset U of M we call a representation $f = [f_0 :$

$\cdots : f_n]$ to be a *reduced representation* of f on U if $f_0, \cdots, f_n$ are holomorphic functions on U and have no common zero. We define the reduced representation as $F = (f_0, \cdots, f_n)$ in $\mathbb{C}^{n+1}$.

As stated above, every holpomorphic map of M into $\mathbb{P}^n(\mathbb{C})$ has reduced representation on some neighborhood of each point in M. Let $f = [f_0 : \cdots : f_n]$ be a reduced representation of f. Then, for an arbitrary nowhere zero holomorphic function h, $f = [f_0 h : \cdots : f_n h]$ is also a reduced representatioin of f. Conversely, for every reduced representation $f = [g_0 : \cdots : g_n]$ of f, each g_i can be written as $g_i = hf_i$ with a nowhere zero holomorphic function h.

Definition 3.1.2. For a nonzero meromorphic function h on M, we define the divisor ν_h of h as a map of M into the set of integers such that for $z_0 \in M$

$$\nu_h(z_0) = \begin{cases} m & \text{if } \ h \text{ has a zero of order } m \text{ at } \ z_0, \\ -m & \text{if } \ h \text{ has a pole of order } m \text{ at } \ z_0, \\ 0 & \text{otherwise.} \end{cases}$$

We now take $n+1$ holomorphic functions $f_0, \cdots, f_n$ on M at least one of which does not vanish identically. Take a nonzero holomorphic function g such that $\nu_g(z_0) = \min\{\nu_{f_i}(z_0) : f_i \not\equiv 0,\ 0 \le i \le n\}$ for $z_0 \in M$. Then, f_i/g $(0 \le i \le n)$ are holomorphic functions without common zeros. We can define a holomorphic map f with a reduced representation $f = [f_0/g : \cdots : f_n/g]$, which we call the *holomorphic curve defined by* $f_0, \cdots, f_n$.

Definition 3.1.3. Let $a_0, a_1, \cdots, a_n \in \mathbb{C}$ be scalars not all equal to 0. Then the set

H consisting of all homogeneous coordinates $[w_0 : w_1 : \cdots : w_n]$ in $\mathbb{P}^n(\mathbb{C})$ such that $a_0w_0 + a_1w_1 + \cdots + a_nw_n = 0$ is a subspace of one dimension less than the demension of $\mathbb{P}^n(\mathbb{C})$ is called a *hyperplane* in $\mathbb{P}^n(\mathbb{C})$. In other words,

$$H = \{[w_0 : w_1 : \cdots : w_n] : \ a_0w_0 + a_1w_1 + \cdots + a_nw_n = 0\}.$$

Take a holomorphic map f of M into $\mathbb{P}^n(\mathbb{C})$ and a hyperplane H in $\mathbb{P}^n(\mathbb{C})$ not including the image $f(M)$ of f. For each point $z \in M$ choosing a reduced representation $f = [f_0 : \cdots : f_n]$ or $F = (f_0, \cdots, f_n)$ on a neighborhood on U of z, we consider the holomorphic function $F(H) := a_0f_0 + \cdots + a_nf_n$ on U. Since $\nu_{F(H)}$ depends only on f and H, we can define $\nu_{F(H)}$ on M, which we call *pull-back* of H considered as a divisor.

Proposition 3.1.4. (H. Fujimoto, [9])*Every holomorphic map f of an open Riemann surface M into $\mathbb{P}^n(\mathbb{C})$ has a reduced representation on M.*

Proof. Set $H_i := \{w_i = 0\}$ $(0 \leq i \leq n)$ for a fixed system of homogeneous coordinates $[w_0 : \cdots : w_n]$. Changing indices if necessary, we may assume that $f(M) \not\subset H_0$, and so $\nu_{F(H_0)}$ is well-defined. Then there is a nonzero holomorphic function g such that $\nu_g = \nu_{F(H_0)}$. On the other hand, we can take an open covering $\{U_k : k \in I\}$ of M such that f has a reduced representation $f = [f_{k0} : \cdots : f_{kn}]$ on each U_k. Then, $\nu_g = \nu_{f_{k0}}$. Let $g_{ki} := \frac{g}{f_{k0}} \cdot f_{ki}$ $(0 \leq i \leq n)$, which is holomorphic on U_k. If $U_{k\lambda} := U_k \cap U_\lambda \neq \phi$, then there is a nowhere zero holomorphic function h with $f_{\lambda i} = hf_{ki}$ $(0 \leq i \leq n)$ on $U_{k\lambda}$. We have

$$g_{ki} = \frac{gf_{ki}}{f_{k0}} = \frac{ghf_{ki}}{hf_{k0}} = \frac{gf_{\lambda i}}{f_{\lambda 0}} = g_{\lambda i}$$

on $U_{k\lambda}$ for each i. Therefore, we can define the function g_i on M which equals g_{ki}

on each U_k. For these functions, $f = [g_0 : \cdots : g_n]$ is a reduced representation of M. Q.E.D.

Now, we consider k arbitrarily given holomorphic functions $f_0, \cdots, f_k$ on M. For a holomorphic local coordinate z on an open subset U of M, we denote by $(f_i^{(l)})_z$, or simply by $f_i^{(l)}$, l-th derivative of f_i with respect to z, where we set $(f_i^{(0)})_z := f_i$. By definition, the Wronskian of $f_0, \cdots, f_k$ is given by

$$W(f_0, \cdots, f_k) \equiv W_z(f_0, \cdots, f_k) := \det\left((f_i^{(l)})_z : 0 \leq i, l \leq k\right).$$

Proposition 3.1.5. (H. Fujimoto, [9])*For two holomorphic local coordinate z and ζ, it holds that*

$$W_\zeta(f_0, \cdots, f_k) \equiv W_z(f_0, \cdots, f_k) \left(\frac{dz}{d\zeta}\right)^{k(k+1)/2}.$$

Proof. Set $F = (f_0, \cdots, f_k)$ and $\left(F^{(l)}\right)_z = \left((f_0^{(l)})_z, \ \cdots, (f_k^{(l)})_z\right)$. Then, we have

$$\begin{aligned}
W_\zeta(f_0, \cdots, f_k) &= \det\left({}^t(F^{(0)})_\zeta, {}^t(F^{(1)})_\zeta, \ \cdots, {}^t(F^{(k)})_\zeta\right) \\
&= \det\left({}^t(F^{(0)})_z, {}^t(F^{(1)})_z \frac{dz}{d\zeta}, \ \cdots, {}^t(F^{(k)})_z \left(\frac{dz}{d\zeta}\right)^k\right) \\
&= \det\left({}^t(F^{(0)})_z, {}^t(F^{(1)})_z, \ \cdots, {}^t(F^{(k)})_z\right) \left(\frac{dz}{d\zeta}\right) \left(\frac{dz}{d\zeta}\right)^2 \cdots \left(\frac{dz}{d\zeta}\right)^k \\
&= W_z(f_0, \cdots, f_k) \left(\frac{dz}{d\zeta}\right)^{k(k+1)/2}
\end{aligned}$$

where ${}^tF^{(l)}$ denotes the transpose of the vector $F^{(l)}$. Q.E.D

The following proposition is a well-known property, so we will not prove it.

Proposition 3.1.6. *For holomorphic functions $f_0, \cdots, f_k$ on M, the following conditions are equivalent:*

(i) $f_0, \cdots, f_k$ *are linearly dependent over* $\mathbb{C}$

(ii) $W_z(f_0, \cdots, f_k) \equiv 0$ *for some (or all) holomorphic local coordinate z.*

Definition 3.1.7. A holomorphic map f of M into $\mathbb{P}^n(\mathbb{C})$ is said to be (*linearly*) *non-degenerate* if the image of f is not included in any hyperplane in $\mathbb{P}^n(\mathbb{C})$. If $f = [f_0 : \cdots : f_n]$ is non-degenerate, then $f_0, \cdots, f_n$ are linearly independent over $\mathbb{C}$.

3.2 The Associated Curves

Let f be a linearly non-degenerate holomorphic map of $\Delta(R) := \{z : |z| < R\} (\subset \mathbb{C})$ into $\mathbb{P}^n(\mathbb{C})$ where $0 < R \leq \infty$. Take a reduced representation $F = (f_0, \cdots, f_n)$ of f with $\mathbf{P}(F) = f$ where $\mathbf{P}$ is the canonical projection of $\mathbb{C}^{n+1} \setminus \{0\}$ onto $\mathbb{P}^n(\mathbb{C})$. Denote the k-th derivative of F by $F^{(k)}$ and define

$$F_k = F^{(0)} \wedge \cdots \wedge F^{(k)} : \Delta(R) \to \bigwedge^{k+1} \mathbb{C}^{n+1}$$

for $k = 0, 1, \cdots, n$. Evidently, $F_{n+1} \equiv 0$. Note that $F_0 = F$. Take a basis $\{E_0, \cdots, E_n\}$ of $\mathbb{C}^{n+1}$. Then the set $\{E_{i_0} \wedge \cdots \wedge E_{i_k} : \ 0 \leq i_0 < \cdots < i_k \leq n\}$ gives a basis of $\bigwedge^{k+1} \mathbb{C}^{n+1}$. Then we see that

$$F_k = \sum_{0 \leq i_0 < \cdots < i_k \leq n} W(f_{i_0}, \cdots, f_{i_k}) E_{i_0} \wedge \cdots \wedge E_{i_k} \tag{3.1}$$

and

$$|F_k|^2 = \sum_{0 \leq i_0 < \cdots < i_k \leq n} |W(f_{i_0}, \cdots, f_{i_k})|^2$$

where $W(f_{i_0}, \cdots, f_{i_k})$ is the Wronskian of $f_{i_0}, \cdots, f_{i_k}$. If $f = [f_0 : \cdots : f_n]$ is non-degenerate, then $f_0, \cdots, f_n$ are linearly independent over $\mathbb{C}$. Therefore, by (3.1) and Proposition 3.1.6 we have $F_k \not\equiv 0$ for $0 \leq k \leq n-1$.

Let

$$\mathbf{P} : \bigwedge^{k+1} \mathbb{C}^{n+1} \setminus \{0\} \to \mathbf{P}(\bigwedge^{k+1} \mathbb{C}^{n+1} \setminus \{0\}) \subset \mathbb{P}^{N_k}(\mathbb{C})$$

be the canonical projection where $N_k = {}_{n+1}C_{k+1} - 1 = \frac{(n+1)!}{(n-k)!(k+1)!} - 1$.

Definition 3.2.1. The curve $\mathbf{f}^k := \mathbf{P}(F_k) : \Delta(R) \to \mathbb{P}^{N_k}(\mathbb{C}),\ k = 0, 1, \cdots, n$ is called the k-th *associated curve* of f. Note that $\mathbf{f}^0 = \mathbf{P}(F_0) = \mathbf{P}(F) = f$.

Let ω_k be the Fubini-Study form on $\mathbb{P}^{N_k}(\mathbb{C})$, and let $\Omega_k = (\mathbf{f}^k)^*\omega_k,\ k = 0, 1, \cdots, n$, be the pullback via the k-th associated curve. Then

$$\Omega_k = dd^c \ln |F_k|^2 = \frac{\sqrt{-1}}{2\pi} \frac{|F_{k-1}|^2 |F_{k+1}|^2}{|F_k|^4} dz \wedge d\bar{z} \geq 0$$

for $0 \leq k \leq n$, and by convention $F_{-1} \equiv 1$. Note that $\Omega_0 = dd^c \ln |F|^2$, and $\Omega_n \equiv 0$ since $F_{n+1} \equiv 0$. It follows that

$$\begin{aligned} \text{Ric}\ \Omega_k :&= dd^c \ln \left(\frac{|F_{k-1}|^2 |F_{k+1}|^2}{|F_k|^4} \right) \\ &= dd^c \ln |F_{k-1}|^2 + dd^c \ln |F_{k+1}|^2 - 2dd^c \ln |F_k|^2 \\ &= \Omega_{k-1} + \Omega_{k+1} - 2\Omega_k. \end{aligned}$$

Let $H = \{[z_0 : \cdots : z_n] |\ a_0 z_0 + \cdots + a_n z_n = 0\}$ be a hyperplane in $\mathbb{P}^n(\mathbb{C})$ with $a_0^2 + \cdots + a_n^2 = 1$. Take a basis $\{E_0, \cdots, E_n\}$ of $\mathbb{C}^{n+1}$. Then the set

$$\{E_{i_0} \wedge \cdots \wedge E_{i_k} :\ 0 \leq i_0 < \cdots < i_k \leq n\}$$

gives a basis of $\bigwedge^{k+1}\mathbb{C}^{n+1}$. We define

$$F_k(H) = \sum_{0\le i_1<\cdots<i_k\le n}\left(\sum_{i_0\ne i_1,\cdots i_n} a_{i_0}W(f_{i_0},\cdots,f_{i_k})\right)E_{i_1}\wedge\cdots\wedge E_{i_k}.$$

Then

$$F_0(H) = a_0f_0+\cdots+a_nf_n = F(H).$$

Note that

$$|F_k(H)|^2 = \sum_{0\le i_1<\cdots<i_k\le n}\left|\sum_{i_0\ne i_1,\cdots i_n} a_{i_0}W(f_{i_0},\cdots,f_{i_k})\right|^2.$$

and

$$|F(H)|^2 = |a_0f_0+\cdots+a_nf_n|^2.$$

Definition 3.2.2.

$$\frac{|F_k(H)(z)|}{|F_k(z)|}$$

is said to be the *projective distance* from the k-th associated curve $\mathbf{f}^k(z)$ to the hyperplane H.

Definition 3.2.3. We define the *k-th contact function* of f for H by

$$\phi_k(H) := \frac{|F_k(H)|^2}{|F_k|^2}.$$

Then $\phi_0(H) = \frac{|F(H)|^2}{|F|^2}$ and $\phi_n(H) = \frac{|W(f_0,\cdots,f_n)|^2\sum_{i=0}^n|a_i|^2}{|F_n|^2} = \frac{|W(f_0,\cdots,f_n)|^2}{|F_n|^2} = 1$. Note that $0\le\phi_k(H)\le\phi_{k+1}(H)\le 1$ for $0\le k\le n-1$.

3.3 Results in Nevanlinna Theory

We shall introduce a few portions of the Nevalinna Theory in order to enforce the proof of the Main Theorem.

Definition 3.3.1. Let f be a meromorphic function on $\Delta(R)$, where $0 \leq R \leq \infty$ and let $r < R$. Denote the number of poles of f on the closed disc $\overline{\Delta(R)}$ by $n_f(r, \infty)$, counting multiplicity. We then define the *counting function* $N_f(r, \infty)$ to be

$$N_f(r, \infty) = \int_0^r \frac{n_f(t, \infty) - n_f(0, \infty)}{t} dt + n_f(0, \infty) \ln r,$$

here $n_f(0, \infty)$ is the multiplicity if f has a pole at $z = 0$. For each complex number a, we then define the *counting function* $N_f(r, a)$ to be

$$N_f(r, a) = N_{\frac{1}{f-a}}(r, \infty).$$

Definition 3.3.2. The *Nevanlinna's proximity function* $m_f(r, \infty)$ is defined by

$$m_f(r, \infty) = \frac{1}{2\pi} \int_0^{2\pi} \ln^+ |f(re^{i\theta})| d\theta$$

where $\ln^+ x = \max\{0, \ln x\}$. For any complex number a, the *poximity function* $m_f(r, a)$ of f with respect to a is then defined by

$$m_f(r, a) = m_{\frac{1}{f-a}}(r, \infty).$$

Definition 3.3.3. The *Nevanlinna's characteristic function* of f is defined by

$$T_f(r) = m_f(r, \infty) + N_f(r, \infty).$$

Here, $T_f(r)$ measures the growth of f.

Example 3.3.4. (K.S. Charak [27]) Consider the rational function

$$f(z) = \frac{P(z)}{Q(z)} = \frac{a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0}{b_m z^m + b_{m-1} z^{m-1} + \cdots + b_0}, \quad a_n,\ b_m \neq 0.$$

Distingquish the following two cases:

Case 1. When $m \geq n$. In this case $\lim_{|z| \to \infty} f(z)$ is finite, so there is a positive real number r_0 such that $n_f(r, \infty) = m$ for all $r \geq r_0$. Thus

$$\begin{aligned}
N_f(r,\infty) &= \int_0^{r_0} \frac{n_f(t,\infty) - n_f(0,\infty)}{t} dt + \int_{r_0}^{r} \frac{n_f(t,\infty) - n_f(0,\infty)}{t} dt + n_f(0,\infty) \ln r \\
&= (m - n_f(0,\infty)) (\ln r - \ln r_0) + n_f(0,\infty) \ln r + \mathcal{O}(1) \\
&= m \ln r - m \ln r_0 + n_f(0,\infty) \ln r_0 + \mathcal{O}(1) \\
&= m \ln r + \mathcal{O}(1)
\end{aligned}$$

Next, note that for polynomial $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$ with $a_n \neq 0$, given positive ϵ there is an $r_0 > 0$ such that for all $r = |z| > r_0$ we have

$$(1 - \epsilon)|a_n| r^n \leq |P(z)| \leq (1 + \epsilon)|a_n| r^n.$$

Thus, for all $r \geq r_0$ we can assume that $|P(z)| = |a_n| r^n (1 + o(1))$ and $|Q(z)| = |b_m| r^m (1 + o(1))$. This implies that $\ln^+ |f| = \mathcal{O}(1)$, and so $m_f(r, \infty) = \mathcal{O}(1)$. Hence in this case

$$T_f(r) = m \ln r + \mathcal{O}(1) = \mathcal{O}(\ln r).$$

Case 2. When $m < n$. by the same arguments used in *Case 1* we get

$$T_f(r) = T_{\frac{1}{f}}(r) + \mathcal{O}(1) = n \ln r + \mathcal{O}(1) = \mathcal{O}(\ln r).$$

Thus for a rational function f we have $T_f(r) = \mathcal{O}(\ln r)$. Also, the converse of this statement holds. That is, if f is a meromorphic function with $T_f(r) = \mathcal{O}(\ln r)$, then f is a rational function. Furthermore, by letting $m = 0$ and $n = 0$, we have that $T_f(r) = \mathcal{O}(1)$ if and only if f is constant.

Example 3.3.5. (K.S. Charak [27]) Consider $f(z) = e^z$. Then for all r we have

$$\begin{aligned}
m_f(r,\infty) &= \frac{1}{2\pi}\int_0^{2\pi} \ln^+ |e^{re^{i\theta}}| d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \ln^+ e^{r\cos\theta} d\theta \\
&= \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} r\cos\theta d\theta \\
&= \frac{r}{\pi}.
\end{aligned}$$

Since e^z is an entire function, for all r $N_f(r,\infty) = 0$ and so

$$T_f(r) = \frac{r}{\pi}.$$

Theorem 3.3.6. (Nevanlinna's First Main Theorem) *Let f be a non-constant meromorphic function on $\overline{\Delta(R)}$, $R \le \infty$. Then, for any r ($0 \le r < R$) and $a \in \mathbb{C}$*

$$T_f(r) = m_f(r,a) + N_f(r,a) + \mathcal{O}(1)$$

or

$$T_f(r) = T_{\frac{1}{f-a}}(r) + \mathcal{O}(1)$$

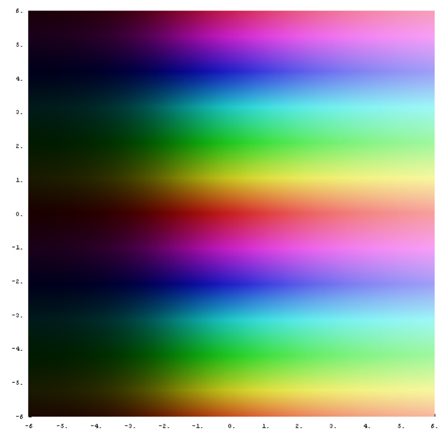
holds. This theorem implies $N_f(r,a) \le T_f(r)$.

Example 3.3.7. Consider the function in *Example 3.3.5.* If $a = 0$, then by the same arguments in *Example 3.3.5.* we have

$$m_f(r,0) = m_{\frac{1}{f}}(r,\infty) = \frac{r}{\pi} \quad \text{and} \quad N_f(r,0) = N_{\frac{1}{f}}(r,\infty) = 0.$$

for all r. Thus we get $T_f(r) = \frac{r}{\pi} = m_f(r,0) + N_f(r,0) = T_{\frac{1}{f}}(r)$. Therefore, the result in Theorem 3.3.6 holds. In this case, letting $r \to \infty$ the Nevanlinna's First Main Theorem tells us in a sense that f is close to 0 on the left half plane and close to ∞ on the right half plane. We can see this fact from the Color Wheel Graph of the exponential function $f(z) = e^z$ on the right. The transition from dark to light colors shows that the magnitude of the exponential function is increasing to the right. The periodic horizontal bands indicate that the exponential function is periodic in the imaginary part of its argument.

Figure 3.1: Color Wheel Graph of $f(z) = e^z$



en.wikipedia.org/wiki/Exponential_function

Now we define $\bar{N}_f(r, a)$ in the same way as $N_f(r, a)$ but without taking multiplicity into account.

Theorem 3.3.8. (Nevanlinna's Second Main Theorem) *Let f be a non-constant meromorphic function on $\overline{\Delta(R)}$, $R \leq \infty$, and let $a_1, \cdots, a_q$ be distinct complex numbers in $\mathbb{C} \cup \{\infty\}$. Then, the inequality*

$$(q-2)T_f(r) \leq_{exc} \sum_{j=1}^{q} \bar{N}_f(r, a_j) + S(r, f)$$

holds where $S(r, f)$ is the small error term and $\leq_{exc}$ means that the inequality holds for all $r \in [0, \infty)$ outside of a set of finite Lebesgue measure.

Following are some of the known estimates of $S(r, f)$.

Theorem 3.3.9. (W.K. Hayman [28], K.S. Charak [27]) *Let f be a non-constant meromorphic function $|z| < R \leq +\infty$. Then*

(a) *if* $R = +\infty$,

$$S(r, f) = \mathcal{O}(\ln^+ T_f(r)) + o(\ln r)$$

as $r \to \infty$ *through all values if* f *is of finite order, and as* $r \to \infty$ *outside a set* E *of finite linear measure otherwise.*

(b) *if* $0 < R < +\infty$,

$$S(r, f) = \mathcal{O}\left(\ln^+ T_f(r)) + \ln \frac{1}{R - r}\right)$$

as $r \to \infty$ *outside a set* E *with* $\int_E \frac{dr}{R-r} < +\infty$.

As an immediate deduction from Theorem 3.3.9, we have

Theorem 3.3.10. (W.K. Hayman [28], K.S. Charak [27]) *Let* f *be a non-constant meromorphic function* $|z| < R \leq +\infty$. *Then*

$$\frac{S(r, f)}{T_f(r)} \to 0 \quad \text{as } r \to R, \tag{3.2}$$

with the following provisos:

(a) *if* $R = +\infty$ *and* f *is of finite order, then (3.2) holds without any restriction.*

(b) *if* $R = +\infty$ *and* f *is of infinite order, then (3.2) holds as* $r \to \infty$ *outside a set* E *of finite length.*

(c) *if* $R = +\infty$ *and*

$$\liminf_{r \to R} \frac{T_f(r)}{\ln(\frac{1}{R-r})} = +\infty,$$

then (3.2) holds as $r \to R$ *through a suittable sequence of values of* r.

From Theorem 3.3.10 we see that $S(r, f)$ is the small error term with the property that $S(r, f) = o(T_f(r))$ as $r \to \infty$. Let $\epsilon > 0$. Then by replacing $o(T_f(r))$ by $\epsilon\, T_f(r)$

we can restate the inequality

$$(q-2-\epsilon)T_f(r) \leq_{exc} \sum_{j=1}^{q} \bar{N}_f(r, a_j) \quad \text{as } r \to R.$$

Definition 3.3.11. We say that q hyperplanes in $\mathbb{P}^n(\mathbb{C})$ are in *general position* if for $1 \leq k \leq n$, k hyperplanes of them intersect in an $(n-k)$-dimensional plane, and for $k > n$, any k hyperplanes of them have empty intersection. In other words, q hyperplanes are in general position if any subset of k normal vectors of the q hyperplanes is linearly independent whenever $1 \leq k \leq n$.

Theorem 3.3.12. (M. Ru [24], Cartan's Second Main Theorem with Truncated Counting Functions) *Let $H_1, \cdots, H_q$ be hyperplanes in $\mathbb{P}^n(\mathbb{C})$ in general position. Let $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be a linearly non-degenerate holomorphic curve. Then, the inequality*

$$(q-(n+1))T_f(r) \leq_{exc} \sum_{j=1}^{q} N_f^{(n)}(r, H_j) + \mathcal{O}(\ln^+ T_f(r)) + o(\ln r)$$

holds where $\leq_{exc}$ means that the inequality holds for all $r \in [0, \infty)$ outside of a set of finite Lebesgue measure.

As we see before, $S(r,f) = \mathcal{O}(\ln^+ T_f(r)) + o(\ln r)$ is the small error term with the property that $S(r,f) \to o(T_f(r))$ as $r \to \infty$. Thus by replacing $o(T_f(r))$ by $\epsilon\, T_f(r)$ we can restate the inequality

$$(q-(n+1)-\epsilon)T_f(r) \leq_{exc} \sum_{j=1}^{q} N_f^{(n)}(r, H_j).$$

Chapter 4

Previous Studies of the Unicity Results

The uniqueness theory mainly studies conditions under which there exists essentially only one function. Here we shall introduce two types of unicity theorems.

4.1 Unicity Theorem for meromorpic functions of $\mathbb{C}$

The Finnish mathematician Rolf Nevanlinna is the person who made the decisive contribution to the development of the theory of value distribution by introducing the charicteristic function $T_f(r)$ for the meromorphic function f and proved the first unicity theorem for meromorpic functions in 1926. The theorem is as follows:

Theorem 4.1.1. (R. Nevanlinna [2], 1926) *If two non-constant meromorphic functions $f, g : \mathbb{C} \to \mathbb{P}^1(\mathbb{C})$ have the same inverse images ignoring multiplicities for five distinct complex values, then $f \equiv g$.*

Proof. Suppose that $f \not\equiv g$. Let $a_1, \cdots, a_q$ be distinct complex numbers in $\mathbb{P}^1(\mathbb{C})$ and define $\chi = f - g$. We apply Nevanlinna's Second Main Theorem to f and g. Then, for $\epsilon > 0$ we have

$$(q - 2 - \epsilon)T_f(r) \leq_{exc} \sum_{j=1}^{q} \bar{N}_f(r, a_j) \tag{4.1}$$

$$(q - 2 - \epsilon)T_g(r) \leq_{exc} \sum_{j=1}^{q} \bar{N}_g(r, a_j) \tag{4.2}$$

as $r \to \infty$. Let $T(r) = T_f(r) + T_g(r)$. Since $f^{-1}(a_j) = g^{-1}(a_j)$ for $j = 1, \cdots, q$, there exists a z such that $f(z) = a_j = g(z)$ for each j. From this, we get the equation $\chi(z) = f(z) - g(z) = 0$. Now by combining (4.1) and (4.2) and applying Nevanlinna's First Main Theorem, we can get the inequality

$$\begin{aligned}
(q - 2 - \epsilon)T(r) &\leq_{exc} \sum_{j=1}^{q} \big(\bar{N}_f(r, a_j) + \bar{N}_g(r, a_j)\big) \\
&\leq_{exc} 2\; N_\chi(r, 0) \\
&\leq_{exc} 2\; T_\chi(r) \\
&\leq_{exc} 2\; T(r)
\end{aligned}$$

This is equivalent to

$$(q - 4 - \epsilon)T(r) \leq_{exc} 0.$$

From this we obtain $q \leq 4 + \epsilon$, which contradicts the assumption $q = 5$. Therefore, $f \equiv g$. Q.E.D.

This theorem, tells that any non-constant meromorphic functions can be uniquely determined by five values. However, the 'five' in the theorem cannot be reduced to 'four'. For example, e^z and e^{-z} share four values which are $0,\ 1,\ -1$, and ∞, but they are not identical.

Hirotaka Fujimoto, in 1975, extended Nevanlinna's result to non-degenerate holomorphic curves $f, g : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$. During the four decades following 1975, this problem has been studied intensively by H. Fujimoto, W. Stoll, L. Smiley, S. Ji, M. Ru, S.D. Quang, Z.H. Chen, Q.M. Yan and other mathematicians. In 1983, Smiley considered holomorphic maps $f, g : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ which share $3n+2$ hyperplanes in $\mathbb{P}^n(\mathbb{C})$ without counting multiplicity and proved the following theorem.

Theorem 4.1.3. (L. Smiley [11], 1983) *Let $f, g : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be linearly non-degenerate meromorphic mappings. Assume that there are q hyperplanes $H_1, \cdots, H_q$ in $\mathbb{P}^n(\mathbb{C})$ located in general position satisfying*

(i) $f^{-1}(H_j) = g^{-1}(H_j)$ *for all* $1 \le j \le q$,

(ii) $f^{-1}(H_i) \cap f^{-1}(H_j) = \phi$ *for all* $1 \le i < j \le q$,

(iii) $f = g$ *on* $\displaystyle\bigcup_{j=1}^{q} f^{-1}(H_j)$.

If $q \ge 3n+2$, then $f \equiv g$.

Proof. Suppose that $f \not\equiv g$. Fix reduced (global) representations $F = (f_0, \cdots, f_n)$ with $T_{f_i}(r) \le T_f(r)$ and $G := (g_0, \cdots, g_n)$ with $T_{g_i}(r) \le T_g(r)$ such that $f = \mathbf{P}(F)$ and $g = \mathbf{P}(G)$ where $\mathbf{P}$ denotes the canonical projection of $\mathbb{C}^n \backslash \{0\}$ onto $\mathbb{P}^n(\mathbb{C})$. Since $f \not\equiv g$, there exist indices $1 \le i, j \le q$ such that $\chi = F(H_i)G(H_j) - G(H_i)F(H_j) \not\equiv 0$.

We now apply Cartan's Second Main Theorem to f and g. Then, for $\epsilon > 0$ we have

$$(q-(n+1)-\epsilon)T_f(r) \leq_{exc} \sum_{j=1}^{q} N_f^{(n)}(r, H_j) \tag{4.3}$$

$$(q-(n+1)-\epsilon)T_g(r) \leq_{exc} \sum_{j=1}^{q} N_g^{(n)}(r, H_j) \tag{4.4}$$

as $r \to \infty$. Let $T(r) = T_f(r) + T_g(r)$. Then, by combining (4.3) and (4.4), we can get the inequality

$$\begin{aligned}(q-(n+1)-\epsilon)T(r) &\leq_{exc} \sum_{j=1}^{q} \left(N_f^{(n)}(r, H_j) + N_g^{(n)}(r, H_j)\right) \\ &\leq_{exc} n \sum_{j=1}^{q} \left(\bar{N}_f(r, H_j) + \bar{N}_g(r, H_j)\right)\end{aligned}$$

If $F(H_i)(z_0) = 0$, then $z_0 \in \bigcup_{j=1}^{q} f^{-1}(H_i)$, so $f(z_0) = g(z_0)$ by the assumption (iii), and hence $\chi(z_0) = 0$. Also, note that $F(H_j)(z_0) \neq 0$ for all $j \neq i$, so $\sum_{j=1}^{q} \bar{N}_f(r, H_j) \leq N_\chi(r, 0)$. Similarly, $\sum_{j=1}^{q} \bar{N}_g(r, H_j) \leq N_\chi(r, 0)$. By these, together with by Nevanlinna's First Main Theorem, $N_\chi(r, 0) \leq T_\chi(r) \leq T(r)$, implies

$$\begin{aligned}(q-(n+1)-\epsilon)T(r) &\leq_{exc} 2n\, N_\chi(r, 0) \\ &\leq_{exc} 2n\, T_\chi(r) \\ &\leq_{exc} 2n\, T(r)\end{aligned}$$

This is equivalent to

$$(q-(3n+1)-\epsilon)T(r) \leq_{exc} 0.$$

From this we obtain $q \leq 3n + 1 + \epsilon$, which contradicts the assumption $q \geq 3n + 2$. Therefore, $f \equiv g$. Q.E.D.

In 2009, Z.H. Chen and Q.M. Yan [14] improved Smiley's result for $q \geq 3n + 2$

to $2n+3$ with the same assumption.

Theorem 4.1.4. (Z.H. Chen and Q.M. Yan [14], 2009) *Let f and g be two linearly non-degenerate holomorphic maps of $\mathbb{C}$ into $\mathbb{P}^n(\mathbb{C})$ over $\mathbb{C}$ and let $H_1, \cdots, H_q (q \geq 2n)$ be hyperplanes in $\mathbb{P}^n(\mathbb{C})$ in general position. Assume that*

(i) $f^{-1}(H_j) = g^{-1}(H_j)$ for all $1 \leq j \leq q$,

(ii) $f^{-1}(H_i) \cap f^{-1}(H_j) = \phi$ for all $1 \leq i < j \leq q$,

(iii) $f = g$ on $\bigcup_{j=1}^{q} f^{-1}(H_j)$.

If $q \geq 2n+3$, then $f \equiv g$.

In 2012, H. Giang, L. Quynh and S. Quang [15] generalized previous results by changing the condition (ii) to a more general one using the auxiliary function

$$\chi = \sum_{i=1}^{q} \left[F(H_i)G(H_{\sigma(i)}) - G(H_i)F(H_{\sigma(i)})\right] \not\equiv 0$$

Their theorem is stated as follows:

Theorem 4.1.5. (H. Giang, L. Quynh and S. Quang [15], 2012) *Let f and g be two linearly non-degenerate holomorphic maps of $\mathbb{C}$ into $\mathbb{P}^n(\mathbb{C})$ and let $H_1, \cdots, H_q$ $(q \geq 2n)$ be hyperplanes in $\mathbb{P}^n(\mathbb{C})$ in general position. Assume that for a positive integer k with $1 \leq k \leq n$,*

(i) $f^{-1}(H_j) = g^{-1}(H_j)$ for all $1 \leq j \leq q$,

(ii) $f^{-1}(\bigcap_{j=1}^{k+1} H_{i_j}) = \phi$ for all $1 \leq i_1 < \cdots < i_{k+1} \leq q$,

(iii) $f = g$ on $\displaystyle\bigcup_{j=1}^{q} f^{-1}(H_j)$.

If $q \geq (n+1)k + n + 2$, then $f \equiv g$.

Also in 2012, after [15] was published, using the same auxiliary function

$$\chi = \sum_{i=1}^{q} \left[F(H_i)G(H_{\sigma(i)}) - G(H_i)F(H_{\sigma(i)})\right] \not\equiv 0,$$

F. Lü [16] reduced the number of hyperplanes in the theorem of Giang, Quynh and Quang to

$$q \geq \frac{\sqrt{1+8k}+1}{2}n + k + \frac{\sqrt{1+8k}+2}{4},$$

which is derived from $q < \frac{q-2k+2kn}{2kn}[q-(n+1)]$. A multitude of unicity theorems have been proved, but only those theorems that are relevant to this thesis were mentioned here.

4.2 Unicity Theorem for Gauss Maps of complete Minimal Surfaces

Hirotaka Fujimoto is the first person who used the above technique to prove the unicity theorem for Gauss maps of minimal surfaces immersed in $\mathbb{R}^m$. He also showed that generalized Gauss maps of complete minimal surfaces in $\mathbb{R}^m$ are holomorphic curves in $\mathbb{P}^{m-1}(\mathbb{C})$, so many value-distribution-theoretic properties of holomorphic curves in the complex projective space could be used. The following theorem is his first unicity theorem submitted in 1992, which is for Gauss maps of minimal surfaces immersed in $\mathbb{R}^3$.

Theorem 4.2.1. (H. Fujimoto [8], submitted in 1992) *Let $x := (x_1,\ x_2,\ x_3) : M \to \mathbb{R}^3$ and $\tilde{x} := (\tilde{x}_1,\ \tilde{x}_2,\ \tilde{x}_3) : \tilde{M} \to \mathbb{R}^3$ be two non-flat minimal surfaces immersed in $\mathbb{R}^3$ and assume that there is a conformal diffeomorphism Φ of M onto $\tilde{M}$. Consider the maps $f := \pi \circ \mathsf{G}$ and $g := \pi \circ \tilde{\mathsf{G}} \circ \Phi$ where π is the stereographic projection and G and $\tilde{\mathsf{G}}$ are the Gauss maps of M and $\tilde{M}$ respectively. Suppose that there are q distinct points $\alpha_1, \alpha_2, \cdots, \alpha_q$ such that $f^{-1}(\alpha_j) = g^{-1}(\alpha_j)$ for $1 \leq j \leq q$.*

(a) If $q \geq 7$ and either M or $\tilde{M}$ is complete, then $f \equiv g$.

(b) If $q \geq 6$ and both of M or $\tilde{M}$ are complete and have finite total curvature, then $f \equiv g$.

Since it will take several pages to prove this theorem completely, we just give the outline of this proof. The proving process is very similar to the process of proving our Main Theorem.

Outline of the proof of (a) in Theorem 4.2.1. First, suppose that $f \not\equiv g$ in order to get a contradiction, and assume that $\alpha_q = \infty$.

Second, we define a pseudo-metric $d\eta^2$ and show that $d\eta^2$ is continuous on M and has strictly negative curvature on the set $\{d\eta^2 \neq 0\}$. $d\eta^2$ is defined as follows: Set

$$\lambda := \left(\prod_{j=1}^{q} |f, \alpha_j| \ln \left(\frac{a_0}{|f, \alpha_j|^2} \right) \right)^{-1+\epsilon}, \ \tilde{\lambda} := \left(\prod_{j=1}^{q} |g, \alpha_j| \ln \left(\frac{a_0}{|g, \alpha_j|^2} \right) \right)^{-1+\epsilon}$$

for $a_0 > 0$ and ϵ with $q - 4 > q\epsilon > 0$ and define

$$d\eta^2 := |f, g|^2 \lambda \tilde{\lambda} \frac{f'}{1 + |f|^2} \frac{g'}{1 + |g|^2} |dz|^2$$

outside the set $E := \bigcup_{j=1}^{q} f^{-1}(\alpha_j)$ and $d\eta^2 = 0$ on E where $|\cdot, \cdot|$ is the cordal distance between two complex values.

Third, we apply Ahlfors-schwartz Lemma for $d\eta^2$ in order to get

$$d\eta^2 \leq C \frac{4R^2}{(R^2 - |z|^2)^2} |dz|^2 \tag{4.5}$$

for a constant C.

Fourth, we assume that M is complete, and we consider M as open Riemann surfaces with induced metric $ds^2 = |h|^2(1 + |f|^2)(1 + |g|^2)|dz|^2$ from $\mathbb{R}^3$ where h is a nowhere zero holomorphic function.

Fifth, we take some δ with $q-6>q\delta>0$, set

$$\tau := \frac{2}{q-4-q\delta} < 1 \quad \text{for } q \geq 7,$$

and define the pseudo-metric $d\sigma^2$ by

$$d\sigma^2 := |h|^{\frac{2}{1-\tau}} \left(\frac{\prod_{j=1}^{q-1}(|f-\alpha_j||g-\alpha_j|)^{1-\delta}}{|f-g|^2|f'||g'|\prod_{j=1}^{q-1}(1+|\alpha_j|^2)^{1-\delta}} \right)^{\frac{\tau}{1-\tau}} |dz|^2 \tag{4.6}$$

which does not depend on a choice of holomorphic local coordinate z and so well-defined on $M' := M \setminus D$ where

$$D := \{z \in M \ : \ f'(z)=0, \ g'(z)=0, \text{ or } f(z)=g(z)=\alpha_j \text{ for some } j\}.$$

Note that $d\sigma^2$ is flat on M'.

Sixth, take an arbitrary point z in M'. Since $d\sigma^2$ is flat on M', we can take R ($\leq \infty$) such that there is a holomorphic map $B : \Delta(R) \to M'$ with $B(0)=z$ which is a local isometry with respect to the standard metric on $\Delta(R)$ and the metric $d\sigma^2$ on M'. Then the pseudo-metric $B^*d\eta^2$ on $\Delta(R)$ also has strictly negative curvature. Since there is no metric with strictly negative curvature on C, we have necessarily $R<\infty$.

Seventh, we choose a point a with $|a|=R$ such that for the line segment

$$L_a : w = ta \ (0 \leq t < 1),$$

the image Γ_a of L_a by B tends to the boundary of M' as t tends to 1. Then choosing the suitable δ in the definition τ, we can actually show that Γ_a tends to the boundary of M as $t \to 1$.

Eighth, Since B is a local isometry, we may take the coordinate w as a holomorphic local coordinate on M' and we may write $d\sigma^2 = |dw|^2$. Then, from (4.6) we obtain

$$|h|^2 = \left(\frac{|f-g|^2|f'||g'|\prod_{j=1}^{q-1}(1+|\alpha_j|^2)^{1-\delta}}{\prod_{j=1}^{q-1}(|f-\alpha_j||g-\alpha_j|)^{1-\delta}} \right)^{\tau}.$$

Ninth, by (4.5) with $\epsilon = \delta/2$ we have the inequality

$$
\begin{aligned}
ds^2 &= |h|^2(1+|f|^2)(1+|g|^2)|dz|^2 \\
&\leq C' \left(\frac{|f,g|^2|f'||g'|\lambda\tilde{\lambda}}{(1+|f|^2)(1+|g|^2)} \right)^{\tau} |dw|^2 \\
&\leq C \left(\frac{2R}{R^2-|w|^2} \right)^{2\tau} |dw|^2
\end{aligned}
$$

Finally, we calculate the length of Γ_a

$$
\int_{\Gamma_a} ds = \int_{L_a} B^* ds \leq C \int_0^R \left(\frac{2R}{R^2-|w|^2} \right)^{\tau} |dw| < \infty \quad \text{since } \tau < 1,
$$

which contradicts the assumption of completeness of M. Therefore, we have necessarily $f = g$. Q.E.D.

Fujimoto, in his paper [8], gave an example which shows that the number seven in (a) is the best possible number. Here is an example:

Take a number α with $\alpha \neq 0, 1, -1$ and consider the homomorphic functions

$$
h(z) := \frac{1}{z(z-\alpha)(\alpha z-1)}, \quad g(z) = z
$$

and the universal covering surface M of $\mathbb{C} \setminus \{0, \alpha, 1/\alpha\}$. Then by the Enneper-Weierstrass representations we can construct a minimal surface $x = (x_1,\ x_2,\ x_3)$ using the following formulas:

$$
x_1 := 2\,\mathbf{Re} \int_0^z (1-g^2)hdz, \;\; x_2 := 2\,\mathbf{Re} \int_0^z \sqrt{-1}(1+g^2)hdz, \;\; x_3 := 2\,\mathbf{Re} \int_0^z ghdz
$$

As we have seen in chapter 2, the map g is the classical Gauss map of M. It is easily seen that M is complete (the proof is similar to Example 2.4.5.). We can also construct another minimal surface $x = (\tilde{x}_1,\ \tilde{x}_2,\ \tilde{x}_3)$ in a similar manner from

$$
h(z) := \frac{1}{z(z-\alpha)(\alpha z-1)}, \quad \tilde{g}(z) = \frac{1}{z}
$$

Since $\tilde{M}$ is isometric with M, the identity map $\Phi : z \in M \to z \in \tilde{M}$ is a conformal diffeomorphism. For the classical Gauss maps g and $\tilde{g}$, we have $g^{-1}(\alpha_j) = \tilde{g}^{-1}(\alpha_j)$ for six values

$$\alpha_1 := 0, \quad \alpha_2 := \infty, \quad \alpha_3 := \alpha, \quad \alpha_4 := \frac{1}{\alpha}, \quad \alpha_5 := 1, \quad \alpha_6 := -1.$$

However, $g \not\equiv \tilde{g}$. Hence, the number seven in Theorem 4.1.1. cannot be reduced to six.

In 1993, Fujimoto [10] generalized Theorem 4.2.1 into $\mathbb{R}^m$. The generalized theorem is stated as follows.

Theorem 4.2.2. (H. Fujimoto [10], 1993) *Let $x := (x_1, \cdots, x_m) : M \to \mathbb{R}^m$ and $\tilde{x} := (\tilde{x}_1, \cdots, \tilde{x}_m) : \tilde{M} \to \mathbb{R}^m$ be two oriented non-flat complete minimal surfaces immersed in $\mathbb{R}^m$ and let $\mathsf{G} : M \to \mathbb{P}^{m-1}(\mathbb{C})$ and $\tilde{\mathsf{G}} : \tilde{M} \to \mathbb{P}^{m-1}(\mathbb{C})$ be their generalized Gauss maps. Assume that there is a conformal diffeomorphism Φ of M onto $\tilde{M}$. Then the Gauss map of the minimal surface $\tilde{x} \circ \Phi : M \to \mathbb{R}^m$ is given by $\tilde{\mathsf{G}} \circ \Phi$. Consider the holomorphic maps $f = \mathsf{G} : M \to \mathbb{P}^{m-1}(\mathbb{C})$, $g = \tilde{\mathsf{G}} \circ \Phi : M \to \mathbb{P}^{m-1}(\mathbb{C})$. Assume that there exist hyperplanes $H_1, \cdots, H_q$ in $\mathbb{P}^{m-1}(\mathbb{C})$ located in general position such that*

(i) *$f^{-1}(H_j) = g^{-1}(H_j)$ for every $j = 1, \cdots, q$,*

(ii) *$f = g$ on $\bigcup_{j=1}^{q} f^{-1}(H_j)$.*

If $q > m^2 + m(m-1)/2$, then $f \equiv g$.

The unicity theorem for meromorphic functions in the value distribution theory has been improved for several decades by many mathematicians, but there has been no improvement of the unicity theorem for Gauss maps in the minimal surface theory since Theorem 4.2.2 was published. Thus we are going to improve upon the theorem through this thesis.

4.3 Comparison of the Unicity Theorem for holomorphic curves of $\mathbb{C}$ and the Unicity Theorem for Gauss Maps of Complete Minimal Surfaces

In order to get some idea to make an improvement of Theorem 4.2.2, we now compare the recent unicity theorem for holomorphic maps on $\mathbb{C}$ to the recent unicity theorem for Gauss maps on complete minimal Surface M. See the table below.

Table 4.1: Comparison of the Recent Results of Unicity Theorems

<table>
<tr><td colspan="2">Holomorphic Curves $f, g : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$</td><td colspan="2">Gauss maps $f, g : M \to \mathbb{P}^n(\mathbb{C})$</td></tr>
<tr><td>with $f^{-1}(H_i \cap H_j) \neq \phi$</td><td>without</td><td>with $f^{-1}(H_i \cap H_j) \neq \phi$</td><td>without</td></tr>
<tr><td>$q > 2n+2$
(Chen and Yan [14], 2009)</td><td>$q > (n+1)^2$
(Giang, Quynh, and Quang [15], 2012)</td><td>(No result in the previous studies)</td><td>$q > (n+1)^2 + \frac{n(n+1)}{2}$
(Fujimoto [10], 1993)</td></tr>
<tr><td colspan="2">with $f^{-1}\left(\bigcap_{j=1}^{k+1} H_{i_j}\right) \neq \phi$</td><td colspan="2">with $f^{-1}\left(\bigcap_{j=1}^{k+1} H_{i_j}\right) \neq \phi$</td></tr>
<tr><td colspan="2">$q > (n+1) + \frac{2knq}{q-2k+2kn}$
(F. Lü [16], 2012)</td><td colspan="2">$q > (n+1) + \frac{2knq}{q-2k+2kn} + \frac{n(n+1)}{2}$
(The result in this thesis, 2016)</td></tr>
</table>

As we see the table above, the unicity theorem for holomorphic curves and the unicity theorm for Gauss maps are very related and the term $n(n+1)/2$ is the key.

Chapter 5

The Auxiliary Function and the Main Lemma

5.1 The Auxiliary Function

In this section, we construct a (new) auxiliary function, similar to the auxiliary function used in Chen-Yan [14] (see also [15] or [16]), which will be used later. Let M be a simply connected Riemann surface and $f,\ g : M \to \mathbb{P}^n(\mathbb{C})$ be two linearly non-degenerate holomorphic maps. Fix a reduced (global) representations $F = (f_0, \cdots, f_n)$ and $G := (g_0, \cdots, g_n)$, i.e. $f = \mathbf{P}(F),\ g = \mathbf{P}(G)$ where $\mathbf{P}$ denotes the canonical projection of $\mathbb{C}^n \setminus \{0\}$ onto $\mathbb{P}^n(\mathbb{C})$ and $f_0, \cdots, f_n$ (resp. $g_0, \cdots, g_n$) are holomorphic functions on M without common zeros such that $T_{f_i}(r) \leq T_f(r)$ and $T_{g_i}(r) \leq T_g(r)$. For a hyperplane $H = \{[z_0 : \cdots : z_n] |\, a_0 z_0 + \cdots + a_n z_n = 0\}$ in $\mathbb{P}^n(\mathbb{C})$, we define

$$F(H) := a_0 f_0 + \cdots + a_n f_n \quad \text{and} \quad G(H) := a_0 g_0 + \cdots + a_n g_n.$$

Assume that $f \not\equiv g$ on M, and let $H_1, \cdots, H_q$ be hyperplanes in general position in $\mathbb{P}^n(\mathbb{C})$. We define an equivalence relation on $L := \{1, \cdots, q\}$ as $i \sim j$ if and only if $\frac{F(H_i)}{G(H_i)} - \frac{F(H_j)}{G(H_j)} \equiv 0$. Set $\{L_1, \cdots, L_s\} = L/\sim$. Since $f \not\equiv g$ and $H_1, \cdots, H_q$ are in general position, we have that $\#(L_k) \leq n$ for all $k = \{1, \cdots, s\}$ where $\#(L_k)$ is the number of elements in L_k. Without loss of generality, we assume that $L_k := \{i_{k-1}+1,\ i_{k-1}+2, \cdots, i_k\}$ for $1 \leq k \leq s$, where $1 < i_1 < \cdots < i_s = q$, i.e.

$$\underbrace{\frac{F(H_1)}{G(H_1)} \equiv \frac{F(H_2)}{G(H_2)} \equiv \cdots \equiv \frac{F(H_{i_1})}{G(H_{i_1})}}_{L_1 \text{ group}} \not\equiv \underbrace{\frac{F(H_{i_1+1})}{G(H_{i_1+1})} \equiv \frac{F(H_{i_1+2})}{G(H_{i_1+2})} \equiv \cdots \equiv \frac{F(H_{i_2})}{G(H_{i_2})}}_{L_2 \text{ group}}$$
$$\not\equiv \underbrace{\frac{F(H_{i_2+1})}{G(H_{i_2+1})} \equiv \frac{F(H_{i_2+2})}{G(H_{i_2+2})} \equiv \cdots = \frac{F(H_{i_3})}{G(H_{i_3})}}_{L_3 \text{ group}} \not\equiv \cdots \not\equiv \underbrace{\frac{F(H_{i_{s-1}+1})}{G(H_{i_{s-1}+1})} \equiv \cdots \equiv \frac{F(H_{i_s})}{G(H_{i_s})}}_{L_s \text{ group}}.$$

Define the map $\sigma : \{1, \cdots, q\} \to \{1, \cdots, q\}$ by

$$\sigma(i) = \begin{cases} i+n & \text{if } i+n \leq q, \\ i+n-q & \text{if } i+n > q. \end{cases}$$

Then obviously σ is bijective and $|\sigma(i) - i| \geq n$ assuming $q \geq 2n$. This implies that i and $\sigma(i)$ belong two distinct sets of $\{L_1, \cdots, L_s\}$, so

$$\frac{F(H_i)}{G(H_i)} - \frac{F(H_{\sigma(i)})}{G(H_{\sigma(i)})} \not\equiv 0.$$

Let $\chi_i := F(H_i)G(H_{\sigma(i)}) - G(H_i)F(H_{\sigma(i)})$. Then $\chi_i \not\equiv 0$. Define

$$\chi := \prod_{i=1}^{q} \chi_i = \prod_{i=1}^{q} [F(H_i)G(H_{\sigma(i)}) - G(H_i)F(H_{\sigma(i)})] \not\equiv 0. \tag{5.1}$$

For a nonzero meromorphic function h on M, we define the divisor ν_h of h as a map of M into the set of integers such that for $z_0 \in M$

$$\nu_h(z_0) = \begin{cases} m & \text{if } \ h \text{ has a zero of order } m \text{ at } \ z_0, \\ -m & \text{if } \ h \text{ has a pole of order } m \text{ at } \ z_0, \\ 0 & \text{otherwise.} \end{cases}$$

We note that $\nu_{f(H)}(z)$ is the intersection multiplicity of the images of f and the hyperplane H at $f(z)$.

Lemma 5.1.1. (F. Lü [16], see also [14] and [15]) *Let $f, g : M \to \mathbb{P}^n(\mathbb{C})$ be two linearly nondegenerate holomorphic mappings. Let $H_1, \cdots, H_q$ be hyperplanes in $\mathbb{P}^n(\mathbb{C})$ located in general position with $q \geq 2n$ such that*

(i) $f^{-1}(H_j) = g^{-1}(H_j)$ for every $j = 1, \cdots, q$,

(ii) $\bigcap_{j=1}^{k+1} f^{-1}(H_{i_j}) = \varnothing$ for any $\{i_1, \cdots, i_{k+1}\} \subset \{1, \cdots, q\}$,

(iii) $f = g$ on $\displaystyle\bigcup_{j=1}^{q} f^{-1}(H_j)$.

Then the following holds on the domain of each holomorphic local coordinate z of M:

$$\nu_\chi(z) \geq \left(\frac{q - 2k + 2kn}{2kn}\right) \sum_{j=1}^{q} \left(\nu^n_{F(H_j)}(z) + \nu^n_{G(H_j)}(z)\right)$$

where $\nu^n_{F(H_j)}(z) = \min\{n, \nu_{F(H_j)}(z)\}$.

Proof. If $z \notin \bigcup_{j=1}^q f^{-1}(H_j)$, then $\nu_{F(H_j)}(z) = 0$, so this lemma is obviously true. Thus, we only need to consider the case when $z \in \bigcup_{j=1}^q f^{-1}(H_j)$. Let $J = \{j : F(H_j)(z) = 0, 1 \leq j \leq q\}$ and denote by $\#(J)$ the number of elements of J. Then, $\#(J) \leq k$ by the assumption (ii). If $j \in J$, then z is a zero of $F(H_j)$, and hence z

is a zero of χ_j with multiplicity at least $\min\{\nu_{F(H_j)}(z),\ \nu_{G(H_j)}(z)\}$. We now define $\sigma^{-1}(J) = \{j : \sigma(j) \in J\}$. If $l \in \{1, \cdots, q\} \setminus (J \bigcup \sigma^{-1}(J))$, then z is a zero of χ_l with multiplicity at least 1 by the assumption (iii), and so $\nu_{\chi_l} \geq 1$. Therefore,

$$
\begin{aligned}
\nu_\chi(z) &\geq \sum_{j\in J,\, j\in\sigma^{-1}(J)} \min\{\nu_{F(H_j)}(z), \nu_{G(H_j)}(z)\} + \sum_{l\in\{1,\cdots,q\}\setminus(J\bigcup\sigma^{-1}(J))} \nu_{\chi_l} \\
&\geq \sum_{j\in J,\, j\in\sigma^{-1}(J)} \min\{\nu_{F(H_j)}(z), \nu_{G(H_j)}(z)\} + \sum_{l\in\{1,\cdots,q\}\setminus(J\bigcup\sigma^{-1}(J))} 1 \\
&\geq 2\sum_{j\in J} \min\{\nu_{F(H_j)}(z), \nu_{G(H_j)}(z)\} + q - \#(J \cup \sigma^{-1}(J)) \\
&\geq 2\sum_{j\in J} \min\{\nu_{F(H_j)}(z), \nu_{G(H_j)}(z)\} + q - 2k \\
&\geq 2\sum_{j\in J} [\min\{n, \nu_{F(H_j)}(z)\} + \min\{n, \nu_{G(H_j)}(z)\} - n] + q - 2k. \\
&= 2\sum_{j\in J} [\nu^n_{F(H_j)}(z) + \nu^n_{G(H_j)}(z) - n \cdot \nu^1_{F(H_j)}(z)] + \frac{q-2k}{2k} \cdot 2k \\
&\geq 2\sum_{j=1}^{q} [\nu^n_{F(H_j)}(z) + \nu^n_{G(H_j)}(z) - n \cdot \nu^1_{F(H_j)}(z)] \\
&\quad + \frac{q-2k}{2k} \sum_{j=1}^{q} [\nu^1_{F(H_j)}(z) + \nu^1_{G(H_j)}(z)] \\
&\geq 2\sum_{j=1}^{q} [\nu^n_{F(H_j)}(z) + \nu^n_{G(H_j)}(z)] + [\frac{q-2k}{2k} - n] \sum_{j=1}^{q} [\nu^1_{F(H_j)}(z) + \nu^1_{G(H_j)}(z)] \\
&\geq 2\sum_{j=1}^{q} [\nu^n_{F(H_j)}(z) + \nu^n_{G(H_j)}(z)] + [\frac{q-2k}{2kn} - 1] \sum_{j=1}^{q} [\nu^n_{F(H_j)}(z) + \nu^n_{G(H_j)}(z)] \\
&= \left(\frac{q-2k+2kn}{2kn}\right) \sum_{j=1}^{q} \left(\nu^n_{F(H_j)}(z) + \nu^n_{G(H_j)}(z)\right),
\end{aligned}
$$

where, in above, we use the facts that $\min\{a, b\} \geq \min\{a, n\} + \min\{b, n\} - n$ for all positive integers a and b, $\nu^1_{F(H_j)}(z) = 1$ for $j \in J$, as well as $k \geq \#(J) \geq \sum_{j=1}^q \min\{1, \nu_{F(H_j)}(z)\} = \sum_{j=1}^q \nu^1_{F(H_j)}(z)$. This finishes the proof. Q.E.D.

Lemma 5.1.2. (M. Ru [21]) *Let $H_1, \cdots, H_q$ be hyperplanes $\mathbb{P}^n(\mathbb{C})$ in general position and let $F = (f_0, \cdots, f_n)$ be the reduced representation of a linearly non-degenerate holomorphic map $f : M \to \mathbb{P}^n(\mathbb{C})$. Then the following holds on the domain of each holomorphic local coordinate z of M:*

$$\sum_{j=1}^{q} \nu_{F(H_j)}(z) - \nu_{F_n}(z) \leq \sum_{j=1}^{q} \nu^n_{F(H_j)}(z)$$

where $F_n = W(f_0, \cdots, f_n)$ which is the Wronskian of $f_0, \cdots, f_n$.

Proof. If $z_0 \notin \bigcup_{j=1}^{q} f^{-1}(H_j)$, then $\nu_{F(H_j)}(z_0) = 0$, so this lemma is obviously true. Thus, we only need to consider the case when $z_0 \in \cup_{j=1}^{q} f^{-1}(H_j)$. Then there are integers $h_j \geq 0$ and nowhere vanishing holomorphic function g_j in a neighborhood U of z_0 such that

$$F(H_j) = (z - z_0)^{h_j} g_j \quad \text{for } \ j = 1, \cdots, q$$

Let $J = \{j : F(H_j)(z_0) = 0, 1 \leq j \leq q\}$. Then $\#(J) \leq n$ because $H_1, \cdots, H_q$ are in general position. We re-order $H_1, \cdots, H_q$ so that the multiplicity $h_j,\ 1 \leq j \leq q$ are in descending order. Without loss of generality, we assume that $h_j \geq h_{j+1}$ for $1 \leq j \leq j_1$ and $h_j = 0$ for $j_1 + 1 \leq j \leq q$ where $j_1 \leq n$. Assume that $h_1 \geq h_2 \geq \cdots \geq h_{j_0} \geq n$ for $1 \leq j_0 \leq j_1$. Since each $F(H_j),\ 1 \leq j \leq q$ is a linear combination of $f_0, \cdots, f_n$, by the property of the wronskian,

$$F_n = W(f_0, \cdots, f_n) = cW(F(H_1), \cdots, F(H_{n+1})) = \prod_{j=1}^{j_0} (z - z_0)^{h_j - n} \xi(z)$$

where $\xi(z)$ is a holomorphic function defined on U and c is a constant. Therefore,

$$\nu_{F_n}(z_0) \geq \sum_{j=1}^{j_0} (h_j - n).$$

Thus,

$$\begin{aligned}\sum_{j=1}^{q}\nu_{F(H_j)}(z_0)-\nu_{F_n}(z_0) &= \sum_{j=1}^{n}\nu_{F(H_j)}(z_0)-\nu_{F_n}(z_0)\\ &\leq \sum_{j=1}^{n}\nu_{F(H_j)}(z_0)-\sum_{j=1}^{j_0}(h_j-n)\\ &= \sum_{j=j_0+1}^{n}\nu_{F(H_j)}(z_0)+\sum_{j=1}^{j_0}[\nu_{F(H_j)}(z_0)-(h_j-n)]\\ &= \sum_{j=j_0+1}^{n}\nu_{F(H_j)}(z_0)+\sum_{j=1}^{j_0}n\\ &= \sum_{j=1}^{n}\nu^n_{F(H_j)}(z_0)=\sum_{j=1}^{q}\nu^n_{F(H_j)}(z_0). \qquad \text{Q.E.D.}\end{aligned}$$

Proposition 5.1.3. *The function*

$$\frac{|\chi|^{\frac{2kn}{q-2k+2kn}}|F_nG_n|}{\prod_{j=1}^{q}|F(H_j)G(H_j)|}$$

is continuous on the domain of each holomorphic local coordinate z of M, where χ is defined in (5.1).

Proof. Denote by

$$P:=\frac{|\chi|^{\frac{2kn}{q-2k+2kn}}|F_nG_n|}{\prod_{j=1}^{q}|F(H_j)G(H_j)|}$$

and let $E:=\bigcup_{j=1}^{q}f^{-1}(H_j)$. Then P is obviously continuous at z_0 for all $z_0\notin E$. Now assume that $z_0\in E$. By Lemma 5.1.1. and Lemma 5.1.2., we get

$$\begin{aligned}\nu_P(z_0) &\geq \frac{2kn}{q-2k+2kn}\nu_\chi(z_0)+\nu_{F_n}(z_0)+\nu_{G_n}(z_0)-\sum_{j=1}^{q}\nu_{F(H_j)}(z_0)-\sum_{j=1}^{q}\nu_{G(H_j)}(z_0)\\ &\geq \sum_{j=1}^{q}\left(\nu^n_{F(H_j)}(z_0)+\nu^n_{G(H_j)}(z_0)\right)-\sum_{j=1}^{q}\nu^n_{F(H_j)}(z_0)-\sum_{j=1}^{q}\nu^n_{G(H_j)}(z_0)=0.\end{aligned}$$

Therefore, P is continuous. Q.E.D.

5.2 Pseudo-metric on $\Delta(R)$ with Negative Curvature

Let $f : \Delta(R) \to \mathbb{P}^n(\mathbb{C})$ be a linearly non-degenerate holomorphic map where $\Delta(R) := \{z : |z| < R\}$ with $0 < R < \infty$. Let $H_1, \cdots, H_q$ be hyperplanes located in general position in $\mathbb{P}^n(\mathbb{C})$. Min Ru (see M. Ru [19]) constructed a pseudo-metric $\Gamma = \frac{\sqrt{-1}}{2\pi} h(z) dz \wedge d\bar{z}$ on $\Delta(R) \setminus \cup_{j=1}^q \{\phi_0(H_j) = 0\}$ whose Gauss curvature is less than or equal to -1, i.e. Ric $\Gamma \geq \Gamma$ (where Ric $\Gamma := dd^c \ln h$), for $q \geq n+2$ as follows:

Lemma 5.2.1 (M. Ru [19]). *Assume $q \geq n+2$ and $\frac{2q}{N} < \frac{q-(n+1)}{n(n+2)}$ and define*

$$h(z) := c \prod_{j=1}^{q} \left(\frac{1}{\phi_0(H_j)} \right)^{\beta_n} \prod_{j=1}^{q} \left[\prod_{p=0}^{n-1} \frac{h_p^{\beta_n/\lambda_p}}{(N - \ln \phi_p(H_j))^{2\beta_n}} \right]$$

where $h_p = \frac{|F_{p-1}|^2 |F_{p+1}|^2}{|F_p|^4}$, $\beta_n = \frac{1}{\sum_{p=0}^{n-1} \lambda_p^{-1}}$, $\lambda_p = \frac{1}{n-p+(n-p)^2 \frac{2q}{N}}$, and $c > 0$ is some positive constant. Let $\Gamma = \frac{\sqrt{-1}}{2\pi} h(z) dz \wedge d\bar{z}$. Then Ric $\Gamma \geq \Gamma$ on $\Delta(R) \setminus \cup \{\phi_0(H_j) = 0\}$.

(In our case, since hyperplanes $H_1, \cdots, H_q$ are located in general position, $\theta = 1$ and $\sum_{j=1}^q \omega(j) = q$ where θ is the *Nochka constant* and ω is the *Nochka weight*.)

Note that, Min Ru (see at the end of the proof of the Main Lemma in [19]) indeed proved the following slightly stronger result:

$$\textit{Ric}\, \Gamma \geq \beta_n \left[\left(q - (n+1) - (n^2+2n) \frac{2q}{N} \right) \Omega_0 + \sum_{p=1}^{n-2} \frac{2q}{N} \Omega_p + \frac{2q}{N} \Omega_{n-1} \right] + \Gamma. \quad (5.2)$$

It is obvious that the right-hand side of (5.2) is bigger than Γ because $q - (n+1) - (n^2+2n)\frac{2q}{N} > 0$ by the assumption. We will use (5.2) in our proof.

Noticing that

$$\begin{aligned} \prod_{p=0}^{n-1} h_p^{\frac{1}{\lambda_p}} &= \prod_{p=0}^{n-1} \left(\frac{|F_{p-1}|^2 |F_{p+1}|^2}{|F_p|^4} \right)^{n-p+(n-p)^2 \frac{2q}{N}} \\ &= |F_0|^{-2(n+1)-2(n^2+2n-1)\frac{2q}{N}} |F_1|^{\frac{8q}{N}} \cdots |F_{n-1}|^{\frac{8q}{N}} |F_n|^{2+\frac{4q}{N}}, \end{aligned}$$

$|F_0| = |F|$ and $\phi_0(H_j) = \frac{|F(H_j)|^2}{|F|^2}$, we get

$$h(z) := c\left[\frac{|F|^S(|F_1|\cdots|F_{n-1}|)^{\frac{4q}{N}}|F_n|^{1+\frac{2q}{N}}}{\prod_{j=1}^q |F(H_j)| \prod_{p=0}^{n-1}\prod_{j=1}^q (N - \ln\phi_p(H_j))}\right]^{2\beta_n} \tag{5.3}$$

where $S = q - (n+1) - (n^2 + 2n - 1)\frac{2q}{N}$.

Let $f,\ g : \Delta(R) := \{z : |z| < R\}(\subset \mathbb{C}) \to \mathbb{P}^n(\mathbb{C})$ be two linearly non-degenerate holomorphic maps with reduced representations $F = (f_0, \cdots, f_n)$ and $G = (g_0, \cdots, g_n)$ respectively, where $0 < R < \infty$. Let $H_1, \cdots, H_q$ be hyperplanes in $\mathbb{P}^n(\mathbb{C})$ in general position. Denote by

$$\phi_k^F(H) = \frac{|F_k(H)|^2}{|F_k|^2}, \qquad \phi_k^G(H) = \frac{|G_k(H)|^2}{|G_k|^2}$$

and, according to (5.3), we let

$$h_1 := c\left[\frac{|F|^S(|F_1|\cdots|F_{n-1}|)^{\frac{4q}{N}}|F_n|^{1+\frac{2q}{N}}}{\prod_{j=1}^q |F(H_j)| \prod_{p=0}^{n-1}\prod_{j=1}^q (N - \ln\phi_p^F(H_j))}\right]^{2\beta_n}, \tag{5.4}$$

$$h_2 := c\left[\frac{|G|^S(|G_1|\cdots|G_{n-1}|)^{\frac{4q}{N}}|G_n|^{1+\frac{2q}{N}}}{\prod_{j=1}^q |G(H_j)| \prod_{p=0}^{n-1}\prod_{j=1}^q (N - \ln\phi_p^G(H_j))}\right]^{2\beta_n}. \tag{5.5}$$

Lemma 5.2.2. *Assume $q \geq n+1$, $q - (n+1) - (n^2+2n-1)\frac{2q}{N} - \frac{2knq}{q-2k+2kn} > 0$ and take N such that $\frac{2q}{N} < \frac{q-(n+1)}{n(n+2)}$. Let $\Theta = \frac{\sqrt{-1}}{2\pi}\eta(z)dz \wedge d\bar{z}$ with*

$$\eta(z) := c\left(\frac{|\chi(z)|}{|F(z)|^q|G(z)|^q}\right)^{\frac{2kn\beta_n}{q-2k+2kn}} \sqrt{h_1(z)h_2(z)}, \tag{5.6}$$

where χ is the auxiliary function given in (5.1), h_1, h_2 are given in (5.4) and (5.5) respectively. Then $\eta(z)$ is continuous on $\Delta(R)$ and Ric $\Theta \geq C\Theta$ for some positive *constant* C.

Proof. Plugging $h_1(z)$ and $h_2(z)$ into $\eta(z)$, we have

$$\eta = c\left(\frac{(|F||G|)^{S-\frac{2knq}{q-2k+2kn}}|\chi|^{\frac{2kn}{q-2k+2kn}}\prod_{p=1}^{n-1}(|F_p||G_p|)^{\frac{4q}{N}}(|F_n||G_n|)^{1+\frac{2q}{N}}}{\prod_{j=1}^{q}(|F(H_j)G(H_j)|)\prod_{p=0}^{n-1}\prod_{j=1}^{q}\left[(N-\ln\phi_p^F(H_j))(N-\ln\phi_p^G(H_j))\right]}\right)^{\beta_n}.$$

By Proposition 5.1.3., $\eta(z)$ is continuous on $\Delta(R)$. Now we calculate Ric Θ. Set $\Gamma_1 = \frac{\sqrt{-1}}{2\pi}h_1(z)dz\wedge d\overline{z}$ and $\Gamma_2 = \frac{\sqrt{-1}}{2\pi}h_2(z)dz\wedge d\overline{z}$ where Γ_1, Γ_2 are given in (5.4) and (5.5). Using (5.3) (which holds for h_1, h_2 as well), the assumption

$$q-(n+1)-(n^2+2n-1)\frac{2q}{N}-\frac{2knq}{q-2k+2kn}>0,$$

and the fact that χ is holomorphic, we have

$$\begin{aligned}
&\text{Ric }\Theta = dd^c\ln\eta(z)\\
&=\frac{1}{2}dd^c\ln h_1(z)+\frac{1}{2}dd^c\ln h_2(z)+\frac{kn\beta_n}{q-2k+2kn}\left[dd^c\ln|\chi|^2-q\cdot dd^c(\ln|F|^2+\ln|G|^2)\right]\\
&\geq\frac{1}{2}\text{Ric }\Gamma_1+\frac{1}{2}\text{Ric }\Gamma_2-\frac{knq\beta_n}{q-2k+2kn}\left(dd^c\ln|F|^2+dd^c\ln|G|^2\right)\\
&\geq\frac{1}{2}\beta_n\left[\left(q-(n+1)-(n^2+2n)\frac{2q}{N}\right)\Omega_0^f+\sum_{p=1}^{n-2}\frac{2q}{N}\Omega_p^f+\frac{2q}{N}\Omega_{n-1}^f\right]+\frac{1}{2}\Gamma_1\\
&\quad+\frac{1}{2}\beta_n\left[\left(q-(n+1)-(n^2+2n)\frac{2q}{N}\right)\Omega_0^g+\sum_{p=1}^{n-2}\frac{2q}{N}\Omega_p^g+\frac{2q}{N}\Omega_{n-1}^g\right]+\frac{1}{2}\Gamma_2\\
&\quad-\frac{knq\beta_n}{q-2k+2kn}\Omega_0^f-\frac{knq\beta_n}{q-2k+2kn}\Omega_0^g\\
&\geq\frac{1}{2}\beta_n\left[\left(q-(n+1)-(n^2+2n)\frac{2q}{N}-\frac{2knq}{q-2k+2kn}\right)\Omega_0^f\right]+\frac{1}{2}\Gamma_1\\
&\quad+\frac{1}{2}\beta_n\left[\left(q-(n+1)-(n^2+2n)\frac{2q}{N}-\frac{2knq}{q-2k+2kn}\right)\Omega_0^g\right]+\frac{1}{2}\Gamma_2\\
&\geq\frac{1}{2}\left(\Gamma_1+\Gamma_2\right)\geq\frac{\sqrt{-1}}{2\pi}\sqrt{h_1(z)h_2(z)}dz\wedge d\overline{z}
\end{aligned}$$

$$\geq \frac{\sqrt{-1}}{2\pi} C' \left(\frac{|\chi(z)|}{|F(z)|^q |G(z)|^q} \right)^{\frac{2kn\beta_n}{q-2k+2kn}} \sqrt{h_1(z)h_2(z)} dz \wedge d\bar{z}$$
$$= C\Theta.$$

In the last step, we used the fact that $|\chi(z)| \leq C''|F(z)|^q|G(z)|^q$ for some positive constant C''. Q.E.D.

5.3 The Main Lemma

The following Ahlfors-Schwarz lemma is well-known (see M. Ru [20]):

Ahlfors-Schwarz Lemma. *Let* $\Gamma = \frac{\sqrt{-1}}{2\pi} h(z) dz \wedge d\bar{z}$ *be a continuous pseudo-metric on* $\Delta(R)$ *whose Gaussian curvature is bounded above by* -1 *(i.e.,* $Ric\Gamma \geq \Gamma$*). Then, for all* $z \in \Delta(R)$,

$$h(z) \leq \left(\frac{2R}{R^2 - |z|^2} \right)^2.$$

Now we are ready to prove our Main Lemma. Let

$$\psi^F_{j,p} = \sum_{l \neq i_1, \cdots, i_p} a_{j,l} W(f_l, f_{i_1}, \cdots, f_{i_p}) \text{ and } \psi^G_{j,p} = \sum_{l \neq i_1, \cdots, i_p} a_{j,l} W(g_l, g_{i_1}, \cdots, g_{i_p}).$$

By the assumption that f and g are linearly non-degenerate, $\psi^F_{j,p}$ and $\psi^G_{j,p}$ do not vanish identically, and thus have only isolated zeros since they are both holomorphic. Note that, from definition,

$$|\psi^F_{j,p}| < |F_p(H_j)| \quad \text{and} \quad |\psi^G_{j,p}| < |G_p(H_j)|. \tag{5.7}$$

Main Lemma. *Assume* $q \geq n+1$, $S - \frac{2knq}{q-2k+2kn} > 0$ *with* $S = q-(n+1)-(n^2+2n-1)\frac{2q}{N}$, *and take* N *such that* $\frac{2q}{N} < \frac{q-(n+1)}{n(n+1)}$. *Then there exists some positive constant* C_0 *such that, for all* $z \in \Delta(R)$,

$$\frac{|F(z)G(z)|^{S-\frac{2knq}{q-2k+2kn}}|\chi(z)|^{\frac{2kn}{q-2k+2kn}}|F_n(z)G_n(z)|^{1+\frac{2q}{N}}\prod_{p=0}^{n-1}\prod_{j=1}^{q}\left(|\psi_{j,p}^F(z)\psi_{j,p}^G(z)|\right)^{\frac{4}{N}}}{\prod_{j=1}^{q}|F(H_j)(z)G(H_j)(z)|}$$

$$\leq C_0\left(\frac{2R}{R^2-|z|^2}\right)^{2[\frac{n(n+1)}{2}+\sum_{p=0}^{n-1}(n-p)^2\frac{2q}{N}]}.$$

Proof. From Lemma 4, we know that $\eta(z)$ is a continuous map and Ric $\Theta \geq C\Theta$. Applying the Schwarz lemma for Θ, we get

$$\eta(z) \leq C_0\left(\frac{2R}{R^2-|z|^2}\right)^2. \tag{5.8}$$

for some constant $C_0 > 0$. On the other hand, from (5.4), (5.5), the definition of η, and the fact that $\beta_n = \frac{1}{\sum_{p=0}^{n-1}\lambda_p^{-1}} = \frac{1}{\sum_{p=0}^{n-1}\left(n-p+(n-p)^2\frac{2q}{N}\right)} = \frac{1}{\frac{n(n+1)}{2}+\sum_{p=0}^{n-1}(n-p)^2\frac{2q}{N}}$, we get

$$\eta^{[\frac{n(n+1)}{2}+\sum_{p=0}^{n-1}(n-p)^2\frac{2q}{N}]}$$
$$= \frac{|FG|^{S-\frac{2knq}{q-2k+2kn}}|\chi|^{\frac{2kn}{q-2k+2kn}}(|F_1|\cdots|F_{n-1}||G_1|\cdots|G_{n-1}|)^{\frac{4q}{N}}|F_nG_n|^{1+\frac{2q}{N}}}{\prod_{j=1}^{q}|F(H_j)G(H_j)|\prod_{p=0}^{n-1}\prod_{j=1}^{q}\left[(N-\ln\phi_p^F(H_j))(N-\ln\phi_p^G(H_j))\right]}.$$

For a given $2/N > 0$, it holds that $\lim_{x\to 0}x^{2/N}(N-\ln x) < \infty$, so we can set $K := \sup_{0<x\leq 1}x^{2/N}(N-\ln x)$. Since $0 < \phi_p^F(H_j) < 1$ for all p and j, we have, by using (5.7),

$$\frac{1}{N-\ln\phi_p^F(H_j)} \geq \frac{1}{K}\phi_p^F(H_j)^{2/N} = \frac{1}{K}\frac{|F_p(H_j)|^{4/N}}{|F_p|^{4/N}} \geq \frac{1}{K}\frac{|\psi_{j,p}^F|^{4/N}}{|F_p|^{4/N}}.$$

Similarly,

$$\frac{1}{N-\ln\phi_p^G(H_j)} \geq \frac{1}{K}\frac{|\psi_{j,p}^G|^{4/N}}{|G_p|^{4/N}}.$$

Hence,

$$\eta^{[\frac{n(n+1)}{2}+\sum_{p=0}^{n-1}(n-p)^2\frac{2q}{N}]}$$

$$\geq C\frac{|FG|^{S-\frac{2knq}{q-2k+2kn}}|\chi|^{\frac{2kn}{q-2k+2kn}}|F_nG_n|^{1+\frac{2q}{N}}\prod_{p=0}^{n-1}\prod_{j=1}^{q}\left(|\psi_{j,p}^F|^{4/N}|\psi_{j,p}^G|^{4/N}\right)}{\prod_{j=1}^{q}|F(H_j)|\prod_{j=1}^{q}|G(H_j)|}$$

for some constant $C > 0$. This, together with (5.8), proves the lemma. Q.E.D.

Chapter 6

The Main Theorem

6.1 Proof of the Main Theorem

Main Theorem. *Assume that both* $f = \mathsf{G} : M \to \mathbb{P}^{m-1}(\mathbb{C})$ *and* $g = \tilde{\mathsf{G}} \circ \Phi : M \to \mathbb{P}^{m-1}(\mathbb{C})$ *are linearly non-degenerate (i.e. the images of* f *and* g *are not contained in any linear subspaces of* $\mathbb{P}^{m-1}(\mathbb{C})$*) and that there exist hyperplanes* $H_1, \cdots, H_q$ *in* $\mathbb{P}^{m-1}(\mathbb{C})$ *located in general position and a positive integer* $k > 0$ *such that*

(i) $f^{-1}(H_j) = g^{-1}(H_j)$ *for every* $j = 1, \cdots, q$,

(ii) $\displaystyle\bigcap_{j=1}^{k+1} f^{-1}(H_{i_j}) = \varnothing$ *for any* $\{i_1, \cdots, i_{k+1}\} \subset \{1, \cdots, q\}$,

(iii) $f = g$ *on* $\displaystyle\bigcup_{j=1}^{q} f^{-1}(H_j)$.

$$\text{If } q > \frac{(m^2+m+4k) + \sqrt{(m^2+m+4k)^2 + 16k(m-2)m(m+1)}}{4}, \quad \text{then } f \equiv g.$$

Proof. First, we let $n = m - 1$ for simplicity. Then we obtain the equivalent inequility from the assumption for our $q > 0$ as the following:

$$\begin{aligned} & q > \frac{(m^2+m+4k) + \sqrt{(m^2+m+4k)^2 + 16k(m-2)m(m+1)}}{4} \\ \Leftrightarrow\ & q > \frac{(n^2+3n+2+4k) + \sqrt{(n^2+3n+2+4k)^2 + 16k(n-1)(n+1)(n+2)}}{4} \\ \Leftrightarrow\ & q - (n+1) - \frac{n(n+1)}{2} - \frac{2knq}{q-2k+2kn} > 0. \end{aligned} \tag{6.1}$$

Let $x : M \to \mathbb{R}^m$ and $\tilde{x} : \tilde{M} \to \mathbb{R}^m$ be two oriented non-flat minimal surfaces immersed in $\mathbb{R}^m$ and let $\mathsf{G} = [\frac{\partial x_1}{\partial z} : \cdots : \frac{\partial x_m}{\partial z}] : M \to \mathbb{P}^{m-1}(\mathbb{C})$ and $\tilde{\mathsf{G}} = [\frac{\partial \tilde{x}_1}{\partial z} : \cdots : \frac{\partial \tilde{x}_m}{\partial z}] : \tilde{M} \to \mathbb{P}^{m-1}(\mathbb{C})$ be their generalized Gauss maps of M and $\tilde{M}$ respectively. Assume that there is a conformal diffeomorphism Φ of M onto $\tilde{M}$. Then the induced metric on M and on $\tilde{M}$ from the standard metric on $\mathbb{R}^m$ are given by $ds^2 = 2\left(\left|\frac{\partial x_1}{\partial z}\right|^2 + \cdots + \left|\frac{\partial x_m}{\partial z}\right|^2\right)|dz|^2$ and $d\tilde{s}^2 = 2\left(\left|\frac{\partial \tilde{x}_1}{\partial z}\right|^2 + \cdots + \left|\frac{\partial \tilde{x}_m}{\partial z}\right|^2\right)|dz|^2$ respectively. Assume that M and $\tilde{M}$ are complete with respect to the induced metrics. Note that we may regard M and $\tilde{M}$ as Riemann surfaces and Φ as a conformal diffeomorphism between M and $\tilde{M}$. Consider the linearly non-degenerate holomorpic maps

$$f := \mathsf{G} : M \to \mathbb{P}^n(\mathbb{C}) \quad \text{and} \quad g := \tilde{\mathsf{G}} \circ \Phi : M \to \mathbb{P}^n(\mathbb{C})$$

with $n = m - 1$ and take their reduced (global) representations $F := (f_0, \cdots, f_n)$ and $G := (g_0, \cdots, g_n)$ respectively with $f = \mathbf{P}(F)$, $g = \mathbf{P}(G)$ where $\mathbf{P}$ is the canonical projection of $\mathbb{C}^{n+1} \setminus \{0\}$ onto $\mathbb{P}^n(\mathbb{C})$. By Proposition 2.2.2. $f_0, \cdots, f_n$ (resp. $g_0, \cdots, g_n$) are holomorphic functions on M without common zeros. Set

$|F|^2 = \left|\frac{\partial x_1}{\partial z}\right|^2 + \cdots + \left|\frac{\partial x_m}{\partial z}\right|^2$ and $|G|^2 = \left|\frac{\partial \tilde{x}_1}{\partial z}\right|^2 + \cdots + \left|\frac{\partial \tilde{x}_m}{\partial z}\right|^2$. Then, since $\mathsf{G} = \tilde{\mathsf{G}} \circ \Phi$, we can rewrite the (induced) metric by

$$\begin{aligned} ds^2 &= 2|F|^2|dz|^2 = 2|F||\mathbf{P}^{-1}(f)||dz|^2 = 2|F||\mathbf{P}^{-1}(\mathsf{G})||dz|^2 \\ &= 2|F||\mathbf{P}^{-1}(\tilde{\mathsf{G}} \circ \Phi)||dz|^2 = 2|F||\mathbf{P}^{-1}(g)||dz|^2 \\ &= 2|F||G||dz|^2 \end{aligned}$$

which is, by the assumption, complete. In order to prove the unicity theorem, we will use $ds^2 = 2|F||G||dz|^2$ because it has both F and G. We are now ready to prove the Main Theorem. By taking the universal cover of M and lifting the maps, if necessary, we can assume that M is simply connected. Then, by the Unifomization theorem, the Riemann surface M is conformally equivalent to either $\mathbb{C}$ or the unit-disc $\Delta := \{z : |z| < 1\}$.

In the case when $M = \mathbb{C}$, it is known that $f \equiv g$ from [8]. We enclose a proof here for the sake of completeness. Assume that $f \not\equiv g$. We will use Cartan's Second Main Theorem to derive a contradiction. To do so, we use some standard notations in Nevanlinna theory. From Lemma 5.1.1., we get

$$N_\chi(r,0) \geq \left(\frac{q - 2k + 2kn}{2kn}\right) \sum_{j=1}^{q} \left(N_f^{(n)}(r, H_j) + N_g^{(n)}(r, H_j)\right),$$

and, by the First Main Theorem, $N_{\chi_i}(r,0) \leq T_f(r) + T_g(r)$ (by choosing and fixing the reduced representation $F = (f_0, \cdots, f_n)$ and $G = (g_0, \cdots, g_n)$ of f and g respectively). On the other hand, from H. Cartan's Second Main Theorem, we have that for $\epsilon > 0$ and all r $(0 \leq r < R,\ R \leq \infty)$ except for a set E with finite Lebesgue measure, the inequality

$$(q-(n+1)-\epsilon)T_f(r) \leq \sum_{j=1}^{q} N_f^{(n)}(r, H_j) + o(T_f(r))$$

hols as $r \to \infty$. It is also holds for g. Hence, we get

$$\begin{aligned} q\left(T_f(r)+T_g(r)\right) &\geq \sum_{j=1}^{q} N_{\chi_j}(r,0) \\ &\geq N_\chi(r,0) \\ &\geq \left(\frac{q-2k+2kn}{2kn}\right)\sum_{j=1}^{q}\left(N_f^{(n)}(r,H_j)+N_g^{(n)}(r,H_j)\right) \\ &\geq \left(\frac{q-2k+2kn}{2kn}\right)(q-(n+1)-\epsilon)(T_f(r)+T_g(r)) \end{aligned}$$

After simplifying, we have

$$q \geq \left(\frac{q-2k+2kn}{2kn}\right)(q-(n+1)) \quad \Longleftrightarrow \quad q-(n+1)-\frac{2knq}{q-2k+2kn} \leq 0$$

which gives a contradiction under the assumption (6.1) for our q.

So we only need to consider the case $M = \Delta := \{z : |z| < 1\}$. We now have two linearly non-degenerate holomorphic maps $f,\ g : \Delta \to \mathbb{P}^n(\mathbb{C})$. Assume that $f \not\equiv g$. We will derive a contradiction. We will use the notations as being introduced in the previous sections. Since $q-(n+1)-\frac{n(n+1)}{2}-\frac{2knq}{q-2k+2kn} > 0$ and $n^2+2n-1+\sum_{p=0}^{n-1}(n-p)^2 = n^2+2n-1+\frac{n(n+1)}{2}+\sum_{p=0}^{n-1}p(p+1)$, we can thus choose $N > 0$ such that

$$\frac{2q\left[n^2+2n-1+\sum_{p=0}^{n-1}(n-p)^2\right]}{q-(n+1)-\frac{n(n+1)}{2}-\frac{2knq}{q-2k+2kn}} < N < \frac{2+2q\left[n^2+2n-1+\frac{n(n+1)}{2}+\sum_{p=0}^{n-1}p(p+1)\right]}{q-(n+1)-\frac{n(n+1)}{2}-\frac{2knq}{q-2k+2kn}}.$$

Let $\rho = S - [\frac{n(n+1)}{2} + \frac{2knq}{q-2k+2kn} + n(n+1)\frac{q}{N} + \sum_{p=0}^{n-1} p(p+1)\frac{2q}{N}]$ where $S = q - (n+1) - (n^2+2n-1)\frac{2q}{N}$. Then

$$\rho > 0 \quad \text{and} \quad \frac{2}{\rho N} > 1. \tag{6.2}$$

Moreover, from

$$N > \frac{2q\left[n^2+2n-1+\sum_{p=0}^{n-1}(n-p)^2\right]}{q-(n+1)-\frac{n(n+1)}{2}-\frac{2knq}{q-2k+2kn}}$$

we get

$$N[q-(n+1)] - 2qn(n+2) > N\left[\frac{n(n+1)}{2} + \frac{2knq}{q-2k+2kn}\right] + 2q\left[-1+\sum_{p=0}^{n-1}(n-p)^2\right] > 0$$

and

$$q-(n+1)-(n^2+2n-1)\frac{2q}{N} - \frac{2knq}{q-2k+2kn} > \frac{n(n+1)}{2} + \frac{2q}{N}\sum_{p=0}^{n-1}(n-p)^2 > 0,$$

and thus

$$\frac{2q}{N} < \frac{q-(n+1)}{n(n+2)} \text{ and } q-(n+1)-(n^2+2n-1)\frac{2q}{N} - \frac{2knq}{q-2k+2kn} > 0.$$

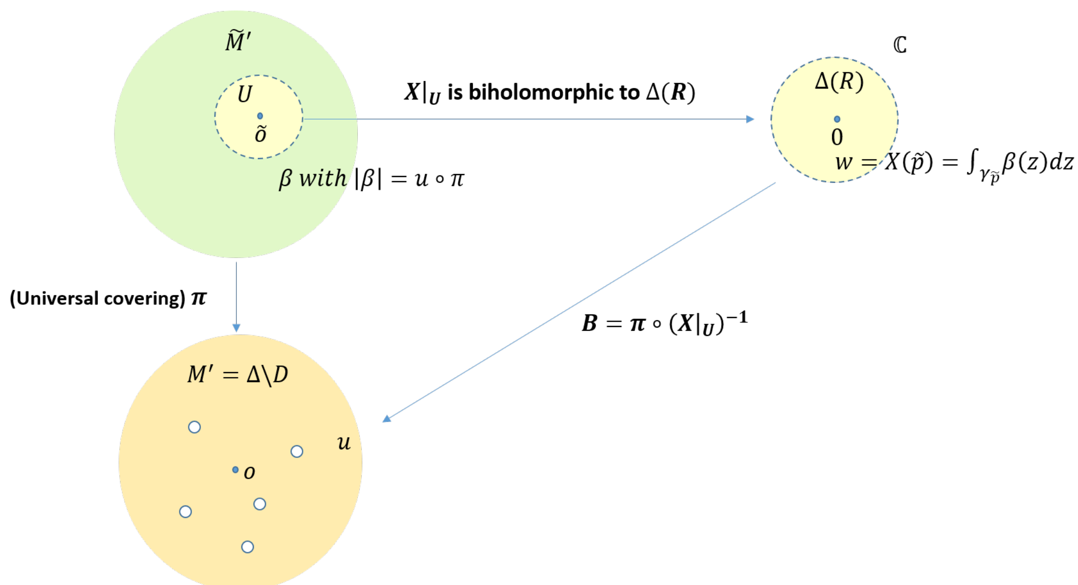
Hence we can apply the Main Lemma. To do so, let $\rho^* = \frac{1}{\rho}$ and let

$$u = \left(\frac{\prod_{j=1}^{q}|F(H_j)G(H_j)|}{|\chi|^{\frac{2kn}{q-2k+2kn}}|F_nG_n|^{1+2q/N}\prod_{p=0}^{n-1}\prod_{j=1}^{q}\left(|\psi_{j,p}^F\psi_{j,p}^G|\right)^{4/N}}\right)^{\frac{\rho^*}{2}} \tag{6.3}$$

which, by Proposition 5.2.3., is a strictly positive continuous function (i.e. it has no zeros) on $M' = \Delta\backslash D$ where

$$D := \{F_n = 0\} \cup \{G_n = 0\} \cup_{j,p} \{z : \ \psi_{j,p}^F(z) = 0 \text{ or } \psi_{j,p}^G(z) = 0\}.$$

Figure 6.1: The relation among M', $\tilde{M}'$, *and* $\Delta(R)$



Let $\pi : \tilde{M}' \to M'$ be the universal covering map of M'. Then $\ln(u \circ \pi)$ is a harmonic function since $F(H_j), G(H_j), F_n, G_n, \psi^F_{j,p}, \psi^G_{j,p}$ and χ are all holomorphic functions and u has no zeros. So we can take a holomorphic function β on $\tilde{M}'$ such that $|\beta| = u \circ \pi$. Without loss of generality, we may assume that M' contains the origin o of $\mathbb{C}$. Let $\tilde{o} \in \tilde{M}'$ denote the point corresponding to $o \in M'$. For each point $\tilde{p}$ of $\tilde{M}'$ we take a continuous curve $\gamma_{\tilde{p}} : [0,1] \to M'$ with $\gamma_{\tilde{p}}(0) = o$ and $\gamma_{\tilde{p}}(1) = \pi(\tilde{p})$, which corresponds to the homotopy class of $\tilde{p}$. Set

$$w = X(\tilde{p}) = \int_{\gamma_{\tilde{p}}} \beta(z)dz$$

where z denotes the holomorphic coordinate on M' induced from the holomorphic global coordinate on $\tilde{M}'$ by π. Then X is a single-valued holomorphic function on $\tilde{M}'$ satisfying the condition $X(\tilde{o}) = 0$ and $dX(\tilde{p}) \neq 0$ for every $\tilde{p} \in \tilde{M}'$. Furthermore, for the metric $|\beta||dz|$ on $\tilde{M}'$ we have the Gaussian curvature $K = -\frac{\Delta \ln|\beta|}{|\beta|^2} = -\frac{\Delta \ln u}{u^2} = 0$, which means that $\tilde{M}'$ is flat. Thus, we can choose the largest $R(\leq \infty)$ such that X maps an open neighborhood U of $\tilde{o}$ biholomorphically onto an open disc $\Delta(R)$ in $\mathbb{C}$, and consider the map $B = \pi \circ (X|_U)^{-1} : \Delta(R) \to M' \subset \Delta$. By Liouville's theorem, $R = \infty$ is impossible. Thus, we conclude that $R < \infty$. Also, by definition of $w = \int_{\gamma_{\tilde{p}}} \beta(z)dz$, we have

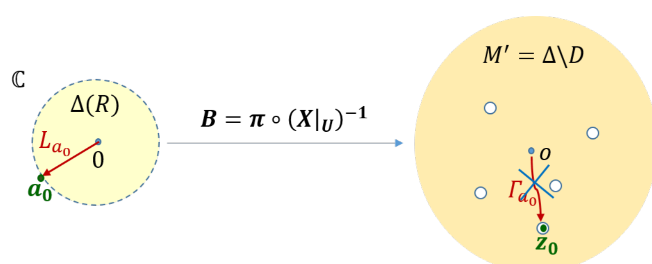
$$|dw| = |\beta(z)||dz| = u(z)|dz|. \tag{6.4}$$

For each $a \in \partial\Delta(R)$ consider the curve

$$L_a : w = ta, 0 \leq t < 1$$

and the image Γ_a of L_a by B. We shall show that there exists a point $a_0 \in \partial\Delta(R)$ such that Γ_{a_0} tends to the boundary of M. To this end, we assume the contrary. Then, for each $a \in \partial\Delta(R)$, there is a sequence $\{t_\nu : \nu = 1, 2, \ldots\}$ such that $\lim_{\nu\to\infty} t_\nu = 1$ and $z_0 = \lim_{\nu\to\infty} B(t_\nu a)$ is in M. Suppose that $z_0 \notin M'$. Then either $F_n(z_0) = 0$,

Figure 6.2: L_{a_0} and its image Γ_{a_0} when $z_0 \notin M'$



$G_n(z_0) = 0$, $\psi^F_{j,p}(z_0) = 0$, or $\psi^G_{j,r}(z_0) = 0$. In any one of these cases, there is a constant C such that

$$u \geq \frac{C}{|z - z_0|^{\frac{2\rho^*}{N}}}$$

in a neighborhood of z_0. Thus,

$$R = \int_{L_a} |dw| = \int_{\Gamma_a} \frac{|dw|}{|dz|}|dz| = \int_{\Gamma_a} u(z)|dz| \geq C \int_{\Gamma_a} \frac{1}{|z - z_0|^{\frac{2\rho^*}{N}}}|dz| = \infty$$

since $\frac{2\rho^*}{N} = \frac{2}{\rho N} > 1$. This is a contradiction because $R < \infty$. Therefore, we have $z_0 \in M'$. Take a simply connected neighborhood V of z_0 which is relatively compact in M'. Set $C' = \min_{z \in V} u(z) > 0$. Then $B(ta) \in V (t_0 < t < 1)$ for some t_0. In fact, if not, Γ_a goes and returns infinitely often from ∂V to a sufficiently small neighborhood of z_0 and so we get the absurd conclusion

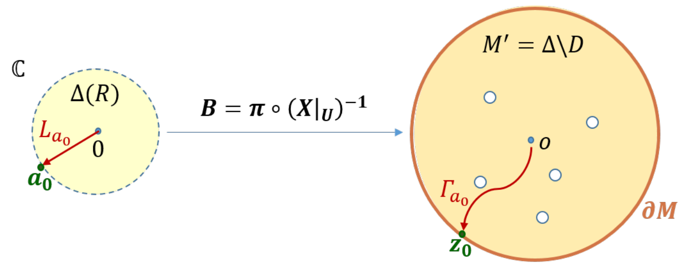
$$R = \int_{L_a} |dw| = \int_{\Gamma_a} u(z)|dz| \geq C' \int_{\Gamma_a} |dz| = \infty.$$

Thus we see that $\lim_{t \to 1} B(ta) = z_0$. Since π maps each connected component of $\pi^{-1}(V)$ bioholomorphically onto V, there exists the limit

$$\tilde{p}_0 = \lim_{t \to 1} (X|_U)^{-1}(ta) \in \tilde{M}'.$$

Thus $(X|_U)^{-1}$ has a biholomorphic extension to a neighborhood of a. Since a is arbitrarily chosen, X maps an open neighborhood $\bar{U}$ biholomorphically onto an open neighborhood of $\overline{\Delta(R)}$. This contradicts the property of R, i.e.,R is the largest radius such that X maps an open neighborhood U of $\tilde{o}$ biholomorphically onto an open disc $\Delta(R)$ in $\mathbb{C}$. Therefore, there exists a point $a_0 \in \partial\Delta(R)$ such that Γ_{a_0} tends to the boundary of M.

Figure 6.3: The image Γ_{a_0} of L_{a_0} tending to the boundary of M



Our goal is to show that Γ_{a_0} has finite length, contradicting the completeness of the given minimal surface M. From (6.3) and (6.4), we have

$$|dw| = u(z)|dz| = \left(\frac{\prod_{j=1}^{q} |F(H_j)(z)G(H_j)(z)|}{|\chi(z)|^{\frac{2kn}{q-2k+2kn}} |F_n(z)G_n(z)|^{1+\frac{2q}{N}} \prod_{p=0}^{n-1} \prod_{j=1}^{q} |\psi_{j,p}^{F}(z)\psi_{j,p}^{G}(z)|^{\frac{4}{N}}} \right)^{\frac{\rho^*}{2}} |dz|.$$

Let $Z_F(w) = F(B(w))$, $Z_G(w) = G(B(w))$, and let $\psi_{j,p}^{Z_F}, \psi_{j,p}^{Z_G}$ be defined from Z_F, Z_G in the same way as $\psi_{j,p}^{F}, \psi_{j,p}^{G}$ from F, G respectively. Then, because

$$Z_F \wedge Z_F' \wedge \cdots \wedge Z_F^{(n)} = (F \wedge F' \wedge \cdots \wedge F^{(n)}) \left(\frac{dz}{dw}\right)^{\frac{n(n+1)}{2}},$$

$$Z_G \wedge Z_G' \wedge \cdots \wedge Z_G^{(n)} = (G \wedge G' \wedge \cdots \wedge G^{(n)}) \left(\frac{dz}{dw}\right)^{\frac{n(n+1)}{2}},$$

$$\psi_{j,p}^{Z_F} = \psi_{j,p}^{F} \cdot \left(\frac{dz}{dw}\right)^{\frac{p(p+1)}{2}} \quad \text{and} \quad \psi_{j,p}^{Z_G} = \psi_{j,p}^{G} \cdot \left(\frac{dz}{dw}\right)^{\frac{p(p+1)}{2}},$$

it is easy to see that, if we let $h = \frac{n(n+1)}{2} + n(n+1)\frac{q}{N} + \sum_{p=0}^{n-1} p(p+1)\frac{2q}{N}$, then

$$\left|\frac{dw}{dz}\right| = \left(\frac{\prod_{j=1}^{q} |Z_F(H_j)Z_G(H_j)|}{|\chi|^{\frac{2kn}{q-2k+2kn}} |(Z_F)_n(Z_G)_n|^{1+2q/N} \prod_{p=0}^{n-1} \prod_{j=1}^{q} |\psi_{j,p}^{Z_F}\psi_{j,p}^{Z_G}|^{4/N}} \right)^{\frac{\rho^*}{2}} \left[\left|\frac{dz}{dw}\right|^{2h}\right]^{\frac{\rho^*}{2}}.$$

In other words,

$$\left|\frac{dw}{dz}\right|^{1+\rho^* h} = \left(\frac{\prod_{j=1}^{q} |Z_F(H_j) Z_G(H_j)|}{|\chi|^{\frac{2kn}{q-2k+2kn}} |(Z_F)_n (Z_G)_n|^{1+2q/N} \prod_{p=0}^{n-1} \prod_{j=1}^{q} |\psi_{j,p}^{Z_F} \psi_{j,p}^{Z_G}|^{4/N}}\right)^{\frac{\rho^*}{2}}.$$

Therefore,

$$\left|\frac{dw}{dz}\right| = \left(\frac{\prod_{j=1}^{q} |Z_F(H_j) Z_G(H_j)|}{|\chi|^{\frac{2kn}{q-2k+2kn}} |(Z_F)_n (Z_G)_n|^{1+2q/N} \prod_{p=0}^{n-1} \prod_{j=1}^{q} |\psi_{j,p}^{Z_F} \psi_{j,p}^{Z_G}|^{4/N}}\right)^{\frac{\rho^*}{2(1+\rho^* h)}}.$$

Note that $\rho^* = \frac{1}{\rho}$ and

$$\rho = S - \left[\frac{n(n+1)}{2} + \frac{2knq}{q-2k+2kn} + n(n+1)\frac{q}{N} + \sum_{p=0}^{n-1} p(p+1)\frac{2q}{N}\right],$$

so

$$\frac{\rho^*}{2(1+\rho^* h)} = \frac{1}{2S - \frac{4knq}{q-2k+2kn}}.$$

Thus

$$\left|\frac{dw}{dz}\right| = \left(\frac{\prod_{j=1}^{q} |Z_F(H_j) Z_G(H_j)|}{|\chi|^{\frac{2kn}{q-2k+2kn}} |(Z_F)_n (Z_G)_n|^{1+2q/N} \prod_{p=0}^{n-1} \prod_{j=1}^{q} |\psi_{j,p}^{Z_F} \psi_{j,p}^{Z_G}|^{4/N}}\right)^{\frac{1}{2S - \frac{4knq}{q-2k+2kn}}}.$$

Now we ready to apply the Main Lemma. Notice that the metric on $\Delta(R)$ through the pull-back of the map B on the induced (complete) metric $ds^2 = 2|F||G||dz|^2$ on M is given by

$$B^* ds = \sqrt{2}|F(B(w))|^{\frac{1}{2}} |G(B(w))|^{\frac{1}{2}} \left|\frac{dz}{dw}\right| |dw| = \sqrt{2}|Z_F|^{\frac{1}{2}} |Z_G|^{\frac{1}{2}} \left|\frac{dz}{dw}\right| |dw|,$$

and, from above,

$$\left|\frac{dz}{dw}\right| = \left(\frac{|\chi|^{\frac{2kn}{q-2k+2kn}} |(Z_F)_n (Z_G)_n|^{1+2q/N} \prod_{p=0}^{n-1} \prod_{j=1}^{q} |\psi_{j,p}^{Z_F} \psi_{j,p}^{Z_G}|^{4/N}}{\prod_{j=1}^{q} |Z_F(H_j) Z_G(H_j)|}\right)^{\frac{1}{2S - \frac{4knq}{q-2k+2kn}}}.$$

Hence,

$$B^* ds = \sqrt{2}\, b(w)^{\frac{1}{2S - \frac{4knq}{q-2k+2kn}}} |dw|$$

where

$$b(w) = \frac{|Z_F Z_G|^{S - \frac{2knq}{q-2k+2kn}} |\chi|^{\frac{2kn}{q-2k+2kn}} |(Z_F)_n (Z_G)_n|^{1+2q/N} \prod_{p=0}^{n-1} \prod_{j=1}^{q} |\psi_{j,p}^{Z_F} \psi_{j,p}^{Z_G}|^{4/N}}{\prod_{j=1}^{q} |Z_F(H_j) Z_G(H_j)|},$$

and, from the Main Lemma,

$$\begin{aligned}\sqrt{2} b(w)^{\frac{1}{2S - \frac{4knq}{q-2k+2kn}}} |dw| &\le C \left(\frac{2R}{R^2 - |w|^2} \right)^{\frac{2\left[\frac{n(n+1)}{2} + \sum_{p=0}^{n-1} (n-p)^2 \frac{2q}{N}\right]}{2S - \frac{4knq}{q-2k+2kn}}} |dw| \\ &= C \left(\frac{2R}{R^2 - |w|^2} \right)^{\tau} |dw|,\end{aligned}$$

where $\tau = [\frac{n(n+1)}{2} + \sum_{p=0}^{n-1} (n-p)^2 \frac{2q}{N}]/[S - \frac{2knq}{q-2k+2kn}]$ and C is a positive constant. From the condition

$$N > \frac{2q \left[n^2 + 2n - 1 + \sum_{p=0}^{n-1} (n-p)^2 \right]}{[q - (n+1) - \frac{n(n+1)}{2} - \frac{2knq}{q-2k+2kn}]},$$

we have $\tau < 1$. Therefore, the length of a divergent path Γ_{a_0} is

$$\int_{\Gamma_{a_0}} ds = \int_{L_{a_0}} B^* ds \le C \int_0^R \left(\frac{2R}{R^2 - |w|^2} \right)^{\tau} |dw| < \infty.$$

This contradicts the fact that M is complete with the metric $ds^2 = 2|F||G||dz|^2$. This finishes the proof. Q.E.D.

Chapter 7

Conclusion

7.1 Main Point

In this thesis, we answer the question, "At least how many hyperplanes must have the same inverse images (with counting multiplicities) in order to make two Gauss maps f and g identical?" Our ultimate goal is to find the smallest number q. There are two main points to take into account in order to achieve this goal.

The first main point is to find the best auxiliary function χ and to calculate ν_χ, which play an essential part in finding the smallest number q. Fujimoto used $\chi := f_i g_j - f_j g_i$ with $\chi \not\equiv 0$ for some distinct indices i, j where f_i and g_i are components of $F = (f_0, \cdots, f_n)$ and $G = (g_0, \cdots, g_n)$ respectively, and we used $\chi := \prod_{i=1}^{q}[F(H_i)G(H_{\sigma(i)}) - G(H_i)F(H_{\sigma(i)})]$ with $\chi \not\equiv 0$ as the auxiliary function. Thanks to the auxiliary function, we were able to get a smaller number than Fujimoto's. Thus, it is shown that finding the best auxiliary function is a crucial step in achieving this goal.

The second main point is to construct a pseudo-metric with a strictly negative curvature associated with two holomorphic maps, f and g, and the auxiliary function, χ, in order to apply the Alfors-Schwarz Lemma. Then, we can get the Main Lemma, which is crucial in proving the Main Theorem.

Therefore, if one was able to find a better auxiliary function to get the smallest number q and construct the pseudo-metric with a strictly negative curvature, the Main Theorem of this thesis would be improved.

7.2 Future Work

I tried to prove the Main Theorem for the case of "degenerate" maps. We had finished all of the other parts of the proof. However, we could not show that the pseudo-metric is continuous, so we could not apply the Ahlfors-Schwartz Lemma, a quintessential portion of the proof. Therefore, the result in this thesis is only for the case of linearly non-degenerate Gauss maps. We hope that an improvement of the unicity theorem for the case of degenerate Gauss maps will be given in the future.

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