SOME RESULTS ON THE DEGENERACY OF ENTIRE CURVES AND INTEGRAL POINTS IN THE COMPLEMENTS OF DIVISORS

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the Faculty of the Department of Mathematics
University of Houston

In Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy

By
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SOME RESULTS ON THE DEGENERACY OF ENTIRE CURVES AND INTEGRAL POINTS IN THE COMPLEMENTS OF DIVISORS

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Abstract

In this dissertation, we first discuss some of the important results in Nevanlinna Theory and Diophantine Approximation Theory. Next, a result by the author and Min Ru [LR14] is presented. In chapter 3, we extend the Second Main Theorem to the case of holomorphic curves into algebraic varieties intersecting numerically equivalent ample divisors. In chapter 4, we improve Ru's defect relation (see [Ru16a]) and the height inequality (see [Ru16b]) in the case when X is a normal projective surface and D_j, $1 \leq j \leq q$, are big and asymptotically free divisors without irreducible common components on X. Lastly, the author and Gordon Heier study a hyperbolicity-type problem involving projections from $\mathbb{P}^{n+2}$ to $\mathbb{P}^n$.

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Chapter 1

Introduction

Nevanlinna theory began with the study of the distribution of values of meromorphic functions. In 1929, Nevanlinna extended the classical little Picard's theorem by introducing the defect (see chapter 2). Right after Nevanlinna, Cartan extended Nevanlinna's result to holomorphic curves in projective spaces and Bloch considered holomorphic curves in Abelian varieties. In 1941, Ahlfors, following Weyl's work, gave a geometric approach to the theory of holomorphic curves in projective spaces. In 1953, Stoll generalized the work of Weyl-Ahlfors to the case of several complex variables. In 1970, Griffiths proved the Second Main Theorem for equi-dimensional holomorphic mappings. The result gave a new insight to the theory in terms of Chern invariant. In 1996, Siu and Yeung settled Lang's conjecture for abelian varieties and made significant progress towards solving Griffiths' conjecture.

Diophantine problems also have a long history. In the first half of the 20th century, Thue and Siegel first obtained important finiteness statements. In 1955, Roth proved the celebrated Roth's theorem. Around 1970 Schmidt extended Roth's result for simultaneous approximation to algebraic numbers. In 1983, Faltings solved Mordell's conjecture: *a smooth algebraic curve of genus $g \geq 2$ defined over $\mathbb{Q}$ has only finitely many rational points.* Vojta derived an alternative proof of Faltings' theorem by Diphantine approximation. In the same year, Faltings extended the theory of Diophantine approximation to Abelian varieties.

There exists a very striking connection between Nevanlinna theory and Diophantine approximation, as discovered by Vojta, Osgood and others. Roughly speaking, the study of holomorphic maps intersecting divisors corresponds to the study of integral points of the complement of the divisors. Vojta even compiled a "dictionary" translating from one to the other. This relation has been proved beneficial for both subjects, as progress in one can provide inspiration for progress in the other.

In recent years, there has been some significant progress in the study of qualitative and quantitative aspects of geometric and arithmetic properties of the complement of divisor in an algebraic projective variety. In 2004, Ru established a defect relation for algebraically nondegenerate holomorphic curves in projective space intersecting curvilinear hypersurfaces which settled a long-standing conjecture of Shiffman. In 2009, he further extended the result to holomorphic curves in complex projective varieties. In the same year, based on the result achieved by Corvaja-Zannier, Levin obtained the sharp qualitative result in the surface case. In 2014, also motivated by Corvaja-Zannier, Ru and I study the Second Main Theorem in the case when divisors are numerically equivalent to an ample divisor (see chapter 3). In 2015, Ru introduced a new notion of Nevanlinna constant which gives an upper bound of the defect. In chapter 4, we will follow Ru's method to derive the quantitative result in the surface case which implies the sharp qualitative result.

To our knowledge, many theorems on integral points apply to V only if D splits into several components, where $V \subset \mathbb{P}^n$ is an affine variety and D is its divisor at infinity. Only few results on integral points are known without such type of hypotheses. An example is provided by the deep theorem of Faltings on sets of integral points on abelian varieties in the complement of an ample effective divisor. Another classic example proved by Siu and Yeung [SY96] is described as follows:
If the degree of a generic (irreducible, smooth) curve C in $\mathbb{P}^2$ is big enough, then $\mathbb{P}^2 \backslash C$ is hyperbolic, i.e., every holomorphic map $f : \mathbb{C} \to \mathbb{P}^2$ is constant.

Following an idea of Faltings, Zannier [Zan05] studied the projection from a hypersurface $\mathcal{X}$ in $\mathbb{P}^{n+1}$ to $\mathbb{P}^n$. Roughly speaking, he used the total ramification locus to control the integral points away from the ramification locus $\mathcal{D}$ of the projection. In this spirit, Gordon Heier and I study a generic projection from the intersection of two generic hypersurfaces in $\mathbb{P}^{n+2}$ to $\mathbb{P}^n$ (see chapter 5). We derived the result that the Zariski closure of any set of S-integral points in $\mathbb{P}^n \setminus \mathcal{D}$ has dimension $\leq \max\{0, n - d_1 d_2 + 2\}$, where d_1, d_2 are degrees of these two hypersurfaces. Note that, as a consequence, if $d_1 d_2 \geq n + 2$, then any set of S-integral points is finite.

Our proof essentially follows Zannier's approach, which is to reduce the problem to a clever application of the finiteness theorem of Siegel-Mahler for solutions of the

S-unit equation. The key difference however is that in Zannier's codimension one case, the divisor $\mathcal{D}$ can be conveniently described by a discriminant. In our case, iterated resultants are required and certain excess vanishing has to be removed in order to identify $\mathcal{D}$.

Chapter 2

Definitions and background materials

In this chapter, the definitions of the counting function, the characteristic function, the proximity function and the defect are given in a geometric way and basic properties are also provided. Let X be a complex projective variety. For a Cartier divisor D on X, the Weil function for D is given by

$$\lambda_D(x) = -\log \|s_D(x)\| \,, \tag{2.1}$$

where s_D is the canonical section of the line bundle $\mathcal{O}_X(D)$, i.e., $(s_D) = D$, and $\|\cdot\|$ is any continuous metric on $\mathcal{O}_X(D)$. The Weil function is well defined, up to a bounded term, independently of the choices of the metric. In the case when $X = \mathbb{P}^n$ and $D = \{Q = 0\} \subset \mathbb{P}^n$ where Q is a homogeneous polynomial of degree d, λ_D can be chosen as, for $x = [x_0 : \cdots : x_n] \in \mathbb{P}^n \backslash \mathrm{supp} D$,

$$\lambda_D(x) = \log \frac{(\max_{0 \le i \le n} |x_i|^d) \cdot \|Q\|}{|Q(x)|} ,$$

where $\|Q\|$ is the maximum of the norm of the coefficients of Q.

Let $f : \mathbb{C} \to X$ be a holomorphic map whose image is not contained in the support of D. The proximity function of f with respect to D is defined by

$$m_f(r, D) = \int_0^{2\pi} \lambda_D(f(re^{i\theta})) \frac{d\theta}{2\pi} . \tag{2.2}$$

The counting function of f is defined by

$$N_f(r, D) = \int_1^r \frac{n_f(t, D)}{t} dt , \tag{2.3}$$

where $n_f(t, D)$ is the number of zeros of $\rho \circ f$ inside $\{|z| < t\}$, counting multiplicities, and ρ is a local defining function of D (note that $n_f(t, D)$ is independent of the choice of ρ). We define the characteristic function by

$$T_{f,D}(r) = \int_1^r \frac{dt}{t} \int_{B(t)} f^*(c_1(\mathcal{O}(D))), \tag{2.4}$$

where $\mathcal{O}(D)$ is the line bundle associated to D and $B(t)$ is an open disk whose radius is t and center is at origin. The first relation between those functions we defined above is called the First Main Theorem which is a consequence of the Green-Jensen formula.

Theorem 2.0.1 (Green-Jensen Formula)**.** *Let α be a function of class $\mathcal{C}^2$ on $\bar{B}(r)$ or a subharmonic function on $\bar{B}(r)$. Then*

$$\int_1^r \frac{dt}{t} \int_{B(t)} [dd^c \alpha] = \frac{1}{2\pi} \int_0^{2\pi} \alpha(re^{\sqrt{-1}\theta}) d\theta + O(1), \tag{2.5}$$

where $[dd^c\alpha] = dd^c\alpha + Sing_\alpha(r)$, $Sing_\alpha(r) = \lim_{\epsilon\to 0}\int_1^r \frac{dt}{t}\int_{S(Z,\epsilon)} d^c\alpha$, $S(Z,\epsilon)$ *is the union of small circles around singularities in* $\bar{B}$ *and* Z *is the set of singularities of* α.

Proof. By Stokes' theorem and $d^c = \frac{1}{2\pi}(r\frac{\partial}{\partial r}\otimes d\theta + \frac{1}{r}\frac{\partial}{\partial\theta}\otimes dr)$,

$$\begin{aligned}
&\int_1^r \frac{dt}{t}\int_{B(t)} dd^c\alpha = \int_1^r \frac{dt}{t}\int_{\partial B(t)} d^c\alpha - \lim_{\epsilon\to 0}\int_1^r \frac{dt}{t}\int_{S(Z,\epsilon)} d^c\alpha \\
&= \int_1^r \frac{dt}{t}\int_0^{2\pi} \frac{1}{2\pi}\left(t\frac{\partial}{\partial t}\otimes d\theta + \frac{1}{t}\frac{\partial}{\partial\theta}\otimes dt\right)\alpha(te^{i\theta}) - \mathrm{Sing}_\alpha(r) \\
&= \frac{1}{2\pi}\int_1^r \frac{dt}{t}\int_0^{2\pi} t\frac{\partial}{\partial t}\alpha(te^{i\theta})d\theta - \mathrm{Sing}_\alpha(r) \\
&= \frac{1}{2\pi}\int_0^{2\pi} \alpha(re^{i\theta})d\theta - \mathrm{Sing}_\alpha(r) + O(1).
\end{aligned}$$

□

Based on the Green-Jensen formula, we have

$$\begin{aligned}
&\int_1^r \frac{dt}{t}\int_{B(t)} f^*(c_1(L)) = -\int_1^r \frac{dt}{t}\int_{B(t)} f^*(dd^c\log||s||) \\
= &\; -\int_1^r \frac{dt}{t}\int_{B(t)} dd^c\log||s\circ f|| \\
= &\; \mathrm{Sing}_{\log||s\circ f||}(t) - \frac{1}{2\pi}\int_0^{2\pi}\log||s\circ f(re^{\sqrt{-1}\theta})||d\theta + O(1).
\end{aligned}$$

Since $\lim_{\epsilon\to 0}\int_{\partial B(\epsilon)} d^c\log|z|^2 = 1$, $\mathrm{Sing}_{\log||s\circ f||}(t) = N_f(r,D)$. Therefore, we have the First Main Theorem.

$$T_{f,D}(r) = m_f(r,D) + N_f(r,D) + O(1)\,. \tag{2.6}$$

Remark 2.0.2. The First Main Theorem is an alternative way to define the characteristic function.

If $X = \mathbb{P}^n$ with homogeneous coordinates $[z_0, \ldots, z_n]$ and $D = \{a_0 z_0 + \ldots + a_n z_n = 0\}$, the Weil function $\lambda_D(x) = \log \frac{\max|z_i| \max|a_i|}{|a_0 z_0 + \ldots + a_n z_n|}$. In particular, when $D = \{z_0 = 0\}$, we simply write $T_f(r) := T_{f,D}(r)$. We call $T_f(r)$ Cartan's characteristic function. By Green-Jensen's formula, we have

$$T_f(r) = \frac{1}{2\pi} \int_0^{2\pi} \log \max_{0 \le i \le n} |f_i(re^{\sqrt{-1}\theta})| d\theta,$$

where $f = [f_0 : \cdots : f_n]$ and $f_0, \ldots, f_n$ are entire functions without common zeros.

To serve our purpose later, we discuss the characteristic function for $f : \mathbb{C} \to X$ with respect to a subspace $V \subset H^0(X, D)$ with $\dim V \ge 2$ where D is base-point free. Let $\Phi : X \to \mathbb{P}^m$ be the canonical rational map associated with V where $m + 1 = \dim V$. Let $\Phi = [\phi_0 : \ldots : \phi_m]$ where $\{\phi_0, \ldots, \phi_m\}$ is a basis of V. Extend $\{\phi_0, \ldots, \phi_m\}$ to $\{\phi_0, \ldots, \phi_l\}$ such that $\{\phi_0, \ldots, \phi_l\}$ is a basis of $H^0(X, D)$, where $l = \dim H^0(X, D) - 1$. Since $|D|$ is base-point free, we can take a reduced representation for $\phi \circ f : \mathbb{C} \to \mathbb{P}^l$, say $\phi \circ f = [h_0, \ldots, h_l]$, where $\phi = [\phi_0 : \ldots : \phi_l]$. Take an entire function g on $\mathbb{C}$ such that $\{h_0/g, \ldots, h_m/g\}$ has no common zeros. Then $[h_0/g, \ldots, h_m/g]$ is a reduced representation of $\Phi \circ f$. We now compare $T_{\Phi \circ f}(r)$ with $T_{f,D}(r)$. Notice that

$$\begin{aligned} T_{\Phi \circ f}(r) &= \int_0^{2\pi} \log \max_{0 \le i \le m} |(h_i/g)(re^{\sqrt{-1}\theta})| \frac{d\theta}{2\pi} \\ &= \int_0^{2\pi} \log \max_{0 \le i \le m} |h_i(re^{\sqrt{-1}\theta})| \frac{d\theta}{2\pi} - N_g(r, 0), \end{aligned}$$

where the second identity holds due to Green-Jensen's formula. On the other hand, by definition,

$$T_{f,D}(r) = T_{\phi \circ f}(r) = \int_0^{2\pi} \log \max_{0 \le i \le l} |h_i(re^{\sqrt{-1}\theta})| \frac{d\theta}{2\pi}.$$

Hence,

$$\begin{aligned} T_{\Phi\circ f}(r) &= \int_0^{2\pi} \log \max_{0\le i\le m} |h_i(re^{\sqrt{-1}\theta})| \frac{d\theta}{2\pi} - N_g(r,0) \\ &\le T_{f,D}(r) - N_g(r,0) \le T_{f,D}(r). \end{aligned} \tag{2.7}$$

We have the following basic properties of these functions.

Lemma 2.0.3 ([Voj07], Theorem 8.8)**.** *Weil functions λ_D for Cartier divisors D on a complex projective variety X satisfy the following properties.*

(**a**) **Additivity:** *If λ_1 and λ_2 are Weil functions for Cartier divisors D_1 and D_2 on X, respectively, then $\lambda_1 + \lambda_2$ extends uniquely to a Weil function for $D_1 + D_2$.*

(**b**) **Functoriality:** *If λ is a Weil function for a Cartier divisor D on X, and if $\phi : X' \to X$ is a morphism such that $\phi(X') \nsubseteq suppD$, then $x \mapsto \lambda(\phi(x))$ is a Weil function for the Cartier divisor $\phi^* D$ on X'.*

(**c**) **Normalization:** *If $X = \mathbb{P}^n$, and if $D = \{x_0 = 0\} \subset X$ is the hyperplane at infinity, then the function*

$$\lambda_D([x_0 : \cdots : x_n]) := \log \frac{\max\{|x_0|, \ldots, |x_n|\}}{|x_0|}$$

is a Weil function for D.

(**d**) **Uniqueness:** *If both λ_1 and λ_2 are Weil functions for a Cartier divisor D on X, then $\lambda_1 = \lambda_2 + O(1)$.*

(**e**) **Boundedness from below:** *If D is an effective divisor and λ is a Weil function for D, then λ is bounded from below.*

(**f**) **Principal divisors:** *If D is a principal divisor (f), then $-\log|f|$ is a Weil function for D.*

Proposition 2.0.4. *Let $f : \mathbb{C} \to X$ be a holomorphic map. The proximity function and the counting function of f have the following properties.*

(**a**) **Additivity:** *If D_1 and D_2 are two divisors on X, then*

$$m_f(r, D_1 + D_2) = m_f(r, D_1) + m_f(r, D_2) + O(1)$$

$$N_f(r, D_1 + D_2) = N_f(r, D_1) + N_f(r, D_2) + O(1).$$

(**b**) **Functoriality:** *If $\phi : X \to X'$ is a morphism and D' is a divisor on X' whose support does not contain the image of $\phi \circ f$, then*

$$m_f(r, \phi^* D') = m_{\phi\circ f}(r, D') + O(1)$$

$$N_f(r, \phi^* D') = N_{\phi\circ f}(r, D') + O(1).$$

(**c**) **Effective divisors:** *If D is effective, then $m_f(r, D)$ and $N_f(r, D)$ are bounded below. In each of the above cases, the implied constant in $O(1)$ depends on the varieties, divisors, and morphisms, but not on f and r.*

Lemma 2.0.5 ([Voj87] Ch.10, Prop.3.2)**.** *Let $\lambda_1, \dots, \lambda_n$ be Weil functions for Cartier divisors $D_1, \dots, D_n$, respectively, on a projective variety X. Assume that the D_i are of the form $D_i = D_0 + E_i$, where D_0 is a fixed Cartier divisor and E_i are effective for all i. Assume also that*

$$suppE_1 \cap \cdots \cap suppE_n = \emptyset.$$

Then the function

$$\lambda(x) = \min\{\lambda_i(x) : x \notin suppE_i\}$$

is defined everywhere on $X \backslash suppD_0$, and is a Weil function for D_0.

The divisors $D_1, \ldots, D_q$ on X are said to be *in l-subgeneral position* on X if for any subset of $l+1$ elements $\{i_0, \ldots, i_l\} \subset \{1, \ldots, q\}$,

$$\mathrm{supp}D_{i_0} \cap \cdots \cap \mathrm{supp}D_{i_l} = \emptyset.$$

When $l = \dim X$, then we say that the divisors $D_1, \ldots, D_q$ are *in general position* on X.

The central problem in Nevanlinna theory (or the theory of holomorphic curves) is to study whether a holomorphic mapping $f : \mathbb{C} \to X \backslash D$ is degenerate (i.e., $f(\mathbb{C})$ is contained in a proper subvariety of X), for a given projective variety X and an effective divisor D on X. A more general quantitative problem is to control the defect $\delta_f(D)$ for $f : \mathbb{C} \to X$, where $\delta_f(D) := \liminf_{r \to +\infty} \dfrac{m_f(r, D)}{T_{f,D}(r)}$. By the First Main Theorem, $\delta_f(D) = 1$. However, if $\delta_f(D) < 1$, then $f : \mathbb{C} \to X \backslash D$ must be degenerate. For example, when $X = \mathbb{P}^1$ and $D = \sum_{j=1}^q (a_j)$ for distinct points $a_1, \ldots, a_q \in \mathbb{P}^1$, Nevanlinna, in 1929, proved $\delta_f(D) \leq \frac{2}{q}$. It gives a quantitative extension of the classical result of Little Picard that *every holomorphic mapping $f : \mathbb{C} \to \mathbb{P}^1 \backslash \{$three distinct points$\}$ must be constant.* In 1933, H. Cartan extended Nevanlinna's defect relation to $\delta_f(D) \leq \frac{n+1}{q}$ for any linearly nondegenerate holomorphic mappings $f : \mathbb{C} \to \mathbb{P}^n$ where $D = \sum_{j=1}^q H_j$, and $H_1, \ldots, H_q$ are hyperplanes in general position.

The following version generalized Cartan's result (see [Ru97], [Voj97]).

Theorem 2.0.6. *Let $f = [f_0 : \ldots : f_m] : \mathbb{C} \to \mathbb{P}^m$ be a holomorphic map whose image is not contained in a proper linear subspace. Let $H_1, \ldots, H_q$ be arbitrary hyperplanes in $\mathbb{P}^m$. Then, for every $\epsilon > 0$,*

$$\int_0^{2\pi} \max_J \sum_{j \in J} \lambda_{H_j}(f(re^{\sqrt{-1}\theta}))\frac{d\theta}{2\pi} \leq (m+1+\epsilon)T_f(r) \ ||_E,$$

where the maximum is taken over all subsets J of $\{1, \ldots, q\}$ such that $\{H_j, j \in J\}$ are in general position and $||_E$, throughout the dissertation, means the inequality holds for all $r \in (0, \infty)$ except for a set E with finite Lebesgue measure.

Motivated by the recent breakthrough in Diophantine approximation by Corvaja-Zannier, and Evertse-Ferretti, my advisor Min Ru extended Cartan's result to the case of hypersurfaces.

Theorem 2.0.7 ([Ru04], Main Theorem)**.** *Let $f : \mathbb{C} \to \mathbb{P}^n$ be an algebraic non-degenerate holomorphic map. Let $D_1, \ldots, D_q$ be hypersurfaces in $\mathbb{P}^n$ of degree d_i, located in general position. Then, for every $\epsilon > 0$,*

$$\sum_{j=1}^{q} d_j^{-1} m_f(r, D_j) \leq (n+1+\epsilon)T_f(r) \ ||_E. \tag{2.8}$$

In 2009, he further extended the result to any complex projective variety.

Theorem 2.0.8 ([Ru09], The main result)**.** *Let $X \subset \mathbb{P}^n$ be a complex projective variety. Let $D_1, \ldots, D_q$ be hypersurfaces in $\mathbb{P}^n$ of degree d_i, located in general position on X. Let $f : \mathbb{C} \to X$ be an algebraically non-degenerate holomorphic map. Then, for every $\epsilon > 0$,*

$$\sum_{j=1}^{q} d_j^{-1} m_f(r, D_j) \leq (\dim X + 1 + \epsilon)T_f(r) \ ||_E. \tag{2.9}$$

Remark 2.0.9. The theorem was proved by Min Ru in the case X is smooth. Vojta pointed out (see page 185, [Voj07]) that the same proof goes through when X is not smooth.

According to the definition of the defect, the inequality (2.9) can be written as follows:

$$\sum_{j=1}^{q} \delta_f(D_j) \leq \dim X + 1.$$

The notions of the Weil function and the height function on the arithmetic (Diophantine approximation) side are defined in a similar way (see, for example, [Lan87] or [Voj07]). Let k be a number field and let $\mathcal{O}_k$ denote the ring of integers of k. As usual, we have a set M_k of places of k consisting of one place for each nonzero prime ideal in $\mathcal{O}_k$, one place for each real embedding $\sigma : k \to \mathbb{R}$, and one place for each pair of conjugate embeddings $\sigma, \bar{\sigma} : k \to \mathbb{C}$. k_v denotes the completion of k with respect to $v \in M_k$. We normalize our absolute values so that $\|p\|_v = p^{-[k_v:\mathbb{Q}_v]/[k:\mathbb{Q}]}$ if v corresponds to the prime ideal above the prime $p \in \mathbb{Q}$, $\|x\|_v = |\sigma(x)|^{1/[k:\mathbb{Q}]}$ if v corresponds to the real embedding σ, and $\|x\|_v = |\sigma(x)|^{2/[k:\mathbb{Q}]}$ if v corresponds to the pair of conjugate embeddings $\sigma, \bar{\sigma} : k \to \mathbb{C}$. Let X be a projective variety defined over a number field k. For every Cartier divisor D on X and every place $v \in M_k$, we can associate a local Weil function $\lambda_{D,v} : X \backslash \mathrm{supp} D \to \mathbb{R}$ (see, for example, [Lan87] or [Voj07]), where $\mathrm{supp} D$ is the support of the divisor D. When D is effective, the Weil function $\lambda_{D,v}$ gives a measurement of the v-adic distance of a point to D. If $X = \mathbb{P}^n$ and $D \subset \mathbb{P}^n$ is a hypersurface defined by a homogeneous polynomial Q of degree d, then

$$\lambda_{D,v}([x_0 : \cdots : x_n]) := \log \frac{\max\{\|x_0\|_v^d, \ldots, \|x_n\|_v^d\}}{\|Q(x_0, \ldots, x_n)\|_v}.$$

Let S be a finite set of places in M_k containing the archimedean places. Let $R \subset X(\bar{k})\backslash D$. Then R is defined to be a (D, S)-integral set of points if there exists a global Weil function $\lambda_{D,v}$ and all embeddings $\bar{k} \to \bar{k}_v$, such that for all $v \in M_k \backslash S$, the inequality $\lambda_{D,v}(P) \leq 0$ for all P in R. The height $h_k(x, D)$ for points $x \in X(k)$ is defined as

$$h_k(x, D) = \sum_{v \in M_k} \lambda_{D,v}(x).$$

It is independent of, up to $O(1)$, the choice of Weil functions. In particular, when $X = \mathbb{P}^n, D = \{z_0 = 0\}$, we simply write $h_k(x) := h_k(x, D)$.

Let $S \subset M_k$ be a finite set of places containing all archimedean ones. We define, for $x \in X(k)\backslash \text{supp}D$,

$$m_S(x, D) = \sum_{v \in S} \lambda_{D,v}(x), \quad N_S(x, D) = \sum_{v \notin S} \lambda_{D,v}(x).$$

Similarly, those functions have following properties.

Proposition 2.0.10 ([Voj87], Theorem 9.8)**.** *Let X be a projective variety over a number field k. Then the following properties hold.*

(a) Additivity:

$$m_S(x, D_1 + D_2) = m_S(x, D_1) + m_S(x, D_2) + O(1)$$

$$N_S(x, D_1 + D_2) = N_S(x, D_1) + N_S(x, D_2) + O(1).$$

(**b**) **Functoriality:** *If $\phi : X \to X'$ is a morphism and D' is a divisor on X' whose support does not contain the image of $\phi \circ f$, then*

$$m_S(x, \phi^* D') = m_S(\phi(x), D') + O(1)$$

$$N_S(x, \phi^* D') = N_S(\phi(x), D') + O(1).$$

(**c**) **Effective divisors:** *If D is effective, then $m_S(x, D)$ and $N_S(x, D)$ are bounded below. In each of the above cases, the implied constant in $O(1)$ depends on the varieties and divisors but not on x.*

When working with the proximity function and the height function, the divisor D is almost always assumed to be effective.

The following (generalized) version of Schmidt's Subspace Theorem from [Voj97] is corresponding to Theorem 2.0.6.

Theorem 2.0.11. *Let k be a number field and $S \subset M_k$ be a finite set containing all archimedean places. Let $H_1, \ldots, H_q$ be hyperplanes in $\mathbb{P}^n$ defined over $\bar{k}$ and $\lambda_{H_1}, \ldots, \lambda_{H_q}$ be Weil functions corresponding to $H_1, \ldots, H_q$. Then there exists a finite union of hyperplanes Z, depending only on $H_1, \ldots, H_q$ (and not k, S), such that for any $\epsilon > 0$,*

$$\sum_{v \in S} \max_I \sum_{i \in I} \lambda_{H_i, v}(P) \leq (n + 1 + \epsilon) h_k(P),$$

holds for all but finitely many $P \in \mathbb{P}^n(k) \backslash Z$, where the maximum is taken over subsets $I \subset \{1, \ldots, q\}$ such that the linear forms defining $H_i, i \in I$, are linearly independent.

Remark 2.0.12. The corresponding statements of Theorem 2.0.8 on the arithmetic side can be found in [EF08].

Chapter 3

Holomorphic curves intersecting numerically equivalent ample divisors

In this chapter (also see [LR14]), we reformulate Theorem 2.0.8 and extend it to the same result for divisors which are numerically equivalent to an ample divisor on X.

3.1 Motivation

The idea follows the breakthrough method introduced by Corvaja and Zannier, where they used Schmidt's subspace theorem to give a new proof of Siegel's celebrated theorem that any affine algebraic curve defined over a number field with positive genus or at least three points at infinity has only finitely many S-integral points. In their paper [CZ04a], they applied the method to study integral points on a surface where the divisors are not necessarily linearly equivalent. Later, Levin significantly improved their results and obtained the sharp result in the surface case, as well as extended the results to higher dimensions. However, all results they obtained are of a qualitative nature. One of the main results in [CZ04a] is stated as follows: Let X be a geometrically irreducible nonsingular algebraic surface and $D_1, \ldots, D_q$ be distinct irreducible divisors located in general position on X, i.e., no three of them share a common point, all defined over a number field, such that $\widetilde{X} := X \backslash \{D_1 + \ldots + D_q\}$ is affine. Assume that there exist positive integers $n_1, \ldots, n_q$ such that $(n_i D_i . n_j D_j)$ is a positive constant (independent of i, j for all pairs $1 \leq i, j \leq q$). If $q \geq 4$, then the S-integral points of $\widetilde{X}$ are degenerate, i.e., there is a curve on X containing all the S-integral points in $\widetilde{X}$. In their paper, they made a remark (see the last three lines on page 706, [CZ04a]) that one may prove that the condition that $(n_i D_i . n_j D_j)$ is constant amounts to the $n_i D_i, 1 \leq i \leq q$, being numerically equivalent. This is indeed an easy consequence of the Hodge Index Theorem, as is verified in this chapter. Nevertheless, it gives a strong motivation to study Schmidt's subspace theorem and the Second Main Type Theorem in Nevanlinna theory for numerically equivalent divisors.

On the other hand, on the quantitative side, Evertse and Ferretti, by using a different method, established a Schmidt's subspace-type theorem for the complement of divisors in an arbitrary projective variety $X \subset \mathbb{P}^N$, where the divisors are coming from hypersurfaces in $\mathbb{P}^N$. By a slight reformulation, one actually only needs to assume that the divisors are linearly equivalent on X to a fixed ample divisor. The discussion above thus naturally leads to the question whether the result still holds for divisors which are only numerically equivalent. Such result on the arithmetic side was just established by Levin in his recent paper [Lev14]. The extension of Evertse and Ferretti's result to numerically equivalent divisors immediately implies the (main) result of Corvaja and Zannier in [CZ04a]. The counter-part of Corvaja-Zannier in Nevanlinna theory is due to Liu and Ru. The purpose of this section is to give a quantitative extension of Liu and Ru's result [LiuRu05].

3.2 Preparation

We recall some notations and results in algebraic geometry. Let X be a projective variety. Two divisors D_1 and D_2 are said to be linearly equivalent on X, denoted by $D_1 \sim D_2$, if $D_1 - D_2 = (f)$ for some meromorphic function f on X. This is the same as saying there is a sheaf isomorphism $\mathcal{O}_X(D_1) \cong \mathcal{O}_X(D_2)$, $1 \to f$. Two divisors D_1 and D_2 are said to be numerically equivalent on X, denoted by $D_1 \equiv D_2$, if $D_1.C = D_2.C$ for all irreducible curves C on X. Obviously, linear equivalence implies numerical equivalence.

We need the following result.

Theorem 3.2.1 (Hodge Index Theorem)**.** *Let X be a smooth complex projective surface. Let $h \in \mathbb{H}^{1,1}_{\mathbb{R}}(X)$ with $h^2 > 0$. Then the cup product form is negative definite on $h^{\perp} \subset \mathbb{H}^{1,1}_{\mathbb{R}}(X)$.*

Corollary 3.2.2. *Let X be a smooth complex projective surface. Let D_1, D_2 be two distinct effective divisors. Assume that $D_1.D_2 = D_1^2 = D_2^2 > 0$. Then D_1 and D_2 are numerically equivalent.*

Proof. Let $h = [D_1]$. Then $h^2 = D_1^2 > 0$. Moreover, $D_1.(D_1 - D_2) = D_1^2 - D_1.D_2 = 0$ and $(D_1 - D_2)^2 = D_1^2 - 2D_1.D_2 + D_2^2 = 0$. So the above Hodge Index Theorem implies that $[D_1 - D_2] = 0 \in H^{1,1}_{\mathbb{R}}(X)$ which means that D_1 and D_2 are numerically equivalent. □

3.3 Main Theorem A

We first give a slight reformulation of Theorem 2.0.8.

Theorem 3.3.1 ([LR14], Theorem B)**.** *Let X be a complex projective variety of dimension $n \geq 1$ but not necessarily smooth. Let $D_1, \ldots, D_q$ be effective divisors on X, located in general position. Suppose that there exists an ample Cartier divisor A on X and positive integers d_j such that $D_j \sim d_j A$ (i.e., D_j is linearly equivalent to $d_j A$) for $j = 1, \ldots, q$. Let $f : \mathbb{C} \to X$ be an algebraically non-degenerate holomorphic map. Then, for every $\epsilon > 0$,*

$$\sum_{j=1}^{q} d_j^{-1} m_f(r, D_j) \leq (n + 1 + \epsilon) T_{f,A}(r)||_E. \tag{3.1}$$

Proof. Let N be a positive integer such that NA is very ample and N is divisible by d_j for $j = 1, \ldots, q$. Let $\phi : X \to \mathbb{P}^m$ be the canonical embedding of X into $\mathbb{P}^m$ associated to NA, where $m = \dim H^0(X, \mathcal{O}_X(NA)) - 1$. Then $\frac{N}{d_j} D_j = \phi^* H_j$ for some hyperplanes H_j in $\mathbb{P}^m$. From the assumption that $D_1, \ldots, D_q$ are in general position on X, $H_1, \ldots, H_q$ are in general position on $X \subset \mathbb{P}^m$ (or more precisely on the image of X under ϕ). Moreover from the functoriality and additivity of Weil functions, for $P \in X \backslash \mathrm{Supp} D_j$, we have

$$\lambda_{H_j}(\phi(P)) = \frac{N}{d_j} \lambda_{D_j}(r) + O(1),$$

so

$$m_{\phi \circ f}(r, H_j) = \frac{N}{d_j} m_f(r, D_j) + O(1).$$

Also, from the functoriality of height (characteristic) functions, we have

$$N T_{f,A}(r) = T_{f,NA}(r) = T_{\phi \circ f}(r) + O(1),$$

where $T_{\phi \circ f}(r) := T_{\phi \circ f, \mathcal{O}_{\mathbb{P}^m}(1)}(r)$. Applying Theorem 2.0.8 to the map $\phi \circ f$ and the hyperplanes H_j for $j = 1, \ldots, q$, we have

$$\sum_{j=1}^{q} m_{\phi \circ f}(r, H_j) \leq (n + 1 + \epsilon) T_{\phi \circ f}(r) \ \|_E.$$

The result then follows by substituting the identities above (we note that here the exceptional set E might change, nevertheless it is still of finite Lebesgue measure).

□

Main Theorem A. *Let X be a smooth complex projective variety of dimension $n \geq 1$ but not necessarily smooth. Let $D_1, \ldots, D_q$ be effective divisors on X, located in general position. Suppose that there exists an ample Cartier divisor A on X and positive integers d_j such that $D_j \equiv d_j A$ for $j = 1, \ldots, q$. Let $f : \mathbb{C} \to X$ be an algebraically non-degenerate holomorphic map. Then, for every $\epsilon > 0$,*

$$\sum_{j=1}^{q} d_j^{-1} m_f(r, D_j) \leq (n + 1 + \epsilon) T_{f,A}(r) \ \|_E. \tag{3.2}$$

To prove the theorem, the following result in algebraic geometry, due to Matsusaka, plays an important role.

Theorem 3.3.2 ([Mat58]). *Let A be an ample Cartier divisor on a projective variety X. Then there exists a positive integer N_0 such that for all $N \geq N_0$, and any Cartier divisor D with $D \equiv NA$, D is very ample.*

Lemma 3.3.3 ([Voj87], Proposition 1.2.9). *Let A be an ample Cartier divisor on a projective variety X. Let $f : \mathbb{C} \to X$ be a holomorphic map. Then, for any $\epsilon > 0$ and any effective divisor D with $D \equiv A$,*

$$T_{f,D}(r) \leq (1+\epsilon)T_{f,A}(r) + O(1),$$

where $O(1)$ is a constant which is independent of f and r.

Proof of Main Theorem A. By replacing D_j with $\frac{d}{d_j}D_j$ with $d = \operatorname{lcm}\{d_1, \ldots, d_q\}$, A by dA, and using the additivity of Weil functions and heights (up to bounded functions), we see that it suffices to prove the case where we can assume that $d_1 = d_2 = \cdots = d_q = 1$, i.e., $D_j \equiv A$ for $j = 1, \ldots, q$. For the given $\epsilon > 0$, let N_0 be the integer in Theorem 3.3.2 for our given A. Take N with

$$N_0 < \frac{\epsilon}{4q}N.$$

By the choice of N_0, we have that $NA - (N - N_0)D_j$ is very ample for $j = 1, \ldots, q$. Since the divisors $D_1, \ldots, D_q$ are in general position and $NA - (N - N_0)D_j$ is very ample for all j, there exist effective divisors E_j such that $(N - N_0)D_j + E_j$ is linearly equivalent to NA for all $1 \leq j \leq q$, and the divisors $(N - N_0)D_1 + E_1$,..., $(N -$

$N_0)D_q+E_q$ are in general position. Applying Theorem 3.3.1 to the linearly equivalent divisors $(N-N_0)D_j+E_j$ (which are all linearly equivalent to NA), $j=1,\ldots,q$, we get

$$\sum_{j=1}^{q} m_f(r,(N-N_0)D_j+E_j) \leq \left(n+1+\frac{\epsilon}{2}\right) T_{f,NA}(r) \ \|_E.$$

Using additivity and that the Weil functions λ_{E_j} are bounded from below outside of the support of E_j and $T_{f,NA}(r)=NT_{f,A}(r)$, we obtain

$$\sum_{j=1}^{q}\left(1-\frac{N_0}{N}\right) m_f(r,D_j) \leq \left(n+1+\frac{\epsilon}{2}\right) T_{f,A}(r) \ \|_E,$$

i.e.,

$$\sum_{j=1}^{q} m_f(r,D_j) \leq \frac{N_0}{N}\sum_{j=1}^{q} m_f(r,D_j) + \left(n+1+\frac{\epsilon}{2}\right) T_{f,A}(r) \ \|_E.$$

Note that in the above inequality, the exceptional set E might change, nevertheless it is still of finite Lebesgue measure. On the other hand, by Lemma 3.3.3 with $\epsilon=1$ and the First Main Theorem, we get

$$m_f(r,D_j) \leq T_{f,D_j}(r)+O(1) \leq 2T_{f,A}(r)+O(1).$$

Thus, by the choice of N that $N_0 < \frac{\epsilon}{4q}N$, we obtain

$$\sum_{j=1}^{q} m_f(r,D_j) \leq \frac{2qN_0}{N}T_{f,A}(r) + \left(n+1+\frac{\epsilon}{2}\right) T_{f,A}(r) \leq (n+1+\epsilon)T_{f,A}(r) \ \|_E.$$

This finishes the proof of Main Theorem A. $\square$

Corollary 3.3.4. *Let X be a smooth complex projective surface but not necessarily smooth and $D_1,\ldots,D_q$ be distinct irreducible ample divisors located in general position on X (i.e. no three of them share a common point). Assume that there* *exist positive integers $n_1, \ldots, n_q$ such that $(n_i D_i).(n_j D_j)$ is a positive constant (i.e independent of i, j for all pairs $1 \leq i, j \leq q$). Let $f : \mathbb{C} \to X$ be an algebraically non-degenerate holomorphic map. Then, for every $\epsilon > 0$,*

$$\sum_{j=1}^{q} n_j m_f(r, D_j) \leq (3+\epsilon)\left(\frac{1}{q}\sum_{j=1}^{q} n_j T_{D_j,f}(r)\right) \ \|_E.$$

In particular, with the same assumptions about the divisors $D_1, \ldots, D_q$, if $q \geq 4$, then every holomorphic map $f : \mathbb{C} \to X \setminus \cup_{j=1}^{q} D_j$ must be algebraically degenerate.

Proof. From Corollary 3.2.2, we know that $n_j D_j, 1 \leq j \leq q$, are numerically equivalent. Therefore applying the Main Theorem A to the divisors $n_j D_j$, together with the additivity property of Weil functions and heights (up to bounded functions), gives

$$\sum_{j=1}^{q} n_j m_f(r, D_j) \leq (3+\epsilon)\left(\frac{1}{q}\sum_{j=1}^{q} n_j T_{f,D_j}(r)\right) \ \|_E.$$

Now assume that $f : \mathbb{C} \to X \setminus \cup_{j=1}^{q} D_j$ and that f is algebraically non-degenerate. Since $n_j D_j$ and D_j share the same support and the image of f omits the support of D_j, we have $N_f(r, nD_j) = 0$, thus from the First Main Theorem,

$$m_f(r, n_j D_j) = T_{f, n_j D_j}(r) + O(1).$$

Thus, we get

$$\begin{aligned}
\sum_{j=1}^{q} n_j T_{f,D_j}(r) + O(1) &= \sum_{j=1}^{q} n_j m_f(r, D_j) \\
&\leq \frac{3+\epsilon}{q}\left(\sum_{j=1}^{q} n_j T_{f,D_j}(r)\right) \ \|_E,
\end{aligned}$$

which is a contradiction when $q \geq 4$. $\square$

Chapter 4

Quantitative results on projective surfaces

In this section, we improve the defect relation (see Theorem 4.2.5) and the height inequality (see Theorem 4.2.12) in the case when X is a normal projective surface and D_j, $1 \leq j \leq q$, are big and asymptotically free divisors without irreducible common components on X. As a consequence, we recover a sharp qualitative result due to Levin (see [Lia15]).

4.1 Levin's result

First, we recall some definitions and lemmas from Levin's paper [Lev09].

Definition 4.1.1 ([Lev09], Definition 9.6)**.** Suppose that X is a projective variety of dimension n. Let $D = D_1 + D_2 + \cdots + D_q$ be a divisor on X with D_i being effective. D is said to have equidegree with respect to $D_1, D_2, \ldots, D_q$ if $D_i.D^{n-1} = \frac{1}{q}D^n$ for $1 \leq i \leq q$.

Lemma 4.1.2 ([Lev09], Lemma 9.7)**.** *Let X be a projective variety of dimension n. If $D_j, 1 \leq j \leq q$, are big and nef, then there exist positive real numbers r_j such that $D = \sum_{j=1}^q r_j D_j$ has equidegree with respect to $r_1 D_1, \ldots, r_q D_q$.*

Proof. We follow the simplified proof given by Autissier [Aut09]. Let

$$\triangle := \{(t_1, \ldots, t_q) \in \mathbb{R}_+^q \mid t_1 + \cdots + t_q = 1\}.$$

Define a map $g : \triangle \to \triangle$ by letting, for $t = (t_1, \ldots, t_q) \in \triangle$,

$$g(t) = \left(\frac{\phi(t)}{(\sum_{j=1}^q t_j D_j)^{n-1}.D_1}, \cdots, \frac{\phi(t)}{(\sum_{j=1}^q t_j D_j)^{n-1}.D_q} \right),$$

where $\phi(t) := \left(\sum_{i=1}^q \frac{1}{(\sum_{j=1}^q t_j D_j)^{n-1}.D_i} \right)^{-1}$. By Brouwer's fixed point theorem, there exists a point $x = (r_1, \ldots, r_q)$ such that $g(x) = x$, i.e., $\phi(x) = (\sum_{j=1}^q r_j D_j)^{n-1}.(r_i D_i)$ for $i = 1, \ldots, q$. This implies, by summing all i, that $q\phi(x) = (\sum_{j=1}^q r_j D_j)^n$. Thus

$$\frac{1}{q} \left(\sum_{j=1}^q r_j D_j \right)^n = \phi(x) = (r_i D_i). \left(\sum_{j=1}^q r_j D_j \right)^{n-1},$$

which proves the lemma. $\square$

Lemma 4.1.2 tells us that we can always make the given big and nef divisors to be of equidegree without changing their supports since the divisors $r_j D_j$ and D_j are of the same supports. This means the notion of equidegree, rather than the condition of linear equivalence for the divisors $D = D_1 + \cdots + D_q$, would be a correct (or proper) notion in the study of degeneracy of holomorphic mappings $f : \mathbb{C} \to X\backslash D$. In the surface case, we denote by $(D_i.D_j)$ the intersection number of D_i and D_j. We also denoted $(D.D)$ by D^2. We recall definitions and results from Levin [Lev09].

Definition 4.1.3. Let X be a projective variety, and let D be an effective Cartier divisor on X, both defined over a number field k. Let L be a number field with $L \supset k$, and $\mathcal{S}$ be a finite set of places of L containing the archimedean places. We define the Diophantine exceptional set of $X\backslash D$ with respect to L and $\mathcal{S}$ to be

$$Exc_{Dio,L,\mathcal{S}}(X\backslash D) = \bigcup_R \dim_{>0}(\bar{R}),$$

where the union runs over all sets R of L-rational $(D, \mathcal{S})$-integral points on X and $\dim_{>0}(\bar{R})$ denotes the union of the positive-dimensional irreducible components of the Zariski-closure of R. We define the absolute Diophantine exceptional set of $X\backslash D$ to be

$$Exc_{Dio}(X\backslash D) = \bigcup_{L\supset k,\mathcal{S}} Exc_{Dio,L,\mathcal{S}}(X\backslash D),$$

with L ranging over all number fields containing k and $\mathcal{S}$ ranging over all sets of places of L as above.

These definitions depend only on $X\backslash D$ and not on the choices of X and D.

Definition 4.1.4. Let X be a complex variety. We define the holomorphic exceptional set $Exc_{hol}(X)$ of X to be the union of all images of non-constant holomorphic maps $f : \mathbb{C} \to X$.

Conjecturally, it is expected that $Exc_{Dio}(X\backslash D) = Exc_{hol}(X\backslash D)$.

Definition 4.1.5. Let X be a projective variety defined over a number field k. Let D be an effective Cartier divisor on X. Then we define $X\backslash D$ to be *Mordellic* if $Exc_{Dio}(X\backslash D)$ is empty. We define $X\backslash D$ to be *quasi-Mordellic* if $Exc_{Dio}(X\backslash D)$ is not Zariski-dense in X.

Definition 4.1.6. Let X be a complex variety. We define X to be *Brody hyperbolic* if $Exc_{hol}(X)$ is empty. We define X to be *quasi-Brody hyperbolic* if $Exc_{hol}(X)$ is not Zariski-dense in X.

Remark 4.1.7. Note that X being quasi-Brody hyperbolic is a stronger condition than the non-existence of algebraically non-degenerate holomorphic maps $f : \mathbb{C} \to X$. Similarly, $X\backslash D$ being quasi-Mordellic is stronger than the non-existence of Zariski-dense sets of D-integral points on X.

Theorem 4.1.8 ([Lev09], Theorem 11.5A)**.** *Let X be a smooth projective surface. Let $D = D_1 + D_2 + \cdots + D_q$ be a divisor on X with D_i being effective. Suppose that D_i have no irreducible components in common, and are in m-subgeneral position.*

(a) If D_i is big for all i and $q \geq 4[(m+1)/2]$, then $X\backslash D$ is quasi-Mordellic.

(b) If D_i is ample for all i and either m is even and $q > 2m$ or m is odd and $q > 2m+1$, then $X\backslash D$ is Mordellic.

Theorem 4.1.9 ([Lev09], Theorem 11.5B)**.** *Let X be a smooth projective surface. Let $D = D_1 + D_2 + \cdots + D_q$ be a divisor on X with D_i being effective. Suppose that D_i have no irreducible components in common, and are in m-subgeneral position.*

(a) If D_i is big for all i and $q \geq 4[(m+1)/2]$, then $X\backslash D$ is quasi-Brody hyperbolic.

(b) If D_i is ample for all i and either m is even and $q > 2m$ or m is odd and $q > 2m+1$, then $X \backslash D$ is Brody-hyperbolic.

4.2 Ru's master result

Encouraged by Corvaja and Zannier, Ru defined so called *μ-growth* divisors. He (see [Ru16a]) also introduced the notion of Nevanlinna constant, denoted by $\mathrm{Nev}(D)$, for an effective Cartier divisor D on a normal projective variety X. He then derived a new defect relation $\delta_f(D) \leq \mathrm{Nev}(D)$ for any algebraically non-degenerate holomorphic mapping $f : \mathbb{C} \to X$. Let X be a normal projective variety and D be an effective Cartier divisor on X. Note that the condition of normality of X is assumed so that $\mathrm{ord}_E D$ (called the *coefficient of D in E*) is defined for any prime divisor E and any effective Cartier divisor D on X (see [Laz04], Remark 1.1.4). For any section $s \in H^0(X, D)$, we use $\mathrm{ord}_E\, s$ or $\mathrm{ord}_E(s)$ to denote the coefficients of (s) in E, where (s) is the divisor on X associated to s.

Definition 4.2.1. Let X be a normal complex projective variety, and D be an effective Cartier divisor on X. The divisor D is said to have *μ-growth* with respect to $V \subset H^0(X, \mathcal{O}(D))$ with $\dim V \geq 2$, such that for all $P \in \mathrm{supp} D$, there exists a basis B of V with

$$\sum_{s \in B} \mathrm{ord}_E(s) \geq \mu \, \mathrm{ord}_E \, D \, ,$$

for all irreducible component E of D passing through P.

The *Nevanlinna constant* of D, denoted by $\mathrm{Nev}(D)$, is given by

$$\mathrm{Nev}(D) := \inf_N \left(\inf_{\{\mu_N, V_N\}} \frac{\dim V_N}{\mu_N} \right), \tag{4.1}$$

where the infimum "$\inf_N$" is taken over all positive integers N and the infimum "$\inf_{\{\mu_N, V_N\}}$" is taken over all pairs $\{\mu_N, V_N\}$, where μ_N is a positive real number and $V_N \subseteq H^0(X, ND)$ is a linear subspace with $\dim V_N \geq 2$ such that, for all $P \in \mathrm{supp} D$, there exists a basis B of V_N with

$$\sum_{s \in B} \mathrm{ord}_E(s) \geq \mu_N \, \mathrm{ord}_E(ND) \, ,$$

for every irreducible component E of D passing through P. If $h^0(ND) \leq 1$ for all positive integers N, we define $\mathrm{Nev}(D) = +\infty$.

Theorem 4.2.2 ([Ru16a], Proposition 3.1)**.** *Let X be a normal complex projective variety and D be an effective Cartier divisor on X. Assume that there exists a positive number $\mu > 0$ and a linear subspace $V \subset H^0(X, \mathcal{O}(D))$ with $\dim V \geq 2$, such that for all $P \in suppD$, there exists a basis B of V with*

$$\sum_{s \in B} \mathrm{ord}_E(s) \geq \mu \, \mathrm{ord}_E D \, ,$$

for all irreducible component E of D passing through P. Let $f : \mathbb{C} \to X$ be an algebraically non-degenerate holomorphic map. Then, for every $\epsilon > 0$,

$$m_f(r, D) \leq \left(\frac{\dim V}{\mu} + \epsilon \right) T_{f,D}(r) \quad \|_E.$$

In order to prove the theorem, the following proposition is important.

Proposition 4.2.3. *Let $\phi : X' \to X$ be a proper birational morphism of normal projective varieties, and let D be a Cartier divisor on X whose support doesn't contain $\phi(X')$. If D has μ-growth with respect to a subspace $V \subset H^0(X, \mathcal{O}(D))$, then $\phi^* D$ also has μ-growth with respect to the corresponding subspace of the same dimension.*

Proof of Theorem 4.2.2. Let $\Phi : X \to \mathbb{P}^m$ be the canonical rational map associated to V where $V \subset H^0(X, \mathcal{O}(D))$ with $\dim V = m + 1 \geq 1$.

We may assume that Φ is a morphism. Indeed, let X' be a desingularization of closure of the graph of Φ. Replace X with X' and D with its pull-back. By previous Proposition 4.2.3, the pull-back still has μ-growth with respect to the corresponding vector space of the same dimension. Moreover, by functoriality of Weil functions, the corresponding Weil function and the height function remain.

Let σ_0 be the set of all prime divisors occurring in D, so we can write

$$D = \sum_{E \in \sigma_0} \mathrm{ord}_E(D) E.$$

Let

$$\Sigma := \{\sigma \subset \sigma_0 | \bigcap_{E \in \sigma} E \neq \emptyset\}.$$

For each $\sigma \in \Sigma$, write

$$D = D_{\sigma,1} + D_{\sigma,2},$$

where

$$D_{\sigma,1} := \sum_{E \in \sigma} \mathrm{ord}_E(D) E, \;\; D_{\sigma,2} := \sum_{E \notin \sigma} \mathrm{ord}_E(D) E.$$

Pick a Weil function for each divisor $D, D_{\sigma,1}, D_{\sigma,2}$. We first claim that *there exists a constant* C*, depending only on* X *and* D*, such that* $\min_{\sigma\in\Sigma}\lambda_{D_{\sigma,2}} \leq C$ *for all* $x \in X$. Indeed, the definition of the set Σ implies that

$$\bigcap_{\sigma\in\Sigma} suppD_{\sigma,2} = \emptyset,$$

since, for all $x \in X$, the set $\sigma := \{E \in \sigma_0 | x \in E\}$ is an element of Σ, and then $x \notin suppD_{\sigma,2}$. Our claim then follows from Lemma 2.0.5 since Σ is a finite set.

Now for each $\sigma \in \Sigma$, since D has μ-growth with respect to V, let B_σ be a basis of V that satisfies

$$\sum_{s\in B_\sigma} \operatorname{ord}_E(s) \geq \mu \operatorname{ord}_E(D)$$

at some (and hence all) points $P \in \cap_{E\in\sigma} E$. Since Σ is finite, $\{B_\sigma | \sigma \in \Sigma\}$ is a finite collection of bases of V. Thus, the distinct hyperplanes in $\mathbb{P}^m$ corresponding to elements of the union $\cup_{\sigma\in\Sigma} B_\sigma$ is finite, say they are $H_1, \ldots, H_q$. Choose a Weil function λ_{H_j} for each $H_j, 1 \leq j \leq q$.

For an arbitrary $x \in X$, from the claim above, pick $\sigma \in \Sigma$ (depends on x) such that

$$\lambda_{D_{\sigma,2}}(x) \leq C,$$

where C is the constant which occurs in the claim. Let $J \subset \{1, \ldots, q\}$ be the subset for which $\{H_j, j \in J\}$ are hyperplanes corresponding to the elements of B_σ. Then Proposition 4.2.3 implies that

$$\sum_{j\in J} \operatorname{ord}_E \Phi^* H_j \geq \mu \operatorname{ord}_E D,$$

for all $E \in \sigma$; and therefore, by the "boundedness from below" property of the Weil functions for effective divisors,

$$\sum_{j\in J}(\mathrm{ord}_E \Phi^* H_j)\lambda_E(x) \geq \mu(\mathrm{ord}_E D)\lambda_E(x) + O(1), \tag{4.2}$$

for all $E \in \sigma$. Now, since

$$D = \sum_{E\in\sigma}(\mathrm{ord}_E D)E + D_{\sigma,2},$$

we have, by using the lemma 2.0.3,

$$\lambda_D(x) = \lambda_{D_{\sigma,1}}(x) + \lambda_{D_{\sigma,2}}(x) = \sum_{E\in\sigma}(\mathrm{ord}_E D)\lambda_E(x) + O(1).$$

Therefore, together with inequality (4.2), we have

$$\begin{aligned}
\sum_{j\in J}\lambda_{H_j}(\Phi(x)) &\geq \sum_{j\in J}\sum_{E\in\sigma}(\mathrm{ord}_E \Phi^* H_j)\lambda_E(x) + O(1) \\
&\geq \mu\sum_{E\in\sigma}(\mathrm{ord}_E D)\lambda_E(x) + O(1) \\
&\geq \mu\lambda_D(x) + O(1).
\end{aligned}$$

Note that, since $\{H_j, j \in J\}$ are the hyperplanes corresponding to subsets of B_σ, we see that $\{H_j, j \in J\}$ are in general position. Thus, for any $x \in X$,

$$\lambda_D(x) \leq \frac{1}{\mu}\left(\max_J \sum_{j\in J}\lambda_{H_j}(\Phi(x)) + O(1)\right), \tag{4.3}$$

where J varies over all subsets of $\{1, \ldots, q\}$ corresponding to the elements of $\{H_1, \ldots, H_q\}$ that lie in general position. Note that, although $O(1)$ that appears above depends on the choices of B_σ (thus depends on σ), it is a constant independent of x since Σ is a finite set (so there are only finitely many choices of σ).

Now for any algebraically non-degenerate holomorphic map $f : \mathbb{C} \to X$, applying (4.3) with $x = f(z)$, and then integrating over $|z| = r$, we obtain, by the definition of $m_f(r, D)$,

$$m_f(r, D) \leq \frac{1}{\mu} \int_0^{2\pi} \max_J \sum_{j \in J} \lambda_{H_j}((\Phi \circ f)(re^{i\theta})) \frac{d\theta}{2\pi} + O(1),$$

where J varies over all subsets of $\{1, \ldots, q\}$ corresponding to subsets of $\{H_1, \ldots, H_q\}$ that lie in general position. Applying Theorem 2.0.6, for any $\epsilon > 0$

$$m_f(r, D) \leq \frac{\dim V + \epsilon}{\mu} T_{\Phi \circ f}(r) \ ||_E.$$

By the inequality (2.7), we know that $T_{\Phi \circ f}(r) \leq T_{f,D}(r)$. Thus,

$$m_f(r, D) \leq \frac{\dim V + \epsilon}{\mu} T_{f,D}(r) ||_E,$$

which proves the theorem. $\square$

Remark 4.2.4. From this theorem, larger μ implies the smaller (better) defect.

As a consequence of theorem 4.2.3, we have

Theorem 4.2.5 ([Ru16a], Main Theorem)**.** *(a) Let X be a complex normal projective variety and D be an effective Cartier divisor on X. Then, for every $\epsilon > 0$,*

$$m_f(r, D) \leq (Nev(D) + \epsilon)\, T_{f,D}(r) \quad ||_E$$

holds for any algebraically non-degenerate holomorphic mapping $f : \mathbb{C} \to X$.

(b) If X is a complex projective variety but not normal, and D is an effective Cartier divisor on X. Let $\pi : \tilde{X} \to X$ be the normalization of X. Then, for every

$\epsilon > 0$,

$$m_f(r, D) \leq (Nev(\pi^* D) + \epsilon)\, T_{f,D}(r) \quad \|_E$$

holds for any algebraically non-degenerate holomorphic mapping $f : \mathbb{C} \to X$.

The above theorem gives the following defect relation.

Corollary 4.2.6 (Defect relation)**.** *Let D be an effective Cartier divisor on a normal complex projective variety X. Then*

$$\delta_f(D) \leq Nev(D)$$

holds for any algebraically non-degenerate holomorphic map $f : \mathbb{C} \to X$.

Corollary 4.2.7. *Let $X = \mathbb{P}^n$ and $D = H_1 + \cdots + H_q$, where $H_1, \ldots, H_q$ are hyperplanes in $\mathbb{P}^n$ in general position. Then*

$$Nev(D) \leq \frac{n+1}{q}.$$

Proof. We take $N = 1$ and consider $V_1 := H^0(\mathbb{P}^n, \mathcal{O}(D)) \cong H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(q))$. Then the $\dim V_1 = \binom{n+q}{n}$. For each $P \in \mathrm{Supp}D$, since $H_1, \ldots, H_q$ are in general position, $P \in H_{i_0} \cap H_{i_1} \cdots \cap H_{i_l}$ with $\{i_0, i_1, \ldots, i_l\} \subset \{1, 2, \ldots, q\}$ and $l+1 \leq n$. W.l.o.g., we can just assume $H_{i_0} = \{z_0 = 0\}, H_{i_1} = \{z_1 = 0\}, \ldots, H_{i_l} = \{z_l = 0\}$ by taking proper coordinates for $\mathbb{P}^n$. Now we take the basis $B = \{z_0^{i_0} z_1^{i_1} \ldots z_n^{i_n} | i_0 + i_1 + \cdots + i_n = q\}$ for $V_1 = H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(q))$. Then, for each irreducible component E of D containing P, say $E = \{z_{j_0} = 0\}$ with $1 \leq j_0 \leq l$, we have $\mathrm{ord}_E\{z_j = 0\} = 0$ for $j \neq j_0$, $\mathrm{ord}_E\{z_{j_0} = 0\} = 1$ and thus $\mathrm{ord}_E D = 1$. Therefore,

$$\sum_{s \in B} \mathrm{ord}_E\, s = \sum_{\mathbf{i}} i_{j_0} = \frac{1}{n+1} \sum_{\mathbf{i}} (i_0 + \ldots + i_n) = \frac{q}{n+1} \binom{q+n}{n} = \frac{q}{n+1} \dim V_1,$$

where the sum is taken for all $\mathbf{i} = (i_0, \dots, i_n)$ with $i_0 + \dots + i_n = q$. Thus we can take $\mu = \frac{q}{n+1} \dim V_1$, and hence,

$$\mathrm{Nev}(D) \leq \frac{\dim V_1}{\mu} = \frac{n+1}{q}.$$

$\square$

Corollary 4.2.8. *Let $X = \mathbb{P}^n$ and $D_1, \dots, D_q$ be hypersurfaces in $\mathbb{P}^n$ of degree d_i, located in general position on X. Then*

$$Nev(D) \leq \frac{n+1}{q}.$$

Proof. Let $D_i \sim A$ and $D_i = \{Q_i = 0\}$, where Q_i is a homogeneous polynomial of degree $d := \deg A$ for $i = 1, \dots, q$. Let $P \in \mathrm{supp} D$. The condition that $D_1, \dots, D_q$ are in general position implies that $P \in \cap_{i=1}^{l} \{\gamma_i = 0\}$ for some $\gamma_1, \dots, \gamma_l \in \{Q_1, \dots, Q_q\}$ and $l \leq n$. We can assume that $l = n$ since we can add more polynomials. Choose a positive integer N which is divisible by qd and $\tilde{N} = \frac{N}{qd}$. Let $V_{\tilde{N}} := H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(\tilde{N}D)) \cong H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(\tilde{N}qd)) \cong H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(N))$, $\mathbf{i} = (i_1, \dots, i_n)$ be a n-tube with lexicographical order and $\sigma(\mathbf{i}) := \sum_{j=1}^{n} i_j \leq \frac{N}{d}$, we obtain a filtration on $V_{\tilde{N}}$ given by

$$W_{\mathbf{i}} = \sum_{\mathbf{e}=(e_1,\dots,e_n)\geq \mathbf{i}} \gamma_1^{e_1} \dots \gamma_n^{e_n} V_{N-d\sigma(\mathbf{e})}.$$

Note that $W_{\mathbf{0}} = V_N$ and $W_{\mathbf{i}} \supset W_{\mathbf{i}'}$ for $\mathbf{i}' \geq \mathbf{i}$. Choose a basis $s_1, s_2, \dots, s_m$, where $m = \binom{N+n}{n}$, for $V_{\tilde{N}}$ with respect to the above filtration. With this choice of the basis, we compute the Nevanlinna constant. We recall the following lemma.

Lemma 4.2.9 ([Ru04], Lemma 3.3)**.** *Fix any $N > n(d-1)$ and any $\mathbf{i} = (i_1, \dots, i_n)$ with $d\sigma(\mathbf{i}) < N - n(d-1)$. Then*

$$\Delta_{\mathbf{i}} := \dim \frac{W_{\mathbf{i}}}{W_{\mathbf{i}'}} = d^n,$$

where $W_{\mathbf{i}} \supset W_{\mathbf{i}'}$ with $\mathbf{i}'$ is next to $\mathbf{i}$.

Let, for $\mu = 1, \dots, m$, $s_\mu = \gamma_1^{i_1}\gamma_2^{i_2}\dots\gamma_n^{i_n}\gamma^\mu$ for some $\gamma^\mu \in V_{N-d\sigma(\mathbf{i})}$ based on its place in the filtration. For any irreducible component E in D with $P \in E$, we may assume that E is contained $\{\gamma_{j_0} = 0\}$ for some $1 \le j_0 \le n$. Then, for N is big enough, we have,

$$\begin{aligned}
\sum_{\mu=1}^m \operatorname{ord}_E s_\mu &\ge \left(\sum_{i_1+\dots+i_n \le \frac{N}{d}-n} \Delta_{\mathbf{i}} i_{j_0} \right) \operatorname{ord}_E D \\
&= \left(\sum_{i_1+\dots+i_n \le \frac{N}{d}-n} i_{j_0} \right) d^n \operatorname{ord}_E D \\
&= \left(\sum_{i_0+i_1+\dots+i_n = \frac{N}{d}-n} i_{j_0} \right) d^n \operatorname{ord}_E D \\
&= \frac{1}{n+1} \left(\sum_{i_1+\dots+i_n = \frac{N}{d}-n} \sum_{\tau=0}^n i_\tau \right) d^n \operatorname{ord}_E D \\
&= \frac{d^n}{n+1} \left(\sum_{i_1+\dots+i_n = \frac{N}{d}-n} \frac{N}{d} \right) \operatorname{ord}_E D \\
&= \frac{d^n}{n+1} \binom{N/d}{n} \frac{N}{d} \operatorname{ord}_E D = \left(\frac{N^{n+1}}{d(n+1)!} + O(N^n) \right) \operatorname{ord}_E D \\
&= \frac{\tilde{N}q}{n+1} \left(\frac{N^n}{n!} + O(N^{n-1}) \right) \operatorname{ord}_E D \\
&= \frac{q}{n+1} \left(\frac{N^n}{n!} + O(N^{n-1}) \right) \operatorname{ord}_E(\tilde{N}D).
\end{aligned}$$

Hence, from the definition of $\mathrm{Nev}(D)$, we have

$$\mathrm{Nev}(D) \leq \liminf_{\tilde{N}\to+\infty} \frac{\dim V_{\tilde{N}}}{\frac{q}{n+1}(\frac{N^n}{n!}+O(N^{n-1}))} = \liminf_{\tilde{N}\to+\infty} \frac{\frac{N^n}{n!}+O(N^{n-1})}{\frac{q}{n+1}(\frac{N^n}{n!}+O(N^{n-1}))} = \frac{n+1}{q}.$$

This concludes the proof of the Corollary. $\square$

Furthermore, Ru obtained the following result.

Theorem 4.2.10 ([Ru16a], Theorem 5.6)**.** *Let X be a complex normal projective variety of dimension ≥ 2, and $D_1, \ldots, D_q$ be effective and big Cartier divisors in l-subgeneral position on X. Let $r_i > 0$ be real numbers such that $D := \sum_{i=1}^q r_i D_i$ is equidegree (such numbers exist due to Lemma 4.1.2). We further assume that the linear system $|ND_i|$ $(i = 1, \ldots, q)$ is base-point free for $N \geq N_0$. Let $f : \mathbb{C} \to X$ be an algebraically non-degenerate holomorphic map. Then, for $\epsilon_0 > 0$ small enough,*

$$\sum_{j=1}^q r_j m_f(r, D_j) < \left(\frac{2l \dim X}{q} - \epsilon_0\right)\left(\sum_{j=1}^q r_j T_{f,D_j}(r)\right) \ \|_E .$$

On the arithmetic side, the counterpart of Theorem 4.2.5 in Diophantine approximation is stated as follows.

Theorem 4.2.11 ([Ru16b], Main Theorem)**.** (a) *Let k be a number field and M_k be the set of places of k. Let $S \subset M_k$ be a finite set of places containing all archimedean ones. Let X be a normal projective variety and D be an effective Cartier divisor on X, both defined over k (we further assume that all irreducible components of D are Cartier divisors). Then, for every $\varepsilon > 0$, the inequality*

$$m_S(x, D) \leq (Nev(D) + \epsilon)\, h(x, D) \tag{4.4}$$

holds for all $x \in X(k)$ outside a Zariski closed subset Z of X.

(*b*) *Suppose that the projective variety X is not normal. Let $\pi : \tilde{X} \to X$ be the normalization of X. Then, for every $\varepsilon > 0$, the inequality*

$$m_S(x, D) \leq (Nev(\pi^* D) + \epsilon)\, h(x, D) \tag{4.5}$$

holds for all $x \in X(k)$ outside a Zariski closed subset Z of X.

As a consequence, the counterpart of Theorem 4.2.10 in Diophantine approximation is stated as follows.

Theorem 4.2.12 ([Ru16b], Theorem 4.1)**.** *Let k be a number field and $S \subset M_k$ be a finite set containing all archimedean places. Let X be a normal projective variety with $\dim X \geq 2$, $D_1, \ldots, D_q$ be effective and big Cartier divisors in l-subgeneral position on X, both defined over k. Let $r_i > 0$ be real numbers such that $D := \sum_{i=1}^q r_i D_i$ is equidegree (such numbers exist due to Lemma 4.1.2). We further assume that the linear system $|ND_i|$ $(i = 1, \ldots, q)$ is base-point free for $N \geq N_0$. Let $\lambda_{D_j,v}(x), 1 \leq j \leq q$, be the Weil function associated to D_j for $v \in S$. Then, for $\epsilon_0 > 0$ small enough,*

$$\sum_{j=1}^{q} \sum_{v \in S} r_j \lambda_{D_j,v}(x) < \left(\frac{2l \dim X}{q} - \epsilon_0 \right) \left(\sum_{j=1}^{q} r_j h(x, D_j) \right),$$

holds for all $x \in X$ outside a Zariski closed subset Z of X.

4.3 The statement of Main Theorem B and Main Theorem C

The purpose of this section is to improve Theorem 4.2.12 in the case when $\dim X = 2$ with an additional condition that the divisors $D_1, \ldots, D_q$ have no common irreducible components. The precise statement is as follows.

Main Theorem B (Complex Part)**.** *Let X be a normal complex projective surface. Let $D_1, \ldots, D_q$ be effective, big Cartier divisors on X, and the linear system $|ND_i|$ ($i = 1, \ldots, q$) be base-point free for $N \geq N_0$. Assume that $D_1, \ldots, D_q$ have no irreducible components in common, and are in l-subgeneral position. We further assume that $D := \sum_{j=1}^{q} r_j D_j$ is equidegree for some positive real numbers r_j (such r_j always exist by Lemma 4.1.2). Let $f : \mathbb{C} \to X$ be holomorphic and algebraically non-degenerate. Then*

$$\sum_{j=1}^{q} r_j m_f(r, D_j) \leq \frac{4[(l+1)/2]}{q(1+\alpha)} \left(\sum_{j=1}^{q} r_j T_{f,D_j}(r) \right) \|_E ,$$

where $\alpha = \frac{\min_{1 \leq j \leq q}(r_j^2 D_j^2)}{384 q D^2}$, $[x]$ denotes the greatest integer less than or equal to x.

Note that our techniques used to prove the above theorem are similar to Ru's method (see [Ru16a]). The main contribution is to use the joint filtrations lemma due to Corvaja and Zannier (see [CZ04a], Lemma 3.2) to lower the upper bound of the defect, under the additional assumption that $D_1, \ldots, D_q$ have no irreducible components in common. Furthermore, we give the explicit computation of the ϵ which appeared in Theorem 4.2.5. Note that our result also holds for any dimension of X. The reason we only focus on the case when $\dim X = 2$ is the following sharp Corollary of our Main Theorem B, which is due to Levin [Lev09].

Corollary 4.3.1 ([Lev09], Theorem 11.5B)**.** *Let X be a smooth projective surface and $D_1, D_2, \ldots, D_q$ be effective and big divisors on X. Assume that $D_1, \ldots, D_q$ have no irreducible components in common, and are in general position. If $q \geq 4$, then every holomorphic mapping $f : \mathbb{C} \to X \backslash \cup_{j=1}^{q} D_j$ must be algebraically degenerate.*

Proof. According to the proof of Theorem 11.5(a) in [Lev09], we can reduce it to the case when $|D_i|$ is base-point free (in particular, $D_i, 1 \leq i \leq q$, are nef), and D_i is big for all i. If f is not algebraically degenerate, then the above Main Theorem B implies that $\delta_f(D) < 1$, where $D = D_1 + D_2 + \cdots + D_q$. On the other hand, from the first main theorem, we have $\delta_f(D) = 1$. This gives a contradiction. Therefore, f must be algebraically degenerate. □

Our theorem on the arithmetic side which improves Theorem 4.2.12 is as follows.

Main Theorem C (Arithmetic Part)**.** *Let k be a number field and $S \subset M_k$ be a finite set containing all archimedean places. Let X be a normal projective surface, and let $D_1, \ldots, D_q$ be effective and big Cartier divisors on X, all defined over k. Assume that the linear system $|ND_i|$ $(i = 1, \ldots, q)$ is base-point free for $N \geq N_0$. Assume that $D_1, \ldots, D_q$ have no irreducible components in common, and are in l-subgeneral position. We further assume that $D := \sum_{i=1}^{q} r_j D_i$ is equidegree for some positive real numbers r_j (such r_j always exist by Lemma 4.1.2). Let $\lambda_{D_j,v}(x), 1 \leq j \leq q$, be the*

Weil function associated to D_j for $v \in S$. Then

$$\sum_{j=1}^{q}\sum_{v\in S} r_j\lambda_{D_j,v}(x) \leq \frac{4[(l+1)/2]}{q(1+\alpha)}\left(\sum_{j=1}^{q} r_j h(x,D_j)\right),$$

holds for all $x \in X$ outside a Zariski closed subset Z of X, where $\alpha = \frac{\min_{1\leq j\leq q}(r_j^2 D_j^2)}{384qD^2}$, $[x]$ denotes the greatest integer less than or equal to x, and $h(x, D_j) = \sum_{v\in M_k} \lambda_{D_j,v}(x)$.

The above theorem gives the following Corollary.

Corollary 4.3.2 ([Lev09], Theorem 11.5A)**.** *Let k be a number field and $S \subset M_k$ be a finite set containing all archimedean places. Let X be a smooth projective surface, and $D_1, \ldots, D_q$ be effective and big Cartier divisors on X, both defined over a number field k. Assume that $D_1, \ldots, D_q$ have no irreducible components in common, and are in general position. If $q \geq 4$, then any set of (D, S)-integral points of $X\backslash D$ is contained in a proper subvariety of X.*

The proof of the result in the arithmetic case is similar to the complex case (see [Ru16b]), so the rest of the section will only focus on the complex part.

4.4 More Lemmas

We also need the following results in algebraic geometry. Let X be a projective surface. Let D be a divisor on X. Let $\mathcal{O}(D)$ be the invertible sheaf associated to the divisor D on X. For $i = 0, 1, 2$, let $h^i(D) = \dim H^i(X, \mathcal{O}(D))$ (we sometimes also just write $H^i(X, \mathcal{O}(D))$ as $H^i(X, D)$ for simplicity).

Lemma 4.4.1 ([Laz04], Corollary 1.4.41)**.** *Let X be a projective surface. Suppose that D is nef divisor on a projective surface on X. Then*

$$h^0(ND) = \frac{D^2N^2}{2} + O(N)\ .$$

In particular, D^2 is positive if and only if D is big.

Lemma 4.4.2 ([Aut09], lemma 4.2)**.** *Let X be a projective surface. Let F be a big and base point free Cartier divisor and D be a Cartier divisor such that $D - F$ is also nef. Let $\beta > 0$ be a positive real number. Then for any positive integer N, k with $1 \le k \le \beta N$, we have*

$$h^0(ND - kF) \ge \frac{D^2N^2}{2} - (D.F)Nk + \frac{F^2}{2}\min\{k^2, N^2\} + O(N)\ ,$$

where $O(N)$ depends on β.

Proof. We separate two cases.

Case $k \le N$. By Riemann-Roch,

$$\begin{aligned}\chi(X, ND - kF) &= \frac{1}{2}(ND - kF)^2 \\ &= \frac{D^2N^2}{2} - Nk(D.F) + \frac{F^2k^2}{2}\end{aligned}$$

Since D and $D - F$ are nef, $h^i(X, ND - kF) = O(N^{2-i})$ for all $i \ge 1$. Thus, we derive the result.

Case $k > N$. Let $N \le i \le \beta N$. We have the following short exact sequence.

$$0 \to \mathcal{O}_X(ND - (i+1)F) \to \mathcal{O}_X(ND - iF) \to \mathcal{O}_Z(ND - iF)|_Z) \to 0,$$

where $Z = \mathrm{div}(s)$ for some generic $s \in \Gamma(X, F)$. The short exact sequence implies

$$h^0(X, ND - (i+1)F) \geq h^0(X, ND - iF) - h^0(Z, (ND - iF)|_Z).$$

Since $h^0(Z, (ND - iF)|_Z) \leq h^0(Z, ND|_Z) = (D.F)N + O(1)$, we have

$$\begin{aligned} h^0(X, ND - kF) &\geq h^0(X, ND - NF) - \sum_{i=N}^{k-1} h^0(Z, (ND - iF)|_Z) \\ &\geq \frac{D^2}{2} N^2 - (D.F)Nk + \frac{1}{2} F^2 N^2 + O(N). \end{aligned}$$

The lower bound of $h^0(X, ND - NF)$ comes from the first case. Combining these two cases, we proved the lemma. □

Lemma 4.4.3. *Let D and F be the same as above on a projective surface X. We further assume that $\frac{D^2}{(D.F)} \geq 1$. Then*

$$\sum_{k=1}^{\infty} h^0(ND - kF) \geq \left(\frac{D^2}{4(D.F)} + \frac{F^2}{24D^2} \right) N h^0(ND) + O(N^2) .$$

Proof. Using Lemma 4.4.2 with $\beta = \frac{D^2}{2(D.F)} \geq \frac{1}{2}$, we get, by noticing that $\min\{k^2, N^2\} \geq \min\{k^2, N^2/4\}$,

$$\begin{aligned} & \sum_{k=1}^{\infty} h^0(ND - kF) \\ \geq\ & \sum_{k=1}^{[\beta N]} \left(\frac{D^2 N^2}{2} - (D.F)Nk + \frac{F^2}{2} \min\{k^2, \frac{N^2}{4}\} \right) + O(N^2) \\ \geq\ & \frac{(D^2)^2 N^3}{4(D.F)} - (D.F)\frac{(D^2)^2 N^3}{8(D.F)^2} + \sum_{k=1}^{[N/2]-1} \frac{F^2}{2} k^2 + \sum_{[N/2]}^{[\beta N]} \frac{F^2}{8} N^2 + O(N^2) \\ =\ & \frac{(D^2)^2}{8(D.F)} N^3 + \frac{F^2}{48} N^3 + \frac{F^2}{8} \beta N^3 - \frac{F^2}{16} N^3 + O(N^2) \end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{D^2}{4(D.F)}+\frac{F^2}{D^2}(\frac{\beta}{4}-\frac{1}{12})\right)\frac{D^2}{2}N^3+O(N^2)\\
&\geq \left(\frac{D^2}{4(D.F)}+\frac{F^2}{24D^2}\right)\frac{D^2}{2}N^3+O(N^2)\\
&= \left(\frac{D^2}{4(D.F)}+\frac{F^2}{24D^2}\right)Nh^0(ND)+O(N^2).
\end{aligned}
$$

□

4.5 Proof of Main Theorem B

The following joint filtrations lemma is crucial to the proof of our Main Theorem B.

Lemma 4.5.1 ([CZ04a], Lemma 3.2)**.** *Let V be a vector space of finite dimension d over a field k. Let $V = W_1 \supset W_2 \supset W_3 \supset \ldots \supset W_h$ and $V = W_1^* \supset W_2^* \supset W_3^* \supset \ldots \supset W_{h^*}^*$ be two filtrations on V. Then there exist a basis of V that contains a basis of each W_j and W_j^* .*

Proof of Main Theorem B. The proof uses Theorem 4.2.5, so we need to compute the Nevanlinna constant. By taking $N \geq N_0$, we can assume that $D_j, 1 \leq j \leq q$, are base-point free. By the assumption that D has equidegree with respect to $r_1D_1, \ldots, r_qD_q$, we have, with $D := \sum_{j=1}^q r_jD_j$,

$$(r_iD_i.D) = \frac{1}{q}D^2.$$

To simplify the notation, we write

$$\alpha := \frac{\min_{1\leq j\leq q}(r_j^2D_j^2)}{384qD^2}. \tag{4.6}$$

Choose positive rational numbers $a_j, 1 \leq j \leq q$, such that

$$|a_j - r_j| \leq \frac{\delta_1(\min_{1\leq j\leq q} r_j)}{2} \min\left\{1, \frac{q(1+2\alpha)}{4[(l+1)/2]}\right\} , \tag{4.7}$$

and, for $i = 1, \dots, q$,

$$\left|\frac{(a_i D_i . D')}{D'^2} - \frac{1}{q}\right| < \delta_2 , \tag{4.8}$$

where $D' := \sum_{j=1}^{q} a_j D_j$ and δ_1, δ_2 will be chosen below (see (4.17) and (4.13)). Note that with our choice of δ_1 and $|a_j - r_j| \leq \frac{r_j}{2}$, we have

$$D'^2 \geq \frac{1}{4} D^2 \qquad \text{and} \qquad D^2 \geq \frac{1}{4} D'^2. \tag{4.9}$$

Now, for $P \in \mathrm{supp} D$, let $D'_P := \sum_{\{i : P \in \mathrm{supp} D_i\}} a_i D_i$. Since $D_1, \dots, D_q$ are in l-subgeneral position and any two of $D_1, \dots, D_q$ have no common components, we can write

$$D'_P := D'_{P,1} + D'_{P,2} ,$$

where $D'_{P,1}$ and $D'_{P,2}$ are each a sum of no more than $[(l+1)/2]$ terms of the $a_i D_i$, and $D'_{P,1}$ and $D'_{P,2}$ have no irreducible components in common. Let d be the product of the denominators of $a_1, \dots, a_q$ and consider $V_N := H^0(X, NdD')$. Note that D' is only a $\mathbb{Q}$-divisor, so we need to multiply d to D' to make dD' to be an (integral) divisor. We consider the following two filtrations of V_N:

$$W_m := H^0(X, NdD' - mdD'_{P,1}),\ W_m^* := H^0(X, NdD' - mdD'_{P,2}),\ \ m = 0, 1, \dots .$$

Using the filtration lemma above, we obtain a basis B that contains a basis for each W_m and W_m^*. Let E be an irreducible component of D which contains P. Then

E is contained either in $dD'_{P,1}$ or $dD'_{P,2}$, but not both. Without loss of generality, we assume that E is an irreducible component of $dD'_{P,1}$, and thus $\mathrm{ord}_E(dD') = \mathrm{ord}_E(dD'_{P,1})$. We also note that $\mathrm{ord}_E\, s \geq m\, \mathrm{ord}_E(dD'_{P,1})$ for any $s \in H^0(X, NdD' - mdD'_{P,1})$ (regarded as a subspace of $H^0(X, NdD')$). Hence

$$\begin{aligned} & \frac{1}{\mathrm{ord}_E(NdD')} \sum_{s \in B} \mathrm{ord}_E\, s = \frac{1}{\mathrm{ord}_E(NdD'_{P,1})} \sum_{s \in B} \mathrm{ord}_E\, s \\ \geq\ & \frac{1}{N} \sum_{m=0}^{\infty} m(h^0(NdD' - mdD'_{P,1}) - h^0(NdD' - (m+1)dD'_{P,1})) \\ =\ & \frac{1}{N} \sum_{k=1}^{\infty} h^0(NdD' - kdD'_{P,1}). \end{aligned}$$

Noticing $\frac{D'^2}{(D'.D'_{P,1})} \geq 1$, by Lemma 4.4.3, we get

$$\begin{aligned} & \sum_{k=1}^{\infty} h^0(NdD' - kdD'_{P,1}) \\ \geq\ & \left(\frac{(dD')^2}{4(dD'.dD'_{P,1})} + \frac{(dD'_{P,1})^2}{24(dD')^2} \right) Nh^0(NdD') + O(N^2) \\ =\ & \left(\frac{D'^2}{4(D'.D'_{P,1})} + \frac{D'^2_{P,1}}{24D'^2} \right) Nh^0(NdD') + O(N^2). \end{aligned}$$

Therefore,

$$\frac{1}{\mathrm{ord}_E(NdD')} \sum_{s \in B} \mathrm{ord}_E\, s \geq \left(\frac{D'^2}{4(D'.D'_{P,1})} + \frac{D'^2_{P,1}}{24D'^2} \right) h^0(NdD') + O(N). \qquad (4.10)$$

We now estimate each term which appeared above. Since $D'_{P,1}$ is a sum of no more than $[(l+1)/2]$ terms of the $a_i D_i$ and, by using (4.8),

$$\frac{(D'.D'_{P,1})}{D'^2} \leq \frac{[(l+1)/2] \max_{i, P \in D_i} \{(D'.(a_i D_i))\}}{D'^2} \leq [(l+1)/2](\frac{1}{q} + \delta_2). \qquad (4.11)$$

Also notice that, using (4.6), (4.7) and (4.9),

$$\begin{aligned}(D'_{P,1}.D'_{P,1}) &\geq \min_{1\leq i\leq q}(a_iD_i.a_iD_i) \geq \frac{1}{4}\min_{1\leq i\leq q}(r_iD_i.r_iD_i) \\ &= 96q(D.D)\alpha \geq 24q(D'.D')\alpha\ .\end{aligned} \tag{4.12}$$

Combining (4.10), (4.11), and (4.12), we get

$$\begin{aligned}&\frac{1}{\mathrm{ord}_E(NdD')}\sum_{s\in B}\mathrm{ord}_E\, s \\ \geq\ & \left(\frac{q}{4[(l+1)/2](1+q\delta_2)}+q\alpha\right)\frac{d^2D'^2}{2}N^2+O(N) \\ \geq\ & \frac{q}{4[(l+1)/2]}\left(\frac{1}{1+q\delta_2}+4\alpha\right)h^0(NdD')+O(N) \\ =\ & \frac{q(1+3\alpha)}{4[(l+1)/2]}h^0(NdD')+O(N)\ ,\end{aligned}$$

where the last equality holds when choosing

$$\delta_2 = \frac{\alpha}{q(1-\alpha)}\ . \tag{4.13}$$

Hence, from the definition of Nevanlinna's constant (see (4.1)), we obtain

$$\mathrm{Nev}(dD') \leq \frac{4[(l+1)/2]}{q(1+3\alpha)}.$$

Applying Main Theorem 4.2.5 with $\epsilon = \frac{4[(l+1)/2]\alpha}{q(1+3\alpha)(1+2\alpha)}$, we get

$$\sum_{j=1}^{q} a_j d m_f(r, D_j) \leq \left(\frac{4[(l+1)/2]}{q(1+2\alpha)}\right)\left(\sum_{j=1}^{q} a_j d T_f(r, D_j)\right)\ \|_E\ .$$

By canceling d on both sides, we have

$$\sum_{j=1}^{q} a_j m_f(r, D_j) \leq \left(\frac{4[(l+1)/2]}{q(1+2\alpha)}\right)\left(\sum_{j=1}^{q} a_j T_f(r, D_j)\right)\ \|_E\ . \tag{4.14}$$

We now use this result to derive our desired result. Note that (4.7) gives us

$$\sum_{j=1}^{q} r_j m_f(r, D_j) \leq \sum_{j=1}^{q} a_j m_f(r, D_j) + \frac{\delta_1}{2}\left(\min_{1\leq j\leq q} r_j\right)\sum_{j=1}^{q} m_f(r, D_j) \tag{4.15}$$

and

$$\sum_{j=1}^{q} a_j T_{f,D_j}(r) \leq \sum_{j=1}^{q} r_j T_{f,D_j}(r) + \frac{\delta_1}{2}\left(\min_{1\leq j\leq q} r_j\right)\frac{q(1+2\alpha)}{4[(l+1)/2]}\sum_{j=1}^{q} T_{f,D_j}(r). \tag{4.16}$$

Using (4.14), (4.15), (4.16), together with the First Main Theorem, it gives

$$\begin{aligned}
&\sum_{j=1}^{q} r_j m_f(r, D_j) \\
\leq\ & \frac{4[(l+1)/2]}{q(1+2\alpha)}\left(\sum_{j=1}^{q} a_j T_{f,D_j}(r)\right) + \frac{(\min r_j)\delta_1}{2}\left(\sum_{j=1}^{q} T_{f,D_j}(r)\right) \|_E \\
\leq\ & \frac{4[(l+1)/2]}{q(1+2\alpha)}\left(\sum_{j=1}^{q} r_j T_{f,D_j}(r) + \frac{(\min r_j)\delta_1}{2}\frac{q(1+2\alpha)}{4[(l+1)/2]}\sum_{j=1}^{q} T_{f,D_j}(r)\right) \\
& + \frac{\delta_1}{2}\sum_{j=1}^{q} r_j T_f(r, D_j) \|_E \\
\leq\ & \left(\frac{4[(l+1)/2]}{q(1+2\alpha)} + \delta_1\right)\sum_{j=1}^{q} r_j T_{f,D_j}(r) \quad \|_E \\
\leq\ & \left(\frac{4[(l+1)/2]}{q(1+\alpha)}\right)\sum_{j=1}^{q} r_j T_{f,D_j}(r) \quad \|_E \ ,
\end{aligned}$$

with

$$\delta_1 = \min\left\{1, \frac{4[(l+1)/2]}{q}\frac{\alpha}{(1+\alpha)(1+2\alpha)}\right\}\ . \tag{4.17}$$

This finishes the proof of Main Theorem B.

4.6 Appendix to chapter 4

4.6.1 Motivation of choice of β in the Lemma 4.4.3

The motivation of choosing $\beta := \frac{(D.D)}{2(D.F)}$ in the proof of Lemma 4.4.3 is as follows.

In order to derive a lower bound for

$$\sum_{m=1}^{M} m\left(h^0(X, ND - mF) - h^0(X, ND - (m+1)F)\right),$$

the first step is to estimate M.

Lemma 4.6.1 ([Lev09], Lemma 11.4)**.** *Let X be a smooth projective surface. Let D be a nef divisor on X. Let F be an effective divisor on X such that either F is linearly equivalent to an irreducible curve or $C.F \leq 0$ for every irreducible component C of F. Then for all $m, N \geq 0$, either $h^0(X, ND - mF) = 0$ or $h^0(X, ND - mF) - h^0(X, ND - (m+1)F) \leq (ND - mE).F$.*

Since $h^0(X, ND) = N^2D^2/2 + O(N)$, we have

$$\sum_{m=1}^{M}\left(h^0(X, ND - mF) - h^0(X, ND - (m+1)F)\right) = h^0(X, ND).$$

On the other hand, by the above lemma and also by the lemma 9.12 in [Lev09], if F is linear equivalent to an irreducible curve, then

$$h^0(X, ND - mF) - h^0(X, ND - (m+1)F) \leq ((ND - mF).F).$$

Therefore, by assumption that D is big and nef, we can solve for M such that the above equality holds.

$$\begin{aligned}
&\frac{N^2D^2}{2} + O(N) = h^0(X, ND) \\
&= \sum_{m=0}^{M} h^0(X, ND - mF) - h^0(X, ND - (m+1)F) \\
&\leq \sum_{m=0}^{M} N(D.F) - m(F.F) \leq \sum_{m=0}^{M} ND.F = (M+1)N(D.F).
\end{aligned}$$

Therefore, we have the lower bound of M, $M \geq \frac{(D.D)N}{2(D.F)} + O(1)$. In our case (Lemma 4.4.3), $M = [\beta N/2]$. Thus $[\beta N] \geq \frac{(D.D)N}{2(D.F)} + O(1)$, and therefore, we choose $\beta := \frac{(D.D)}{2(D.F)}$.

4.6.2 Alternative method to derive similar estimate as Lemma 4.4.3

Let $A = (F.F), B = (D.F), C = (D.D)$. By using the Lemma 4.6.1 again,

$$\begin{aligned}
&h^0(X, ND) - h^0(X, ND - F) \leq NB \\
&h^0(X, ND - F) - h^0(X, ND - 2F) \leq NB - A \\
&\qquad\vdots \\
&h^0(X, ND - (k-1)F) - h^0(X, ND - kF) \leq NB - (k-1)A.
\end{aligned}$$

By summing of above equations, we have

$$h^0(X, ND) - h^0(X, ND - kF) \leq kNB - \sum_{i=0}^{k-1} iA.$$

Therefore,

$$\begin{aligned} h^0(X, ND - kF) &\geq h^0(X, ND) - kNB + \frac{(k-1)k}{2}A \\ &= \frac{N^2D^2}{2} - kNB + \frac{k^2A}{2} - \frac{kA}{2} + O(N). \end{aligned} \tag{4.18}$$

Comparing with Lemma 4.4.2, the term $\min\{k^2, N^2\}$ is replaced by $\frac{k^2A}{2} - \frac{kA}{2}$. After summing them up, we can get rid of the second term $(\frac{kA}{2})$ because it belongs to $O(N^2)$. Thus, the result is the same as choosing k^2 in Autissier's lemma. The precise process is as follows.

Since $M \geq \frac{CN}{2B}$, we have

$$\begin{aligned} &\sum_{k=1}^{M} k\left(h^0(X, ND - kF) - h^0(X, ND - (k+1)F\right) \\ &= \sum_{k=0}^{M} h^0(X, ND - kF) \geq \sum_{k=0}^{\frac{CN}{2B}} h^0(X, ND - kF) \\ &\geq \sum_{k=0}^{\frac{CN}{2B}} \frac{N^2D^2}{2} - \sum_{k=0}^{\frac{CN}{2B}} kNB + \sum_{k=0}^{\frac{CN}{2B}} \frac{k^2A}{2} - \sum_{k=0}^{\frac{CN}{2B}} \frac{kA}{2} \\ &= \frac{C^2N^3}{4B} - \left(\frac{1}{2}\right)\left(\frac{CN}{2B}\right)\left(\frac{CN}{2B} + 1\right)NB + \\ &\left(\frac{A}{12}\right)\left(\frac{CN}{2B}\right)\left(\frac{CN}{2B} + 1\right)\left(\frac{CN}{B} + 1\right) + O(N^2) \\ &= \left(\frac{C}{4B} + \frac{1}{24}\left(\frac{A}{B}\right)\left(\frac{C}{B}\right)^2\right)\frac{CN^3}{2} + O(N^2) \\ &= \left(\frac{(D.D)}{4(D.F)} + \frac{1}{24}\frac{(F.F)(D.D)^2}{(D.F)^3}\right)\frac{(D.D)}{2}N^3 + O(N^2). \end{aligned}$$

Chapter 5

Integral points on the complements of ramification divisors and resultants

5.1 Introduction

The main result of this chapter is an estimate of the dimension of the Zariski closure of a set of $\mathcal{S}$-integral points on $\mathbb{P}^n \setminus \mathcal{D}$ where $\mathcal{D}$ is the branch locus of a generic projection from the intersection of two generic hypersurfaces in $\mathbb{P}^{n+2}$ to $\mathbb{P}^n$. As a consequence, a finiteness theorem for integral points on $\mathbb{P}^n \setminus \mathcal{D}$ is obtained. The theorem generalizes the theorem and techniques of Zannier (see [Zan05]) in the codimension one case of the projection from a single hypersurface.

There are two frequently used ways to derive a statement of hyperbolicity when removing a divisor $\mathcal{D}$ from $\mathbb{P}^n$. One approach is to assume that $\mathcal{D}$ has a sufficiently large number of components, which is the case treated so far in this thesis. The other approach is to assume that $\mathcal{D}$ is an irreducible divisor, but is of sufficiently large degree. In the latter case, based on earlier work of Faltings, Zannier used an innovative approach to study the situation where $\mathcal{D}$ arises as the ramification divisor of the projection from a hypersurface $\mathcal{X}$ in $\mathbb{P}^{n+1}$ to $\mathbb{P}^n$. The result in Zannier's paper uses the total ramification points to control the integral points away from the branch locus $\mathcal{D}$ in $\mathbb{P}^n$ defined by a discriminant form $\Delta = 0$. His result can essentially be formulated as follows.

Theorem 5.1.1 ([Zan05], Theorem 2.1)**.** *Let $\mathcal{D} \subset \mathbb{P}^n$ be the branch locus of a projection from a generic hypersurface $\mathcal{X}$ in $\mathbb{P}^{n+1}$ to $\mathbb{P}^n$. Assume that the degree of $\mathcal{X}$ is at least $n+2$. Then any set of $\mathcal{S}$-integral points on $\mathbb{P}^n \setminus \mathcal{D}$ is finite.*

5.2 Definitions, notations and background

Let $\mathcal{X}$ and $\mathcal{Y}$ be two hypersurfaces in $\mathbb{P}^{n+2}$ defined respectively by two generic homogeneous polynomials $f(X_0, \dots, X_n, Y, Z)$ and $g(X_0, \dots, X_n, Y, Z) \in k[X_0, X_1, \dots, Y, Z]$, where k is a number field. Let Π be a projection from $\mathbb{P}^{n+2}$ to $\mathbb{P}^n$ and L be the light source passing through the points $[0, 0, \dots, 0, 1]$ and $[0, 0, \dots, 1, 0]$ with $L \cap \mathcal{X} \cap \mathcal{Y} = \emptyset$. By choosing a screen $H_i \equiv \mathbb{P}^{n+1}$ containing $\mathbb{P}^n$ and a point $q_i \in L$ which does not lie on the screen, we can decompose the projection Π into two projections Π_i and Π'_i such that $\Pi = \Pi'_i \circ \Pi_i$, where Π_i is the projection from $\mathbb{P}^{n+2} \setminus q_i$ to H_i with respect to the light source q_i and Π'_i is the projection from $H_i \setminus r_i$ to $\mathbb{P}^n$ with respect to the light source r_i, where r_i is the intersection of H_i and L. We define $\mathcal{D} \subset \mathbb{P}^n$ to be the branch locus of the original projection of $\mathcal{Z} = \mathcal{X} \cap \mathcal{Y}$ under Π and $\mathbf{T}_i \subset \mathbb{P}^n$ to be the total ramification locus of the projection of $\Pi_i(\mathcal{Z})$ under Π'_i. We will make generic such choices for $i = 1, \ldots, n+1$.

The notion of the resultant is the key ingredient in dealing with the relation between projections and the intersection of two hypersurfaces. The resultant is defined as the determinant of the Sylvester matrix of two polynomials in one variable and the resultant is zero if and only if the two polynomials have a common root in an algebraically closed field containing the coefficients. Therefore, if we normalize $q_i = [0, \ldots, 0, 1]$, we have $\Pi_i(\mathcal{Z}) = \{Res_Z(f,g) = 0\}$, where $Res_Z(f,g)$ means that f and g are considered as one variable polynomials in Z, making $Res_Z(f,g)$ a polynomial in $X_0, \ldots, X_n, Y$. We can represent $Res_Z(f,g)$ as follows.

$$\begin{aligned} Res_Z(f,g) &= F_0(X_0, \ldots, X_n) Y^{d_1 d_2} + F_1(X_0, \ldots, X_n) Y^{d_1 d_2 - 1} \\ &+ \ldots + F_{d_1 d_2}(X_0, \ldots, X_n), \end{aligned} \tag{5.1}$$

where F_b are homogeneous polynomials of degree b.

Similar to the codimension one case of Zannier, $p \in \mathcal{D}$ if and only if there are $\leq d_1 d_2 - 1$ distinct points of $\mathcal{Z}$ on the fiber which is span$\{p, L\}$. Note that $\mathbf{T}_i$ is defined in $\mathbb{P}^n$ by $F_1 = F_2 = \ldots = F_{d_1 d_2} = 0$.

Property 5.2.1. Let Ω be an effective divisor on $\mathbb{P}^n$ defined by a form $\mathbf{I} \in k[X_0, \ldots, X_n]$. Let Σ be a set of $\mathcal{S}$-integral points for the affine variety $\mathbb{P}^n \setminus \Omega$.

Then there exists a finite set of places $\mathcal{S}' \supset \mathcal{S}$ of k such that each point of Σ has projective coordinates $[x_0 : \ldots : x_n]$ with $x_i \in \mathcal{O}^*_{\mathcal{S}'}$ and $\mathbf{I}(x_0, \ldots, x_n) \in \mathcal{O}^*_{\mathcal{S}'}$.

Proof. Let δ be the degree of $\mathbf{I}$. We may assume that the coefficients of $\mathbf{I}$ are in $\mathcal{O}_{\mathcal{S}}$. The rational functions $Q_i := X_i^\delta / \mathbf{I}(X_0, \ldots, X_n)$ for $i = 0, 1, \ldots, n$ are regular on $\mathbb{P}^n \setminus \Omega$. Therefore, (by definition of $\mathcal{S}$) there exists a non-zero $c \in \mathcal{O}_{\mathcal{S}}$ such that the values $cQ_i(p)$ are in $\mathcal{O}_{\mathcal{S}}$ for all $p \in \Sigma$. By finiteness of class-number we may enlarge $\mathcal{S}$ to a finite set $\mathcal{S}'$ such that $\mathcal{O}_{\mathcal{S}'}$ is a unique factorization domain. Then we may write $p = (x_0, \ldots, x_n)$ where the projective coordinates $x_i = x_i(p)$ are co-prime $\mathcal{S}'$-integers. Since $\mathbf{I}(x_0, \ldots, x_n)$ divides cx_i^δ in $\mathcal{O}_{\mathcal{S}'}$ for all i, we can conclude $\mathbf{I}(x_0, \ldots, x_n)$ divides c in $\mathcal{O}_{\mathcal{S}'}$. Enlarging further $\mathcal{S}'$ makes $c \in \mathcal{O}^*_{\mathcal{S}'}$, from where we get the conclusion. □

5.3 Main Theorem D

Main Theorem D. *Let $\mathcal{X}$ and $\mathcal{Y}$ be two generic hypersurfaces in $\mathbb{P}^{n+2}$. In the above-described geometric setting, the Zariski closure of any set of $\mathcal{S}$-integral points in $\mathbb{P}^n \setminus \mathcal{D}$ has dimension at most* $\dim \mathbf{T}_i + 1$ *(which is independent of i).*

Remark 5.3.1. The significance of the word *generic* in the above theorem is that Lemma 5.3.5 holds. Moreover, it assures that any choice of $\min\{n+1, d_1 d_2\}$ of the coefficient polynomials $F_1, F_2, \ldots, F_{d_1 d_2}$ form a regular sequence, which results in the following Corollary.

Corollary 5.3.2. *The Zariski closure of any set of $\mathcal{S}$-integral points in $\mathbb{P}^n \setminus \mathcal{D}$ has dimension* $\leq \max\{0, n+2-d_1d_2\}$.

Remark 5.3.3. In order to get the qualitative statement of hyperbolicity, we need the bound in Corollary 5.3.2 to be 0, as this implies that any set of $\mathcal{S}$-integral points is finite. An important observation here is that the dimension drops quadratically, which means hyperbolicity is obtained rather quickly.

Next, note that

$$\Pi_i'(\text{branch locus of } \Pi_i' \text{ on } \Pi_i(\mathcal{Z})) = \Pi_i' \circ \Pi_i((\text{branch locus of } \Pi \text{ on } \mathcal{Z}) \cup V_i)$$
$$= \{\Delta = 0\} \cup \{h_i = 0\} = \{\Delta \cdot h_i = 0\},$$

where V_i is a hypersurface on $\mathcal{Z}$ which we can think of as a "fake" branch locus and h_i is a homogeneous polynomial defining $\Pi_i' \circ \Pi_i(V_i)$ on $\mathbb{P}^n$. Therefore, we have the following lemma.

Lemma 5.3.4. *The discriminant Δ of the projection Π is a common factor of the defining functions of $\Pi_i' \circ \Pi_i(\{$branch locus of $\Pi\} \cup V_i)$ for all $i = 1, \ldots, n+1$.*

The following is the key lemma to prove Main Theorem D.

Lemma 5.3.5. *In the above setup, there exist Π_i', Π_i, $i = 1, \ldots, n+1$, such that for all $P \in \mathbb{P}^n \setminus \mathcal{D}$ there exists $j \in \{1, 2, \ldots, n+1\}$ with $h_j(P) \neq 0$.*

Proof. Since the hypersurfaces $\mathcal{X}$ and $\mathcal{Y}$ are assumed to be generic, the proof of this lemma comes down to a dimension count. Consider a given point $P \in \mathbb{P}^n \setminus \mathcal{D}$. If

$h_j(P) = 0$ for all $j = 1, \ldots, n+1$, then for all $j = 1, \ldots, n+1$ there exists a secant line L_j of $\mathcal{Z}$ going through q_j which is contained in the projective plane $\overline{PL}$. The degrees of freedom in choosing such an $(n+1)$-tuple of lines is $2n+1$. On the other hand, by the definition of the secant variety, we have $\mathrm{Sec}(\mathcal{Z}) \subset Gr_{\mathbb{P}}(1, n+2) = Gr_{\mathbb{C}}(2, n+3)$ and $\dim \mathrm{Sec}(\mathcal{Z}) = 2n$. Considering $\mathrm{Sec}(\mathcal{Z})^{n+1} \subset Gr_{\mathbb{P}}(1, n+2)^{n+1}$ and letting $\mathcal{L}$ be the set of $(n+1)$-tuples of lines as above, we compare dimensions as follows:

$$\begin{aligned}
& \dim Gr_{\mathbb{P}}(1, n+2)^{n+1} \\
= \;& 2(n+1)^2 = 2n^2 + 4n + 2 \\
> \;& 2n^2 + 4n + 1 = (n+1) \cdot 2n + 2n + 1 \\
= \;& \dim \mathrm{Sec}(\mathcal{Z})^{n+1} + \dim \mathcal{L}.
\end{aligned}$$

Thus, for appropriate generic choices, there will be an index $j \in \{1, 2, \ldots, n+1\}$ with $h_j(P) \neq 0$. $\square$

Remark 5.3.6. By Lemma 5.3.5, we can decompose the set Σ of $\mathcal{S}$-integral points as $\Sigma = \Sigma_1 \cup \Sigma_2 \ldots \cup \Sigma_{n+1}$, where $\Sigma_i = \{P \in \Sigma | h_1(P) = 0, h_2(P) = 0, \ldots, h_{i-1}(P) = 0, h_i(P) \neq 0\}$.

Proof of Main Theorem D. Let Σ be a set of $\mathcal{S}$-integral points in $\mathbb{P}^n \setminus \mathcal{D}$. Let $P \in \Sigma$. We write $P = [x_0, \ldots, x_n]$ with $x_i \in \mathcal{O}_{\mathcal{S}}$. Since the decomposition $\Sigma = \Sigma_1 \cup \Sigma_2 \ldots \cup \Sigma_{n+1}$ is a finite union, we may suppose w.l.o.g. that $P \in \Sigma_1$. By the above lemma and remark, we may change the coordinates such that the projection Π_1 has screen $H_1 = \{X_{n+2} = 0\}$ and light source $[0, 0, \ldots, 1]$ and Π_1' has light source $[0, 0, \ldots, 1, 0]$. First, we assume that F is monic with respect to Y (see following remark) and normalize equation (5.1) by writing $Y-(1/(d_1d_2))F_1$ in place of Y, implying $F_1=0$, which is equivalent to replacing Y by $Y-(1/(d_1d_2))F_1$ in f and g at the beginning. By repeating the resultant,

$$\begin{aligned}h_1(P)\Delta(P) &= Res_Y\left(Res_Z(f,g), \frac{\partial}{\partial Y}Res_Z(f,g)\right)(P)\\ &= \prod_{1\le i<j\le d_1d_2}(\alpha_i-\alpha_j)^2, \end{aligned}\tag{5.2}$$

where $\alpha_i=\alpha_i(X_0,\dots,X_n)$ is a root of the equation (5.1) of Y.

By the propositions (5.2.1), (5.3.5) and carefully enlarging the set $\mathcal{S}$ with respect to h_i (see the remark) and Δ, we may assume that $h_i(P),\Delta(P)\in\mathcal{O}_{\mathcal{S}}^*$. This implies each difference in (5.2) $\alpha_{ij}\equiv\alpha_i-\alpha_j\in\mathcal{O}_{\mathcal{S}}^*$. By alternating the indices, we obtain the identities:

$$\alpha_{ij}+\alpha_{jl}+\alpha_{li}=0. \tag{5.3}$$

Now, by the well-known result of Siegel and Mahler the equation $x+y+z=0$ has only finitely many non-proportional solutions $(x,y,z)\in(\mathcal{O}_{\mathcal{S}}^*)^3$. Applying this to (5.3) with $\{i,j,l\}=\{1,2,l\}$ and then with $\{i,j,l\}=\{1,j,l\}$, we have $\alpha_{12}+\alpha_{2l}+\alpha_{l1}=0$ and $\alpha_{1j}+\alpha_{jl}+\alpha_{l1}=0$. By the first equation, the values of α_{l1}/α_{12} lie in a finite set independent of P. By the second equation, the same holds for α_{jl}/α_{l1}. Therefore, the values of $\alpha_{jl}/\alpha_{12}=(\alpha_{jl}/\alpha_{l1})\cdot(\alpha_{l1}/\alpha_{12})$ lie in a finite set independent of P. Putting $\gamma=\alpha_{12}$, we summarize these observations as follows:

$$\alpha_j-\alpha_l=c_{jl}\gamma, \tag{5.4}$$

where the c_{jl} lie in a finite set independent of P, and γ, which may depend on P, lies in $\mathcal{O}_{\mathcal{S}}^*$. We split Σ_1 into finitely many subsets so that the c_{jl} is fixed for P in a fixed subset of Σ_1. By arguing separately with each subset, we may assume that the c_{j1} are independent of P. From (5.4) and taking $l = 1$,

$$\alpha_j = \alpha_1 + c_{j1}\gamma. \tag{5.5}$$

By the assumption $F_1 = 0$, it implies

$$(d_1d_2)\alpha_1 + \sum_j c_{j1}\gamma = \sum_i \alpha_i = 0.$$

We have $\alpha_1 = c\gamma$, where c is a fixed number only depended on the subset we are working with, not the point P. By (5.5), $\alpha_j = c_j\gamma$. Recalling that $F_b(x_0, \ldots, x_n)$ is the b-th symmetric function of the α_j, Vieta's formulas yield

$$F_b(x_0, \ldots, x_n) = l_b\gamma^b, b = 2, 3, \ldots, d_1d_2, \tag{5.6}$$

where the l_b, $b = 2, 3, \ldots, d_1d_2$, do not depend on P.

Now, consider the variety W defined in $\mathbb{P}^{n+1}$ by the equations $F_b(X_0, \ldots, X_n) = l_bY^b, b = 2, 3, \ldots, d_1d_2$. Note that $\dim W \leq \dim \mathbf{T}_1 + 1$. By (5.6), Σ_1 lies in the projection of W, whose dimension is $\leq \dim W \leq \dim \mathbf{T}_1 + 1$. $\square$

Remark 5.3.7. We follow Zannier's paper to briefly describe the case that F and G have as their resultant a non-monic homogeneous polynomial.

$$\widetilde{F}(X_0, \ldots, X_n, Y) := F_0(X_0, \ldots, X_n)Y^d + F_1(X_0, \ldots, X_n)Y^{d-1} + \cdots + F_d(X_0, \ldots, X_n).$$

Again, we assume $P \in \mathbb{P}^n \setminus \mathcal{D}$ such that $h_1(P) \neq 0$ where $\mathcal{D}$ is now defined by the discriminant

$$\Delta \equiv F_0^{2d-2} \prod_{1 \leq i < j \leq d} (\alpha_i - \alpha_j)^2,$$

where the α_i are the roots. As in the above proof, we may assume $\alpha_i \in k$ for the integral points in question, writing $\alpha_i = \mu_i/\delta_i$ with μ_i, δ_i co-prime elements in $\mathcal{O}_{\mathcal{S}}^*$. Then $\widetilde{F}(X_0, \ldots, X_n, Y)$ is divisible in $\mathcal{O}_{\mathcal{S}}[Y]$ by $\prod_i (\delta_i Y - \mu_i)$, where $\delta_1, \ldots, \delta_d$ divides $F_0(x_0, \ldots, x_n)$ in $\mathcal{O}_{\mathcal{S}}$. Therefore, $\Delta(P)$ is divisible in $\mathcal{O}_{\mathcal{S}}$ by $\prod_{i \neq j} (\delta_j \mu_i - \delta_i \mu_j)$ according to definition where all the factors are in $\mathcal{O}_{\mathcal{S}}^*$ if $P \in \mathbb{P}^n \setminus \mathcal{D}$. As in the proof of (5.3), the philosophy is to find the identity in order to apply Siegel and Mahler theorem which is the cornerstone of the entire argument. Let $x_{ij} \equiv \delta_j \mu_i - \delta_i \mu_j$ and consider the identity $x_{1i}x_{2j} - x_{1j}x_{2i} - x_{12}x_{ij} = 0$. Then $x_{1i}x_{2j}/x_{1j}x_{2i}$ has only finitely many values independently of P. Define

$$\Box_i \equiv \frac{\alpha_1 - \alpha_i}{\alpha_2 - \alpha_i} \Rightarrow \frac{x_{1i}x_{2j}}{x_{1j}x_{2i}} = \frac{(\alpha_1 - \alpha_i)(\alpha_2 - \alpha_j)}{(\alpha_2 - \alpha_i)(\alpha_1 - \alpha_j)} = \frac{\Box_i}{\Box_j}.$$

Since there are only finitely many possibilities, the above relation allows us to write down α_i in terms of α_1 and α_2 as in the previous proof. Then the integral points lie on a suitable subvariety.

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