UNIQUENESS RESULTS OF ALGEBRAIC CURVES AND RELATED TOPICS

A Dissertation Presented to

the Faculty of the Department of Mathematics

University of Houston

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

By

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May 2017

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Acknowledgements

I am using this opportunity to express my gratitude to everyone who supported me throughout the Ph.D. program in the US.

First, I would like to thank my advisor, Dr. Min Ru. I'm extremely grateful for his tireless guidance, valuable suggestions and patience throughout my years as a graduate student. Without his guidance and persistent help, this dissertation would not have been possible.

Besides my advisor, I would like to thank the rest of my committee members, Dr. Gordon Heier, Dr. Shanyu Ji, and Dr. Qianmei Feng for taking the time to be in my committee, read my dissertation and give valuable feedback.

I would also like to thank all of my friends in the department for their help and friendship especially, Ananya, Angelynn, Saud, and Jungim.

I owe a great deal to my friend Murad. He was extremely helpful.

I would like to thank Paulette, Sincere, and Pauline. I had a great time with them here in the US while taking a break. They were absolutely the best.

I would like to thank Melis, Basri, Aytac, Cem, Ugur, Gokhan, and Bedri. I'm so grateful to them for being amazing friends from the beginning to the end.

My sincere thanks also goes to Cansel. She was always by my side.

I would like to thank my mom, dad and my brother for supporting me spiritually. I owe a lot to each and every one of them for always encouraging me to push forward and for their relentless support throughout my studies.

Finally, I would like to thank my husband. He was always there cheering me on and has continued to stand by me through the good and bad times.

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Abstract

It is well-known that if two complex polynomials P and Q share two values without counting multiplicities, then they are the same. Such problem is called the value sharing problem. In this dissertation, we study the value sharing problem for algebraic and holomorphic curves, as well as give its applications. We first improve the previous result of Ru-Xu [XR07] on value sharing for algebraic mappings from a compact Riemann surface into the n-dimensional projective space that agree on the pre-image for given hyperplanes located in general position, by using a new auxiliary function. Second, we study the value sharing problem for holomorphic mappings from punctured compact Riemann surfaces into the n-dimensional projective space. We also work on p-adic holomorphic curves which is similar to the algebraic curves. In the last chapter, we apply our results to the study of minimal surfaces, namely, the uniqueness theorem for Gauss maps of two minimal surfaces in the n-dimensional Euclidean space. The article which contains the majority results of this thesis has been accepted by the International Journal of Mathematics (see [RU17]).

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Chapter 1

Introduction

It is well-known that for any non-constant complex (or any algebraically closed field of characteristic zero) polynomials P and Q, if there are two distinct complex values $a_j \in \mathbb{C}$, $j = 1, 2$, such that $P(z) = a_j$ if and only if $Q(z) = a_j$ (we say that P and Q share the values a_j), then $P = Q$ (see Theorem 2.1 for the proof). For rational functions on $\mathbb{C}$, one can prove (see Theorem 2.6) that if two non-constant rational functions share four distinct values in $\mathbb{C} \cup \{\infty\}$, then they must be the same. Note that rational functions on $\mathbb{C}$ can be regarded as holomorphic (algebraic) maps from $\mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$. A. Sauer [Sau01], A. Scheizer [Sch10], and E. Ballico [Bal05] etc. studied the value sharing problems for holomorphic (algebraic) maps from a general compact Riemann surface S into $\mathbb{P}^1(\mathbb{C})$. Ru and Xu [XR07] later extended their results to holomorphic maps from S into $\mathbb{P}^n(\mathbb{C})$. In the transcendental case, Nevanlinna, in 1929, as an application of his celebrated Second Main Theorem, proved his famous five point-theorem: *If two non-constant meromorphic functions f and g defined on $\mathbb{C}$ share five distinct values in $\mathbb{C}\cup\{\infty\}$, then $f \equiv g$.* Later, H. Fujimoto [Fuj75] extended the result to holomorphic maps from $\mathbb{C}$ into $\mathbb{P}^n(\mathbb{C})$. Values sharing problems were also studied for p-adic meromorphic functions and p-adic holomorphic maps (see [AS71],[Ru01b], and [Ru01a]), as well as the Gauss maps of minimal surfaces (see [Fuj93a], [PR16], and [JR07]). The purpose of this thesis is to improve previous mentioned results, as well as derive new results, by using a new auxiliary function used in [CY09], [HLS12]. The layout of this dissertation is as follows:

In Chapter 2, we review relevant theorems and show well-known uniqueness results, in particular the uniqueness results for polynomials, and for holomorphic maps from compact Riemann surfaces into 1-dimensional projective space, which inspired the development of this research study.

In Chapter 3, we prove our main theorem on the holomorphic mappings from compact Riemann surfaces into the n-dimensional projective space, which generalizes the result of Ru and Xu (see [XR07]).

Theorem 1.1. *Let S be a compact Riemann surface of genus g, and let $H_1, \ldots, H_q$ be hyperplanes in $\mathbb{P}^n(\mathbb{C})$, located in general position. Let $f_1 : S \to \mathbb{P}^n(\mathbb{C})$ and $f_2 : S \to \mathbb{P}^n(\mathbb{C})$ be two linearly non-degenerate holomorphic maps. Assume that*

(i) $f_1^{-1}(H_i) = f_2^{-1}(H_i)$ for $i = 1, \ldots, q$,

(ii) Let $k \leq n$ be a positive integer such that $i \neq j, f_1^{-1}(\cap_{i=1}^{k+1} H_{j_i}) = \emptyset$ $(1 \leq i_1 < \cdots < i_{k+1} \leq q)$,

(iii) $f_1 = f_2$ on $\cup_{j=1}^{q} f_1^{-1}(H_j)$.

Then

(a) If $g = 0$, i.e, $S = \mathbb{P}^1(\mathbb{C})$, and

$q \geq \frac{1}{2}(2k + n + 1 + \sqrt{8kn^2 + 4k^2 + 4kn - 4k + (n+1)^2}$, then $f_1 \equiv f_2$.

(b) If $g = 1$ and $q > \frac{1}{2}(2k + n + 1 + \sqrt{8kn^2 + 4k^2 + 4kn - 4k + (n+1)^2}$, then $f_1 \equiv f_2$.

(c) For a general genus $g > 1$, if $q - (n+1) - \frac{2kn}{q-2k+2kn}q - \frac{(n+1)(g-1)}{q} > 0$ then $f_1 \equiv f_2$.

In Chapter 4, we first work on value sharing for holomorphic maps from punctured (compact) Riemann surfaces into $\mathbb{P}^n(\mathbb{C})$. This includes $\mathbb{C} = \mathbb{P}^1(\mathbb{C})$ - {one point}, which results recover all the known results for holomorphic maps from $\mathbb{C}$ to $\mathbb{P}^n(\mathbb{C})$. We then discuss the value sharing problem for holomorphic maps from non-compact Riemann surfaces Y into $\mathbb{P}^n(\mathbb{C})$ in three cases. The first case is that we assume that the open Riemann surface Y is parabolic. The second case is that Y is the unit-disc with the maps being admissible. Notice that Nevanlinna theory still works for holomorphic maps on the unit-disc, under the assumption that the map is admissible. For the maps on the unit-disc which is not admissible, we still have the result for maps with mild growth condition with respect to a complete metric due to the work of Fujimoto (see [Fuj86]). We consider here a more general case that Y is (instead of the unit-disc) an open Riemann surface with a complete metric.

In Chapter 5, we work on value sharing problem for p-adic holomorphic maps, which improves the result of Ru (see [Ru01c]).

In Chapter 6, as an application of above results, we consider the uniqueness problem of the Gauss map of minimal surfaces in $\mathbb{R}^m$ due to the fact that generalized Gauss map of any minimal surfaces is holomorphic. We first discuss the case of two complete minimal surfaces with finite total Gauss curvature, and then derive the results of Park-Ru (see [PR16]) on the general case by applying our results on uniqueness of holomorphic maps from an open Riemann surface with a complete metric into $\mathbb{P}^n(\mathbb{C})$.

Chapter 2

Uniqueness Results for Mappings into $\mathbb{P}^1(\mathbb{C})$

2.1 Uniqueness Results for Polynomials

Theorem 2.1 ([AS71], p.418)**.** *Let P and Q be two non-constant complex (or any algebraically closed field of characteristic zero) polynomials. Assume that there are two distinct complex values $a_j \in \mathbb{C}$, $j = 1, 2$ such that $P(z) = a_j$ if and only if $Q(z) = a_j$ without counting multiplicities (we say that P and Q share the values a_j). Then $P \equiv Q$.*

Proof. Without loss of generality, we assume that the two values are 0 and 1. Then $P(z) = 0$ if and only if $Q(z) = 0$ and $P(z) = 1$ if and only if $Q(z) = 1$. Suppose that $n = \deg P \geq \deg Q > 0$ and $P \not\equiv Q$. Now P divides $P'(P - Q)$ since every zero of P is a zero of $P - Q$. Also $P - 1$ divides $P'(P - Q)$, and thus, since P and $P - 1$ are relatively prime, $P(P - 1)$ divides $P'(P - Q)$. But $\deg P(P - 1) = 2n$ and $\deg P'(P - Q) \leq 2n - 1$, which is contradiction. So $P - Q \equiv 0$. $\square$

Rational functions. Questions were raised by A.K. Pizer [Piz73] about the rational functions sharing three values on the sphere which was, unfortunately, not true. Here is an example: the functions

$$f(z) := \frac{(z+1)^3(z-1)}{(z+i)^3(z-i)}, \qquad g(z) := \frac{(z+1)(z-1)^3}{(z+i)(z-i)^3},$$

share the values $0, 1, \infty$ on the sphere. The following statement however is true: *if p, q are non-constant rational functions that share four distinct values $a_j, j = 1, \ldots, 4$, in $\mathbb{C} \cup \{\infty\}$, then $p \equiv q$.* The proof is done by using the Riemann-Hurwitz theorem (See Theorem 2.6).

2.2 Riemann Hurwitz Theorem and the Second Main Theorem

We first recall and prove the well-known Riemann-Hurwitz theorem. Let $f : S \to S'$ be a holomorphic map with S and S' being two compact Riemann surfaces. We call $\nu_f(p)$ the *multiplicity of f at $p \in S$* if there are local coordinates z for S at $p \in S$ and w for S' at $f(p)$ respectively such that $w = z^{\nu_f(p)}$.

We now have the following Riemann-Hurwitz Theorem.

Theorem 2.2 ([GH94], p.216)**.** *Let $f : S \to \mathbb{P}^1(\mathbb{C})$ be a non-constant holomorphic* *map, where S is a compact Riemann surface. Then*

$$2g-2=-2\deg(f)+r_f(S),$$

where $r_f(S):=\sum_{p\in S}(\nu_f(p)-1)$ and g is the genus of S.

For the proof of Riemann-Hurwitz theorem, we need the following classical Gauss-Bonnet theorem.

Theorem 2.3 ([GH94], p.216)**.** *Let S be a compact Riemann surface of genus g, then $\deg(K_S)=2g-2$, where K_S is the canonical bundle on S and g is the genus of S.*

We also need

Theorem 2.4. *Let L be a holomorphic line bundle over a compact Riemann surface S and h be pseudo-metric and assume that there exists a non-trivial meromorphic section σ of L. Then*

$-\int_S dd^c\log h=\deg(\sigma=0)-\deg(\sigma=+\infty)+n(h)=\deg(L)+n(h=0)$, or we can write,

$$-\int_S dd^c[\log h]=\deg(L),$$

where $\int_S dd^c[\log h]=\int_S dd^c\log h+n(h=0)$ and $n(h=0)$ is the number of zeros of the metric h, counting multiplicities.

Proof of Theorem 2.2. Take w_{FS} as the Fubini-Study form on $\mathbb{P}^1(\mathbb{C})$. Then f^*w_{FS} is a pseudo-positive $(1,1)$-form on S whose zeros set is the ramification divisor (denoted by $ram(f)$), which induces a pseudo-metric on K_S^* (the holomorphic tangent bundle of S). Write $f^*w_{FS}=h\frac{\sqrt{-1}}{2\pi}dz\wedge d\bar{z}$, then by Theorem 2.3 and Theorem 2.4,

$$\int_S [dd^c \log h] = -\deg(K_S^*) = 2g - 2$$

i.e.,

$$\int_S dd^c \log h + \deg(ram(f)) = 2g - 2.$$

Now

$$w_{FS} = \frac{1}{(1+|w|^2)^2} \frac{\sqrt{-1}}{2\pi} dw \wedge d\bar{w} = dd^c \log(1 + |w|^2),$$

for affine coordinate $(w, 1) \in \mathbb{P}^1(\mathbb{C})$. Thus $\mathrm{Ric}(w_{FS}) = -2w_{FS}$. Hence

$$\begin{aligned} -2\deg f &= -2\int_S f^* w_{FS} = \int_S f^* \mathrm{Ric}(w_{FS}) = \int_S \mathrm{Ric}(f^* w_{FS}) \\ &= \int_S dd^c \log h = 2g - 2 - \deg(ram(f)) \\ &= 2g - 2 - r_f(S). \end{aligned}$$

This finishes the proof of Riemann-Hurwitz theorem.

As a consequence of the Riemann-Hurwitz's theorem, we have the Second Main Theorem for holomorphic maps.

Theorem 2.5 ([JR07], Theorem 2.1)**.** *Let $f : S \to \mathbb{P}^1(\mathbb{C})$ be a non-constant holomorphic map, where S is a compact Riemann surface with genus g, and let $a_1, \ldots, a_q \in \mathbb{P}^1(\mathbb{C})$ be distinct points. Let $E = f^{-1}\{a_1, \ldots, a_q\} \subset S$. Then*

$$(q-2)\deg(f) \leq |E| + 2(g-1),$$

where $|E|$ is the cardinality of E.

Proof. For $E = f^{-1}(\{a_1, \ldots, a_q\})$, define the ramification

$$r(E) := \sum_{p \in E} (\nu(p) - 1).$$

Then, from the definition of the degree, we have $q \deg(f) = |E| + r(E)$. By applying Theorem 2.2, we have $r(E) \leq 2\deg(f) + 2(g-1)$, where g is the genus of S. Hence $(q-2)\deg(f) \leq |E| + 2(g-1)$. This proves the theorem. □

2.3 Uniqueness Results for Holomorphic Mappings from Compact Riemann Surface into $\mathbb{P}^1(\mathbb{C})$

Theorem 2.6 ([Sau01], Proposition 2.2)**.** *Assume that f_1, f_2 are two non-constant complex rational functions that share four distinct values $a_j, j = 1, \ldots, 4$, in $\mathbb{C} \cup \{\infty\}$ without counting multiplicities, then $f_1 \equiv f_2$.*

Proof. By Theorem 2.5, for $i = 1, 2$

$$(q-2)\deg f_i \leq |E| + (2g-2).$$

Now suppose that $f_1 \not\equiv f_2$ and let $E := f_1^{-1}\{a_1, \ldots, a_4\} \equiv f_2^{-1}\{a_1, \ldots, a_4\}$ and $\tilde{E} := (f_1 - f_2)^{-1}(\{0\})$ (note $E \subset \tilde{E}$). Using above with $q = 4$ and genus $g = 0$, we get

$$2(\deg f_1 + \deg f_2) \leq 2(|E| - 2) \leq 2(|\tilde{E}| - 2) \leq 2\deg(f_1 - f_2) - 4,$$

and so $2(\deg f_1 + \deg f_2) \leq 2(\deg f_1 + \deg f_2) - 4$. This gives a contradiction. $\square$

A. Sauer [Sau01] extended the above results to holomorphic maps on compact Riemann surfaces as follows.

Theorem 2.7 ([Sau01], Corollary 2.5). *Let S be a compact Riemann surface of genus g, and let $f_1, f_2 : S \to \mathbb{P}^1(\mathbb{C})$ be two different non-constant holomorphic maps. If f_1, f_2 share q distinct values $a_1, \ldots, a_q \in \mathbb{P}^1(\mathbb{C})$, then we have the following conclusions.*
(i) If $g = 0$, then $q < 4$.
(ii) If $g \geq 1$, then $q \leq 2 + 2\sqrt{g}$.

Proof. Let $E := f_1^{-1}\{a_1, \ldots, a_q\}$. Similar to the above, we can get, by applying Second Main Theorem (Theorem 2.5) for $i = 1, 2$,

$$(q-2)\deg(f_i) \leq |E| + (2g-2).$$

Assume that $\deg(f_2) \leq \deg(f_1) = d$ and by noticing that $E \subset (f_1 - f_2)^{-1}\{0\}$, we get $|E| \leq 2d$. This concludes,

$$q \leq 4 + \frac{(2g-2)}{d}.$$

This proves the case for (i).
Now assume that $g \geq 1$, we further notice that $q \leq |E| \leq 2d$, we get

$$q(q-4) \leq 2(2g-2)$$

which implies that $q \leq 2 + 2\sqrt{g}$. This proves the theorem. $\square$

Chapter 3

Uniqueness Results for Holomorphic Mappings into $\mathbb{P}^n(\mathbb{C})$

3.1 Theory of Algebraic Curves in $\mathbb{P}^n(\mathbb{C})$

To extend the results to holomorphic maps $f : S \to \mathbb{P}^n(\mathbb{C})$, we need the theory of algebraic curves in the projective spaces in chapter 2 of the book of Griffiths-Harris [GH94].

Let S be a compact Riemann surface of genus g. Let $f : S \to \mathbb{P}^n(\mathbb{C})$ be a linearly non-degenerate holomorphic map, i.e. its image is not contained in any proper subspace of $\mathbb{P}^n(\mathbb{C})$. We define the *kth associate curve of f* as $f^k : S \to \mathbb{P}(\wedge^{k+1}\mathbb{C}^{n+1})$ given by $f^k(z) = \mathbb{P}(\mathbf{F}_k)$, where f is given locally by the vector valued function $\mathbf{f}(z) = (f_0(z), ..., f_n(z)) \in \mathbb{C}^{n+1} - \{\mathbf{0}\}$ (called a reduced representation of f), and $\mathbf{F}_k = \mathbf{f} \wedge \mathbf{f}' \wedge \cdots \wedge \mathbf{f}^{(k)}$. Let $\omega_k = dd^c \log \|Z\|^2$ be the Fubini-Study form on

$\mathbb{P}(\wedge^{k+1}\mathbb{C}^{n+1})$ where $d^c = \frac{\sqrt{-1}}{4\pi}(\bar{\partial} - \partial)$. Then

$$f^{k^*}\omega_k = dd^c \log \|\mathbf{F}_k\|^2 = \frac{\sqrt{-1}}{2\pi}\frac{\|\mathbf{F}_{k-1}\|^2\|\mathbf{F}_{k+1}\|^2}{\|\mathbf{F}_k\|^4}dz \wedge d\bar{z}.$$

The *ramification index* $\beta_k(z_0)$ of f^k at z_0 is the unique integer such that $f^{k*}\omega_k = |z - z_0|^{2\beta(z_0)}h(z)\frac{\sqrt{-1}}{2}dz \wedge d\bar{z}$ with $|h(z_0)| > 0$. It is easy to see ([GH94], P. 266) that if we locally write representation of f as

$$\mathbf{f}(z) = (1 + \cdots, z^{1+\nu_1} + \cdots, z^{2+\nu_1+\nu_2} + \cdots ..., z^{n+\nu_1+\cdots+\nu_n} + \cdots),$$

then $\beta_k(z_0) = \nu_{k+1}$. Alternatively, if we write locally

$$\mathbf{f}(z) = (1 + \cdots, z^{\delta_1} + \cdots, z^{\delta_2} + \cdots, \ldots, z^{\delta_n} + \cdots)$$

with $0 \leq \delta_1 \leq \delta_2 \leq \cdots \leq \delta_n$, then

$$\beta_k(z_0) = \delta_{k+1} - \delta_k - 1. \tag{3.1}$$

Let $\beta_k = \sum_{p\in S}\beta_k(p)$ and let $d_i = \deg(f^i)$. Then we have the following Plücker formula.

Theorem 3.1 ([GH94], p. 270)**.** *$d_{k-1} - 2d_k + d_{k+1} = 2g - 2 - \beta_k$ where $d_k = \deg(f^k)$.*

Using the fact that (assuming that $d_{-1} = 0$ and noticing that $d_n = 0$)

$$\sum_{k=0}^{n-1}(n-k)(d_{k-1} - 2d_k + d_{k+1}) = -(n+1)d_0$$

it gives the so-called the Brill-Segre formula (which generalizes the Riemann-Hurwitz theorem) as below.

Theorem 3.2. $n(n+1)(g-1) = -(n+1)\deg(f) + \sum_{k=0}^{n-1}(n-k)\beta_k$.

Definition 3.3. Let $H_1, \ldots, H_q$ be hyperplanes in $\mathbb{P}^n(\mathbb{C})$ defined by the linear forms L_j, $1 \le j \le q$. $H_1, \ldots, H_q$ are said to be *in general position* if for any injective map $\mu : \{0, 1, \ldots, n\} \to \{1, 2, \ldots, q\}$, $L_{\mu(0)}, \ldots, L_{\mu(n)}$ are linearly independent.

Theorem 2.5 can be extended as follows, which is called the Second Main Theorem for holomorphic maps (simple version).

Theorem 3.4 ([JR07], Theorem 2.2)**.** *Let S be a compact Rieamnn surface of genus g. Let $f : S \to \mathbb{P}^n(\mathbb{C})$ be a linearly non-degenerate holomorphic map. Let $H_1, \ldots, H_q$ be the hyperplanes in $\mathbb{P}^n(\mathbb{C})$, located in general position. Let $E = \cup_{j=1}^{q} f^{-1}(H_j)$. Then*

$$(q-(n+1))\deg(f) \le \frac{1}{2}n(n+1)\{2(g(S)-1) + |E|\}.$$

Proof. Let $L_j, 1 \le j \le q$, be the linear forms defining H_j. Take a local representation $\mathbf{f} = (f_0, \ldots, f_n)$ of f, where $f_0, \ldots, f_n$ are holomorphic and have no common zero. Denote by $l_j := L_j(\mathbf{f})$. For each point $P \in E$, denote by $\nu_{l_j}(P)$ the vanishing order of l_j at P. Choose $\{L_{P,i_1}, \ldots, L_{P,i_n}\} \subset \{L_1, \ldots, L_q\}$ such that

$$\nu_{l_{P,i_1}}(P) \ge \nu_{l_{P,i_2}}(P) \ge \cdots \ge \nu_{l_{P,i_{n+1}}}(P) = 0.$$

Since $L_j, 1 \leq j \leq q$, are in general position, $\nu_{l_{P,j}}(P) = 0$ if $j \neq i_1, \ldots, i_n$. We also have, because $L_{P,i_1}, \ldots, L_{P,i_{n+1}}$ are linearly independent, $f = \mathbb{P}[l_{P,i_1} : \cdots : l_{P,i_{n+1}}]$. Thus from (3.1)

$$\beta_k(P) = \nu_{l_{P,i_{n-k}}}(P) - \nu_{l_{P,i_{n-(k-1)}}}(P) - 1,$$

or

$$\sum_{k=0}^{n-1}(n-k)\beta_k(P) = \sum_{t=1}^{n} \nu_{l_{P,i_t}}(P) - \frac{1}{2}n(n+1) = \sum_{j=1}^{q} \nu_{l_j}(P) - \frac{1}{2}n(n+1),$$

where the last identity holds because $\nu_{l_{P,j}}(P) = 0$ for $j \neq i_1, \ldots, i_n$. Applying Theorem 3.2, gives

$$\begin{aligned} \sum_{j=1}^{q}\sum_{P\in E} \nu_{l_j}(P) &\leq \sum_{k=0}^{n-1}(n-k)\beta_k + \frac{1}{2}n(n+1)|E| \\ &= n(n+1)(g-1) + (n+1)\deg(f) + \frac{1}{2}n(n+1)|E|. \end{aligned}$$

Using the fact that, for each j, $\sum_{P\in E} \nu_{l_j}(P) = \deg(f)$, we get

$$(q-(n+1))\deg(f) \leq \frac{1}{2}n(n+1)\{2(g-1) + |E|\}$$

which proves the theorem. □

The following statement concerning the truncations is a more general and much useful.

Theorem 3.5 ([JR07], Theorem 2.4)**.** *Let S be a compact complex Riemann surface of genus g. Let $f : S \to \mathbb{P}^n(\mathbb{C})$ be a linearly non-degenerate holomorphic map. Let $H_1, \ldots, H_q$ be the hyperplanes in $\mathbb{P}^n(\mathbb{C})$, located in general position and let $L_1, \ldots, L_q$*

be the corresponding linear forms. Then, for any finite subset E (can be empty) of S, we have

$$(q-(n+1))\deg(f) \le \sum_{j=1}^{q}\sum_{P\notin E}\min\{n,\nu_P(L_j(f))\} + \frac{1}{2}n(n+1)\{2(g-1)+|E|\},$$

where $\nu_P(L_j(f))$ is the vanishing order of $L_j(f)$ at the point P.

Proof. For each $P\in S$, there exist a distinct subset $\{L_{P,i_1},...,L_{P,i_n}\}\subset\{L_1,...,L_q\}$ such that

$$\nu_P(l_{P,i_1})\ge\nu_P(l_{P,i_2})\ge...\ge\nu_P(l_{P,n})$$

where $l_j=L_j(f)$. Since L_j, $j=1,\dots,n$ are in general position, $\nu_P(l_{P,j})=0$ for all $j\ne i_1,...,i_n$.

Thus, for $P\in E$,

$$\sum_{k=0}^{n-1}(n-k)\beta_k(P)=\sum_{t=1}^{n}\nu_P(l_{P,i_t})-\frac{1}{2}n(n+1)=\sum_{j=1}^{q}\nu_P(l_{P,j})-\frac{1}{2}n(n+1). \tag{3.2}$$

Now, for $P\notin E$, if we let $I_P=\{l_{P,i_j}:\nu_P(l_{P,i_j})>0,1\le j\le n\}$ and use (3.1), then

$$\begin{aligned}\sum_{k=1}^{n-1}(n-k)\beta_k(P) &= \sum_{t=0}^{n}(\nu_P(l_{P,i_t})-t)\\ &\ge \sum_{l_{i_j}\in I_P}\max\{0,\nu_P(l_{P,i_j})-n\}\\ &= \sum_{l_{i_j}\in I_P}\left(\nu_P(l_{P,i_j})-\min\{n,\nu_P(l_{P,i_j})\}\right)\\ &= \sum_{j=1}^{n+1}\left(\nu_P(l_{P,i_j})-\min\{n,\nu_P(l_{P,i_j})\}\right).\end{aligned}$$

Combining the above equation with (3.2) ,

$$\begin{aligned}\sum_{k=0}^{n-1}(n-k)\beta_k(P) &\geq \sum_{k=0}^{n-1}\sum_{P\in E}(n-k)\beta_k(P)+\sum_{k=0}^{n-1}\sum_{P\notin E}(n-k)\beta_k(P)\\ &\geq \sum_{j=1}^{q}\sum_{P\in S}\nu_P(l_{P,j})-\sum_{j=1}^{q}\sum_{P\notin E}\min\{n,\nu_P(l_{P,i_j})\}-\frac{n(n+1)}{2}|E|\\ &= q\deg(f)-\sum_{j=1}^{q}\sum_{P\notin E}\min\{n,\nu_P(l_{P,i_j})\}-\frac{n(n+1)}{2}|E|,\end{aligned}$$

where we used the fact that $\sum_{P\in S}(\nu_P(l_{p,j}))=\deg f$ for all j. By using Brill-Segre formula (Theorem 3.2),

$$n(n+1)(g-1)+(n+1)\deg f\geq q\deg f-\sum_{j=1}^{q}\sum_{P\notin E}\min\{n,\nu_P(l_{P,i_j})\}+\frac{n(n+1)}{2}|E|.$$

So we get

$$(q-(n+1))\deg f\leq\sum_{j=1}^{q}\sum_{P\notin E}\min\{n,\nu_P(l_{P,i_j})\}+\frac{1}{2}n(n+1)\{2(g-1)+|E|\}.$$

This proves the theorem. □

3.2 Preliminary Results on Uniqueness Theorem for Algebraic Curves into the Projective Spaces

The first result in this direction is due to Ru-Xu [XR07].

Theorem 3.6 ([XR07], Main Theorem). *Let S be a compact Riemann surface of genus g, and let $f_1, f_2 : S \to \mathbb{P}^n(\mathbb{C})$ be two linearly non-degenerate holomorphic maps. Let $H_1, \dots, H_q$ be hyperplanes in $\mathbb{P}^n(\mathbb{C})$, located in general position. Assume that*

(i) $f_1^{-1}(H_j) = f_2^{-1}(H_j)$ for $j = 1, \dots, q$,

(ii) $f_1 = f_2$ on $\cup_{j=1}^{q} f_1^{-1}(H_j)$.

Then we have the following conclusions.

(a) If $g = 0$ and $q \geq (n+1)^2$, then $f_1 \equiv f_2$.

(b) For $g \geq 1$, if $q > \frac{1}{2}(n+1)^2 + \sqrt{(n+1)^4 + 4n^2(n+1)(2g-2)}$, then $f_1 \equiv f_2$.

Proof. Lemma 3.1 in [XR07] implies that there exists a hyperplane

$$H_c = \{c_0 x_0 + \cdots + c_n x_n = 0\}$$

such that $f_1^{-1}(H_c \cap H_j) = \emptyset$ and $f_2^{-1}(H_c \cap H_j) = \emptyset$. We fix such H_c. Let $H_j = \{a_{j0} x_0 + \cdots + a_{jn} x_n = 0\}$ and define, for $i = 1, 2$,

$$F_{ij} = \frac{a_{j0} f_{i0} + \cdots + a_{jn} f_{in}}{c_0 f_{i0} + \cdots + c_n f_{in}},$$

where $(f_{i0}, \dots, f_{in})$ is a (local) reduced representation of f_i for $i = 1, 2$. Assume that $f_1 \not\equiv f_2$, then there exists $1 \leq j_0 \leq q$ such that $F_{1j_0} \not\equiv F_{2j_0}$.

Consider the auxiliary function

$$\Phi = F_{1j_0} - F_{2j_0} \not\equiv 0. \tag{3.3}$$

We apply the Second Main Theorem (Theorem 3.4) with $E = \cup_{j=1}^{q} f_1^{-1}(H_j)$, we get

$$(q-(n+1))(\deg(f_1)+\deg(f_2)) \leq n(n+1)(2g-2+|E|).$$

Notice that $E \subset \Phi^{-1}\{0\}$, so $|E| \leq \#\Phi^{-1}\{0\} \leq \deg(f_1)+\deg(f_2)$. Thus we get

$$(q-(n+1)^2)(\deg(f_1)+\deg(f_2)) \leq n(n+1)(2g-2). \tag{3.4}$$

In the case when $g = 0$, we get

$$(q-(n+1)^2))(\deg(f_1)+\deg(f_2)) < 0$$

which proves the case (a). For the case (b) when $g \geq 1$, we further notice that, using the condition that the given hyperplanes are in general position, $q \leq n\#\Phi^{-1}\{0\} \leq n \deg \Phi \leq n(\deg(f_1)+\deg(f_2))$, i.e.

$$q \leq n(\deg(f_1)+\deg(f_2)). \tag{3.5}$$

Thus, by combining (3.4) and (3.5),

$$q(q-(n+1)^2) \leq n^2(n+1)(2g-2)$$

which implies that $q \leq \frac{1}{2}\left((n+1)^2 + \sqrt{(n+1)^4+4n^2(n+1)(2g-2)}\right)$. This derives a contradiction which proves Theorem 3.6. $\square$

Remark: For the case $n = 1$, this theorem recovers the theorem of Schweizer (see [Sch05]).

Under the assumptions in Theorem 3.6 above, if we assume, in addition that, for every $i \neq j$, $f_1^{-1}(H_i) \cap f_1^{-1}(H_j) = \emptyset$, then we have the following theorem.

Theorem 3.7. *Let $H_1, \ldots, H_q$ be hyperplanes in $\mathbb{P}^n(\mathbb{C})$, located in general position. Let S be a compact Riemann surface of genus g. Let $f_1 : S \to \mathbb{P}^n(\mathbb{C})$ and $f_2 : S \to \mathbb{P}^n(\mathbb{C})$ be two linearly non-degenerate holomorphic maps. Assume that*

(i) $f_1^{-1}(H_i) = f_2^{-1}(H_i)$ for $i = 1, \ldots, q$,

(ii) For every $i \neq j$, $f_1^{-1}(H_i) \cap f_1^{-1}(H_j) = \emptyset$,

(iii) $f_1 = f_2$ on $\cup_{j=1}^q f_1^{-1}(H_j)$.

Then we have the following conclusions.

(a) If $g = 0$ and $q \geq 3n+1$, then $f_1 \equiv f_2$.

(b) For $g \geq 1$, if $q > \frac{1}{2}\left((3n+1) + \sqrt{(3n+1)^2 + 4n(n+1)(2g-2)}\right)$, then $f_1 \equiv f_2$.

The proof is similar to the above by using the same auxiliary function given in (3.3), except we now use Truncated Second Main Theorem (Theorem 3.5).

Proof. As above, there exists a hyperplane

$$H_c = \{c_0x_0 + \cdots + c_nx_n = 0\}$$

such that $f_1^{-1}(H_c \cap H_j) = \emptyset$ and $f_2^{-1}(H_c \cap H_j) = \emptyset$. We fix such H_c. Let $H_j = \{a_{j0}x_0 + \cdots + a_{jn}x_n = 0\}$ and define

$$F_{1j} = \frac{a_{j0}f_{10} + \cdots + a_{jn}f_{1n}}{c_0f_{10} + \cdots + c_nf_{1n}},$$

and

$$F_{2j} = \frac{a_{j0}f_{20} + \cdots + a_{jn}f_{2n}}{c_0 f_{20} + \cdots + c_n f_{2n}}.$$

Assume that $f_1 \not\equiv f_2$, then $\Phi := F_{1j_0} - F_{2j_0} \not\equiv 0$ for some $1 \le j_0 \le q$.

Let $E_{f_1,j} = f_1^{-1}(H_j)$ and $E_{f_2,j} = f_2^{-1}(H_j)$.

Then, by the assumption (i), $E_j := E_{f_1,j} = E_{f_2,j}$ and assumption (ii), we have, for every $i \neq j$,

$$E_{f_1,i} \cap E_{f_1,j} = \emptyset.$$

Apply the Truncated Second Main Theorem (Theorem 3.5) with $E = \emptyset$, we get, for $i = 1, 2$

$$\begin{aligned}(q-(n+1))\deg(f_i) &\le \sum_{j=1}^{q}\sum_{P\in S}\min\{n, \nu_P(L_j(f_i))\} + n(n+1)(g-1)\} \\ &\le n\#\Phi^{-1}(\{0\}) + n(n+1)(g-1) \\ &\le n(\deg f_1 + \deg f_2) + n(n+1)(g-1).\end{aligned}$$

Thus by summing it up, we get

$$(q-(3n+1))(\deg(f_1)+\deg(f_2)) \le n(n+1)(2g-2). \tag{3.6}$$

If $g = 0$, then we get

$$q - (3n+1) < 0,$$

which proves the case (a). For the case when $g \ge 1$, noticing that, for every $i \neq j$, $f_1^{-1}(H_i) \cap f_1^{-1}(H_j) = \emptyset$, we have $q \le \deg(f_1) + \deg(f_2)$. Hence (3.6) gives

$$(q-(3n+1))q \le n(n+1)(2g-2)$$

which implies that

$$q \leq \frac{1}{2}\left((3n+1) + \sqrt{(3n+1)^2 + 4n(n+1)(2g-2)}\right).$$

This proves the part (b). $\square$

3.3 Main Results

Instead of using Φ in (3.3), we use a new and more precise auxiliary function introduced in [HLS12] (see also [CY09]) to improve the above theorems and obtain the following more general theorem.

Theorem 3.8. *Let S be a compact Riemann surface of genus g, and let $H_1, \ldots, H_q$ be hyperplanes in $\mathbb{P}^n(\mathbb{C})$, located in general position. Let $f_1 : S \to \mathbb{P}^n(\mathbb{C})$ and $f_2 : S \to \mathbb{P}^n(\mathbb{C})$ be two linearly non-degenerate holomorphic maps. Assume that*

(i) $f_1^{-1}(H_i) = f_2^{-1}(H_i)$ for $i = 1, \ldots, q$,

(ii) Let $k \leq n$ be a positive integer such that $i \neq j, f_1^{-1}(\cap_{i=1}^{k+1} H_{j_i}) = \emptyset$ $(1 \leq i_1 < \cdots < i_{k+1} \leq q)$,

(iii) $f_1 = f_2$ on $\cup_{j=1}^{q} f_1^{-1}(H_j)$.

Then

(a) If $g = 0$, i.e, $S = \mathbb{P}^1(\mathbb{C})$ and $q \geq \frac{1}{2}(2k+n+1+\sqrt{8kn^2+4k^2+4kn-4k+(n+1)^2}$, then $f_1 \equiv f_2$.

(b) If $g = 1$ and $q > \frac{1}{2}(2k+n+1+\sqrt{8kn^2+4k^2+4kn-4k+(n+1)^2}$, then $f_1 \equiv f_2$.

(c) For general genus $g > 1$, if $q-(n+1)-\frac{2kn}{q-2k+2kn}q-\frac{(n+1)(g-1)}{q} > 0$ then $f_1 \equiv f_2$.

Remark:

(1) If $k = n$, then (ii) is automatically true if $H_1, \ldots, H_q$ are in general position. So it recovers Theorem 3.6.

(2) If we take $k = 1$, then we are in the situation of Theorem 3.7, and we actually improved the Theorem 3.7 from $q \geq 3n + 1$ to $2n + 2$.

Proof of Theorem 3.8. As above, assume that $f_1 \not\equiv f_2$, instead of Φ in (3.3), we construct a new auxiliary function. First, by Lemma 3.1 in [XR07], there exists a hyperplane

$$H_c = \{c_0 x_0 + \cdots + c_n x_n = 0\}$$

such that $f_1^{-1}(H_c \cap H_j) = \emptyset$ and $f_2^{-1}(H_c \cap H_j) = \emptyset$. We fix such H_c. Assume that $q \geq 2n$ and let $H_j = \{a_{j0} x_0 + \cdots + a_{jn} x_n = 0\}$, $j = 1, \ldots, q$. Define, for $i = 1, 2$, $F_{i,j} = \frac{a_{j0} f_{i0} + \cdots + a_{jn} f_{in}}{c_0 f_{i0} + \cdots + c_n f_{in}}$, where $(f_{i0}, \ldots, f_{in})$ is a (local) reduced representation of f_i for $i = 1, 2$. In order to re-arrange the hyperplanes $H_1, \ldots, H_q$ into several groups, we define equivalence relation on $\{1, \ldots, q\}$ as $i \sim j$ if and only if

$$\frac{F_{1,i}}{F_{2,i}} - \frac{F_{1,j}}{F_{2,j}} \equiv 0.$$

Group 1:

$$\frac{F_{1,1}}{F_{2,1}} \equiv \cdots \equiv \frac{F_{1,k_1}}{F_{2,k_1}} \not\equiv \frac{F_{1,k_1+1}}{F_{2,k_1+1}}$$

Group 2:

$$\frac{F_{1,k_1+1}}{F_{2,k_1+1}} \equiv \cdots \equiv \frac{F_{1,k_2}}{F_{2,k_2}} \not\equiv \frac{F_{1,k_2+1}}{F_{2,k_2+1}}$$

$\ldots$

Group s:

$$\frac{F_{1,k_{s-1}+1}}{F_{2,k_{s-1}+1}} \equiv \cdots \equiv \frac{F_{1,k_s}}{F_{2,k_s}}$$

where $k_s = q$. The assumption of "in general position" implies that the number of each group cannot exceed n. For each $1 \leq i \leq q$, we set $\sigma(i) = i + n$ if $i + n \leq q$ and $\sigma(i) = i + n - q$ if $i + n > q$. Then obviously σ is bijective and $|\sigma(i) - i| \geq n$ since $q \geq 2n$ so i and $\sigma(i)$ belong to the different groups. Put

$$\chi_i = F_{1,i}F_{2,\sigma(i)} - F_{1,\sigma(i)}F_{2,i}$$

and consider the new auxiliary function

$$\chi := \prod_{i=1}^{q} \chi_i. \tag{3.7}$$

Lemma 3.9 ([Lü12], Theorem 1.5) (See also [CY09] and [HLS12])**.** *Let $H_1, \ldots, H_q$ be hyperplanes in $\mathbb{P}^n(\mathbb{C})$, located in general position with $q \geq 2n$. Let M be an Riemann surface (not necessarily compact). Let $f_1 : M \to \mathbb{P}^n(\mathbb{C})$ and $f_2 : M \to \mathbb{P}^n(\mathbb{C})$ be two different linearly non-degenerate holomorphic maps. Assume that*

(i) $f_1^{-1}(H_i) = f_2^{-1}(H_i)$ for $i = 1, \ldots, q$,

(ii) Let $k \leq n$ be a positive integer such that $i \neq j, f_1^{-1}(\cap_{i=1}^{k+1} H_{j_i}) = \emptyset$ ($1 \leq i_1 < \cdots < i_{k+1} \leq q$),

(iii) $f_1 = f_2$ on $\bigcup_{j=1}^{q} f_1^{-1}(H_j)$.

Then the following holds on the domain of each holomorphic local coordinate z of M:

$$\nu_\chi(z) \geq \left(\frac{q - 2k + 2kn}{2kn}\right) \sum_{j=1}^{q} \left(\nu^n_{L_j(f_1)}(z) + \nu^n_{L_j(f_2)}(z)\right)$$

where $\nu^n_{L_j(f_1)}(z) = \min\{n, \nu_{L_j(f_1)}(z)\}$.

Proof. If $z \notin \bigcup_{j=1}^q f_1^{-1}(H_j)$, then $\nu_{L_j(f_1)}(z) = 0$, so this lemma is obviously true. Thus, we only need to consider the case when $z \in \bigcup_{j=1}^q f_1^{-1}(H_j)$. Define

$$I = \{i : L_j(f_1)(z) = 0, 1 \leq i \leq q\}$$

and denote by $\#(I)$ the number of elements of I. Then $\#(I) \leq k$ by the assumption (ii). If $i \in I$, then z is the zero of $L_j(f_1)$, and hence z is a zero of χ_i with multiplicity at least $\min\{\nu_{L_j(f_1)}(z), \nu_{L_j(f_2)}(z)\}$. Since $I = \{i : L_j(f_1)(z) = 0,\ 1 \leq i \leq q\}$, we have $\sigma^{-1}(I) = \{i : \sigma(i) \in I\}$. If $l \in \{1, 2, \ldots, q\} \setminus (I \cup \sigma^{-1}(I))$, then z is a zero of χ_l with multiplicity at least 1 by the assumption (iii), and so $\nu_{\chi_l} \geq 1$. Therefore, we can write

$$\begin{aligned}
\nu_\chi(z) &\geq \sum_{i \in I} \min\{\nu_{L_j(f_1)}(z), \nu_{L_j(f_2)}(z)\} + \sum_{i \in \sigma(i)} \min\{\nu_{L_j(f_1)}(z), \nu_{L_j(f_2)}(z)\} + \sum_{j \neq i, \sigma(i)} \nu_{\chi_l} \\
&\geq 2 \sum_{i \in I, \sigma(i)} \min\{\nu_{L_j(f_1)}(z), \nu_{L_j(f_2)}(z)\} + \sum_{j \neq i, \sigma(i)} 1 \\
&\geq 2 \sum_{i \in I, \sigma(i)} \min\{\nu_{L_j(f_1)}(z), \nu_{L_j(f_2)}(z)\} + q - \#(I \cup \sigma^{-1}(I)).
\end{aligned}$$

By assumption (iii) and $\#I \leq k$,

$$\nu_\chi(z) \geq 2 \sum_{i \in I, \sigma(i)} \min\{\nu_{L_j(f_1)}(z), \nu_{L_j(f_2)}(z)\} + q - 2k.$$

Using $\min\{a, b\} \geq \min\{a, n\} + \min\{b, n\} - n$ for any $a, b \in \mathbb{Z}^+$,

$$\nu_\chi(z) \geq 2 \sum_{i \in I, \sigma(i)} \{\min\{n, \nu_{L_j(f_1)}(z)\} + \min\{n, \nu_{L_j(f_2)}(z)\} - n\} + q - 2k.$$

By using the fact that $\min\{n, \nu_{L_j(f_1)}(z)\} = \nu^{(n)}_{L_j(f_1)}(z)$ and $k \geq \#(I)$ again,

$$
\begin{aligned}
\nu_\chi(z) \;&\geq\; 2\sum_{i\in I,\sigma(i)} \min\{n, \nu_{L_j(f_1)}(z)\} + \min\{n, \nu_{L_j(f_2)}(z)\} - n\min\{1, \nu_{L_j(f_1)}(z)\} \\
&\quad +q-2k \\
&\geq\; 2\sum_{i\in I,\sigma(i)} \{\nu^{(n)}_{L_j(f_1)}(z) + \nu^{(n)}_{L_j(f_2)}(z) - n\nu^{(1)}_{L_j(f_1)}(z)\} + q - 2k \\
&\geq\; 2\sum_{i\in I,\sigma(i)} \{\nu^{(n)}_{L_j(f_1)}(z) + \nu^{(n)}_{L_j(f_2)}(z) - n\nu^{(1)}_{L_j(f_1)}(z)\} \\
&\quad + \frac{q-2k}{2k}\{\nu^{(1)}_{L_j(f_1)}(z) + \nu^{(1)}_{L_j(f_2)}(z)\}.
\end{aligned}
$$

Since $n\sum \nu^{(1)}_{L_j(f_1)}(z) \geq \sum \nu^{(n)}_{L_j(f_1)}(z)$, we can get

$$
\begin{aligned}
\nu_\chi(z) \;&\geq\; 2\sum_{i\in I,\sigma(i)} \{\nu^{(n)}_{L_j(f_1)}(z) + \nu^{(n)}_{L_j(f_2)}(z) - n\nu^{(1)}_{L_j(f_1)}(z)\} \\
&\quad + \frac{q-2k}{2k}\{\nu^{(1)}_{L_j(f_1)}(z) + \nu^{(1)}_{L_j(f_2)}(z)\} - n\nu^{(1)}_{L_j(f_2)}(z) \\
&\geq\; 2\sum_{i\in I,\sigma(i)} \{\nu^{(n)}_{L_j(f_1)}(z) + \nu^{(n)}_{L_j(f_2)}(z)\} \\
&\quad + \left\{\frac{q-2k}{2k} - n\right\}\frac{1}{n}\sum_{i\in I,\sigma(i)} \{\nu^{(n)}_{L_j(f_1)}(z) + \nu^{(n)}_{L_j(f_2)}(z)\} \\
&\geq\; \left\{\frac{q-2k+2kn}{2kn}\right\}\sum_{i=1}^{q}\{\nu^{(n)}_{L_j(f_1)}(z) + \nu^{(n)}_{L_j(f_2)}(z)\}.
\end{aligned}
$$

This finishes the proof. $\square$

We now continue to prove Theorem 3.8. Let L_j be the linear forms defining the hyperplanes H_j for $j = 1, \ldots, q$. Applying Theorem 3.5 with $E = \emptyset$, we get,

$$(q - (n+1))\deg(f_1) \leq \sum_{j=1}^{q}\sum_{z\in S} \nu^{n}_{L_j(f_1)}(z) + n(n+1)(g-1).$$

This is same for f_2,

$$(q-(n+1))\deg(f_2) \leq \sum_{j=1}^{q}\sum_{z\in S}\nu^n_{L_j(f_2)}(z) + n(n+1)(g-1).$$

Adding the above two inequalities,

$$(q-(n+1))(\deg(f_1)+\deg(f_2)) \leq \sum_{j=1}^{q}\sum_{z\in S}\left(\nu^n_{L_j(f_1)}(z)+\nu^n_{L_j(f_2)}(z)\right) + n(n+1)(2g-2).$$

This, together with the Lemma 3.9, implies

$$\begin{aligned}
&(q-(n+1))(\deg(f_1)+\deg(f_2)) \\
&< \frac{2kn}{q+2kn-2k}\sum_{z\in S}\nu^0_{\chi}(z) + n(n+1)(2g-2) \\
&\leq \frac{2kn}{q-2k+2kn}\#\chi^{-1}\{0\} + n(n+1)(2g-2) \\
&\leq \frac{2knq}{q-2k+2kn}(\deg(f_1)+\deg(f_2)) + n(n+1)(2g-2).
\end{aligned}$$

When $g=0$, it gives

$$\begin{aligned}
&\left\{(q-(n+1))\left(\frac{q-2k+2kn}{2kn}\right) - q\right\}(\deg f_1 + \deg f_2) \\
\leq\ &\left(\frac{q-2k+2kn}{2kn}\right)(-2n(n+1)).
\end{aligned}$$

Thus,

$$\left\{(q-(n+1)) - \frac{2knq}{q-2k+2kn}\right\}(\deg f_1 + \deg f_2) \leq -2n(n+1).$$

Thus, we get

$$(q-(n+1)) - \frac{2knq}{q-2k+2kn} < 0$$

which implies that

$$q < \frac{1}{2}\{2k + n + 1 + \sqrt{8kn^2 + 4k^2 + 4kn - 4k + (n+1)^2}\}.$$

This proves the case (a). The case $g = 1$ is similar. We can get

$$q \leq \frac{1}{2}\{2k + n + 1 + \sqrt{8kn^2 + 4k^2 + 4kn - 4k + (n+1)^2}\}.$$

In the general case of genus g, assume that $\deg(f_2) \leq \deg(f_1) = d$. Then from above,

$$(q - (n+1))d \leq \frac{2kn}{q - 2k + 2kn} qd + n(n+1)(g-1)$$

or

$$(q - (n+1) - \frac{2kn}{q - 2k + 2kn} q \leq \frac{n(n+1)(g-1)}{d}.$$

On the other hand, using $f_1^{-1}(\cap_{i=1}^{k+1} H_{j_i}) = \emptyset$, we get

$$q \leq k(\deg(f_1) + \deg(f_2)). \tag{3.8}$$

So we get

$$q - (n+1) - \frac{2kn}{q - 2k + 2kn} q \ \leq \frac{kn(n+1)(g-1)}{q}.$$

This proves the main theorem.

Chapter 4

Uniqueness Results for Holomorphic Mappings from Punctured Compact Riemann Surfaces into $\mathbb{P}^n(\mathbb{C})$

4.1 Value Sharing for Holomorphic Mappings Around an Essential Singularity

While Nevanlinna obtained his famous five-sharing-points theorem, he also obtained the similar result for meromorphic functions around an essential singularity (see [Nev25]): *Let f_1, f_2 be two meromorphic functions on a punctured disc $\triangle^*(r_0) = \{z \mid 0 < |z| < r_0\}$, where r_0 is a positive number with 0 as their essential singularity. Assume that f_1, f_2 share five distinct complex values (including ∞) without counting* *multiplicities, then* $f_1 \equiv f_2$. Before extending the result to holomorphic maps into $\mathbb{P}^n(\mathbb{C})$, we need to introduce a few notions of the Nevanlinna Theory.

Definition 4.1. Let f be a meromorphic function on $\Delta(R)$, where $0 \leq R \leq \infty$ and let $r < R$. Denote the number of poles of f on the closed disc $\overline{\Delta(R)}$ by $n_f(r, \infty)$, counting multiplicity. We then define the *counting function* $N_f(r, \infty)$ to be

$$N_f(r, \infty) = \int_0^r \frac{n_f(t, \infty) - n_f(0, \infty)}{t} dt + n_f(0, \infty) \log r,$$

here $n_f(0, \infty)$ is the multiplicity if f has a pole $z = 0$. For each complex number a, we then define the *counting function* $N_f(r, a)$ to be

$$N_f(r, a) = N_{\frac{1}{f-a}}(r, \infty).$$

Definition 4.2. The *Nevanlinna's proximity function* $m_f(r, \infty)$ is defined by

$$m_f(r, \infty) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta,$$

where $\log^+ x = max\{0, \log x\}$. For any complex number a, the *proximity function* $m_f(r, a)$ of f with respect to a is then defined by

$$m_f(r, a) = m_{\frac{1}{f-a}}(r, \infty).$$

Definition 4.3. The *Nevanlinna's characteristic function* of f is defined by

$$T_f(r) = m_f(r, \infty) + N_f(r, \infty).$$

Here, $T_f(r)$ measures the growth of f.

Introduction to Nevanlinna's theory can be found in [CY13], [Hay64], and [Nev70] in detail.

The logarithmic derivative lemma holds for meromorphic functions on $\mathbb{C} - \overline{\triangle_{r_0}}$ for $r_0 > 0$ in the following form.

Proposition 4.4 ([Siu15] Proposition 6.2)**.** *Let $r_1 > r_0 > 0$ and F be a meromorphic function on $\mathbb{C} - \overline{\triangle_{r_0}}$. Then*

$$\int_0^{2\pi} \log^+ \left| \frac{F'(re^{i\theta})}{F(re^{i\theta})} \right| \frac{d\theta}{2\pi} = O(\log T_F(r, r_1) + \log r) \ \|$$

for $r > r_1$, where $\|$ means that the inequality holds outside a subset E of $\mathbb{R} \cap \{r > r_1\}$ with finite Lebesgue measure.

Using this Proposition and by the same proof as in H. Cartan's Second Main Theorem (see [Ru01a] and [NO90]), we can prove the following result.

Theorem 4.5 (**Second Main Theorem for Holomorphic Mappings on $\mathbb{C} - \overline{\triangle_{r_0}}$**)**.** *Let $r_1 > r_0 > 0$ and $f : \mathbb{C} - \overline{\triangle_{r_0}} \to \mathbb{P}^n(\mathbb{C})$ be a linearly non-degenerate holomorphic map. Let $H_1, \dots, H_q$ be hyperplanes in $\mathbb{P}^n(\mathbb{C})$, located in general position. Then we have*

$$(q - (n+1))T_f(r, r_1) \leq \sum_{j=1}^{q} N_f^{(n)}(r, r_1, H_j) + O(\log T_f(r, r_1) + \log r) \ \|$$

for $r > r_1$.

Lemma 4.6 ([Siu15], Lemma 5.12). *Let $f\colon \mathbb{C} - \overline{\triangle_{r_0}} \to \mathbb{P}^n(\mathbb{C})$ be holomorphic. If $T_f(r, r_1) = O(\log r)$, then f can be extended to a holomorphic map from $\mathbb{C} \cup \{\infty\} - \overline{\triangle_{r_0}}$ to $\mathbb{P}^n(\mathbb{C})$.*

Theorem 4.7. *Let $f_1, f_2 : \triangle^*(r_0) \to \mathbb{P}^n(\mathbb{C})$ be two linearly non-degenerate holomorphic maps, where $\triangle^*(r_0) = \{z \mid 0 < |z| < r_0\}$ with r_0 being a positive number. Assume that 0 is an essential singularity for both f_1 and f_2. Let $H_1, \ldots, H_q$ be hyperplanes in $\mathbb{P}^n(\mathbb{C})$, located in general position. Assume that*

(i) $f_1^{-1}(H_i) = f_2^{-1}(H_i)$ for $i = 1, \ldots, q$,

(ii) Let $k \le n$ be a positive integer such that $f_1^{-1}(\cap_{i=1}^{k+1} H_{j_i}) = \emptyset$ $(1 \le i_1 < \cdots < i_{k+1} \le q)$,

(iii) $f_1 = f_2$ on $\cup_{j=1}^{q} f_1^{-1}(H_j)$.

If $q > n + 1 + \frac{2knq}{q-2k+2kn}$ (in particular, if $q > (n+1)(k+1)$), then $f_1 \equiv f_2$.

Proof. Assume that $f_1 \not\equiv f_2$ and let χ be defined in (3.7). We let $\zeta := \frac{1}{z}$, then $f_1(\zeta), f_2(\zeta)$ are holomorphic mappings on $\mathbb{C} - \overline{\triangle_{1/r_0}}$. By applying Theorem 4.5, we get, for $i = 1, 2$,

$$(q-(n+1))T_{f_i}(r, r_1) \le \sum_{j=1}^{q} N_{f_i}^{(n)}(r, r_1, H_j) + O(\log T_{f_i}(r, r_1) + \log r) \ \|.$$

Thus, by combining Lemma 3.9 and the First Main Theorem,

$$\begin{aligned} & (q-(n+1))(T_{f_1}(r, r_1) + T_{f_2}(r, r_1)) \\ \le \ & \sum_{j=1}^{q} (N_{f_1}^{(n)}(r, r_1, H_j) + N_{f_2}^{(n)}(r, r_1, H_j)) + O(\max_{1 \le i \le 2} \log T_{f_i}(r, r_1) + \log r) \ \| \end{aligned}$$

$$
\begin{aligned}
&\leq \left(\frac{2kn}{q-2k+2kn}\right) N_{\chi}(r, r_1, 0) + O(\max_{1\leq i\leq 2} \log T_{f_i}(r, r_1) + \log r) \ \| \\
&\leq \left(\frac{2knq}{q-2k+2kn}\right) (T_{f_1}(r, r_1) + T_{f_2}(r, r_1)) + O(\max_{1\leq i\leq 2} \log T_{f_i}(r, r_1) + \log r) \ \|.
\end{aligned}
$$

Hence, by the assumption and Lemma 4.6, we get a contradiction. This proves the theorem. □

4.2 Value Sharing for Holomorphic Mappings from Punctured (compact) Riemann Surfaces into $\mathbb{P}^n(\mathbb{C})$

The most famous result on value-sharing is of course Nevanlinna's five points theorem for meromorphic functions on the complex plane. A true generalization of this result to Riemann surfaces would, for example, be a statement about meromorphic functions on a punctured compact Riemann surface. The goal of this section is thus to apply the theory developed above on the value sharing for holomorphic mappings on compact Riemann surface to holomorphic mappings on the punctured (compact) Riemann surfaces.

Theorem 4.8. *Let S be a compact Riemann surface of genus g, and let $R = S - \{P_1, \ldots, P_l\}$. Let $f_1, f_2 : R \to \mathbb{P}^n(\mathbb{C})$ be two linearly non-degenerate holomorphic maps. Let $H_1, \ldots, H_q$ be hyperplanes in $\mathbb{P}^n(\mathbb{C})$, located in general position. Assume that*

(i) $f_1^{-1}(H_i) = f_2^{-1}(H_i)$ for $i = 1, \ldots, q$,

(ii) Let $k \leq n$ be a positive integer such that $f_1^{-1}(\cap_{i=1}^{k+1} H_{j_i}) = \emptyset$ $(1 \leq i_1 < \cdots < i_{k+1} \leq q)$,

(iii) $f_1 = f_2$ on $\cup_{j=1}^{q} f_1^{-1}(H_j)$.

Then the following conclusions are true:

(a) If $\chi(R) := 2 - 2g - l > 0$, and $q - (n+1) - \frac{2knq}{q-2k+2kn} > 0$, then $f_1 \equiv f_2$.

(b) If $\chi(R) := 2 - 2g - l \leq 0$, and

$$q - (n+1) - \frac{2knq}{q - 2k + 2kn} - \frac{kn(n+1)\{2(g-1)+l\}}{q} > 0,$$

then $f_1 \equiv f_2$.

Proof. Assume that $f_1 \not\equiv f_2$. We consider the following two cases.

Case 1: We consider the case that at least one of the mappings, say f_1, has an essential singularity at one of the points P_i, say P_0. By Big Picard Theorem, in every neighborhood of P_i, f_1 can omit at most $(n+1)$ hyperplanes among H_1, $\ldots$, H_q. Then there are at most $n+1$ hyperplanes f_2 omits, and thus there exists another $n+1$ hyperplanes $f_2(\Delta(\epsilon) - P_0)$ does not omit for $\forall \epsilon > 0$. If P_0 is not the essential singularity of f_2, then $f_2(P_0)$ will be contained in these $n+1$ hyperplanes which contradicts with the fact that these $n+1$ hyperplanes are in general position. Therefore, if f_1 and f_2 on $\Delta^*(P_0) = \Delta - \{P_0\}$ share more than $2n+1$ hyperplanes in general position then P_0 is also an essential singularity of f_2. Note that in both cases (a) and (b), $q - (n+1) - \frac{2knq}{q-2k+2kn} > 0$, i.e.

$$q > \frac{1}{2}(2k + n + 1 + \sqrt{8kn^2 + 4k^2 + 4kn - 4k + (n+1)^2}).$$

Hence $f_1 \equiv f_2$ by Theorem 4.7.

Case 2: Now suppose that none of the points P_i is an essential singularity. Then both f_1 and f_2 extend to holomorphic mappings from S into $\mathbb{P}^n(\mathbb{C})$. Then we are in the familiar situation as before: Assume $f_1 \not\equiv f_2$ and applying Theorem 3.5 with $E = \{P_1, \ldots, P_l\}$ and Lemma 3.9 to get

$$\begin{aligned} & (q-(n+1))(\deg(f_1)+\deg(f_2)) \\ \leq \; & \frac{2kn}{q-2k+2kn} q(\deg(f_1)+\deg(f_2)) + n(n+1)\{2(g-1)+l\}. \end{aligned} \tag{4.1}$$

In the case (a) that $2(g-1)+l < 0$, it gives $q-(n+1)-\frac{2knq}{q-2k+2kn} < 0$, which leads a contradiction. In the case (b) that $2(g-1)+l \geq 0$, using the fact that $q \leq k(\deg(f_1)+\deg(f_2))$ see (3.8), this implies that

$$q-(n+1)-\frac{2knq}{q-2k+2kn} \leq \frac{kn(n+1)\{2(g-1)+l\}}{q}$$

which again gives a contradiction. □

Applying Theorem 4.8 (a) with $R = \mathbb{P}^1(\mathbb{C}) - \{\infty\}$ (i.e. $S = \mathbb{P}^1(\mathbb{C})$ with $g = 0$ and $l = 1$), we recover the following result of [HLS12] which extends the classical result of Nevanlinna (for $n = 1$) (see [Nev26]) and H. Fujimoto (for $n > 1$) (see [Fuj75]).

Corollary 4.9 ([HLS12], Theorem 1)**.** *Let $f_1, f_2 : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ be two linearly non-degenerate holomorphic maps. Let $H_1, \ldots, H_q$ be hyperplanes in $\mathbb{P}^n(\mathbb{C})$, located in general position. Assume that*

(i) $f_1^{-1}(H_i) = f_2^{-1}(H_i)$ for $i = 1, \ldots, q$,

(ii) Let $k \leq n$ be a positive integer such that $f_1^{-1}(\cap_{i=1}^{k+1} H_{j_i}) = \emptyset$ $(1 \leq i_1 < \cdots <$

$i_{k+1} \leq q$),

(iii) $f_1 = f_2$ on $\cup_{j=1}^q f_1^{-1}(H_j)$.

If $q-(n+1)-\frac{2knq}{q-2k+2kn} > 0$ (in particular, if $q > (n+1)(k+1)$), then $f_1 \equiv f_2$.

4.3 Value Sharing for Holomorphic Mappings from non-Compact Riemann Surfaces into $\mathbb{P}^{\mathbf{n}}(\mathbb{C})$

In this section, we study the value sharing problem for holomorphic mappings on open Riemann surfaces. It is known that Nevanlinna theory can be extended to holomorphic mappings on parabolic (open) Riemann surfaces (see [PS14]), or the unit-disc with the maps being admissible, i.e grow fast enough. The theory also was extended by Fujimoto [Fuj86] to holomorphic mappings on the unit-disc (or a general open Riemann surface with a complete metric) which may not be admissible, but still satisfy a (mild) growth condition. We discuss each case.

The parabolic (open) Riemann surface case. A Riemann surface Y is *parabolic* if any bounded subharmonic function defined on Y is constant. This is a large class of surfaces, including e.g. $Y = S\backslash\Lambda$, where S is a compact Riemann surface of arbitrary genus and $\Lambda \subset S$ is any closed polar set. It is well-known (see [AS60]) that a Riemann surface Y is parabolic if and only if it admits a smooth exhaustion function $\sigma : Y \to [1, \infty[$ such that:

- σ is strictly subharmonic in the complement of a compact set,
- $\tau := \log\sigma$ is harmonic in the complement of a compact set of Y. Moreover, we

impose the normalization

$$\int_Y dd^c \tau = 1,$$

where the operator d^c is defined as follows $d^c := \frac{\sqrt{-1}}{4\pi}(\bar{\partial} - \partial)$.

In Nevanlinna theory for parabolic (non-compact) Riemann surfaces, the growth of the Euler characteristic of the balls $\mathbb{B}(r) := \{\sigma < r\}$ will appear. We introduce the following notion.

Definition 4.10 ([PS14], Definition 1.2)**.** Let (Y, σ) be a parabolic Riemann surface, together with an exhaustion function σ as above. For each $t \geq 1$ such that the set $S(t) := \{\sigma = t\}$ is non-singular, we denote $\chi_\sigma(t)$ the Euler characteristic of the domain $\mathbb{B}(t)$, and let

$$\mathfrak{X}_\sigma(r) := \int_1^r |\chi_\sigma(t)| \frac{dt}{t}$$

be the (weighted) mean Euler characteristic of the ball of radius r.

If $Y = \mathbb{C}$, then $\mathfrak{X}_\sigma(r)$ is bounded by $\log r$. The same type of bound is verified if Y is the complement of a finite number of points in $\mathbb{C}$. If $Y = \mathbb{C} \backslash E$, where E is a closed polar set of infinite cardinality, then things are more subtle, depending on the density of the distribution of the points of E in the complex plane. However, an immediate observation is that the surface Y has finite Euler characteristic if and only if $\mathfrak{X}_\sigma(r) = O(\log r)$. To state the logarithmic derivative lemma in the parabolic context, we first recall that the tangent bundle T_Y of a non-compact parabolic surface admits a trivializing global holomorphic section $v \in H^0(Y, T_Y)$ cf. [PS14] (actually, any such Riemann surface admits a submersion into $\mathbb{C}$). We will now suppose that as part of the data we are given a vector field $\xi \in H^0(Y, T_Y)$ which is nowhere vanishing hence it trivializes the tangent bundle of our surface Y. We denote by f' the section $df(\xi)$ of the tangent bundle f^*T_X (for a holomorphic map $f : Y \to X$, where X is any complex manifold). For example, if $Y = \mathbb{C}$, then we can take $\xi = \frac{\partial}{\partial z}$. We recall the following version of the classical logarithmic derivative lemma (see [PS14]).

Lemma 4.11 ([PS14], Theorem 3.7)**.** *Let $f : Y \to \mathbb{P}^1(\mathbb{C})$ be a holomorphic map defined on a parabolic Riemann surface Y. The following inequality holds*

$$m_{f'/f}(r) \leq C(\log T_f(r) + \log r) + \mathfrak{X}_\sigma(r) \ \|.$$

Using this we can prove the following analogy of H. Cartan's Second Main Theorem, for example, see Wong-Stoll [WS94], Theorem 6.3 (note that $ric_\tau(r) = \mathfrak{X}_\sigma(r)$ in our case).

Theorem 4.12 (Second Main Theorem for Parabolic Open Riemann Surfaces). *Let $f : (Y, \sigma) \to \mathbb{P}^n(\mathbb{C})$ be a linearly non-degenerate holomorphic map. Let $H_1, \ldots, H_q$ be hyperplanes in $\mathbb{P}^n(\mathbb{C})$, located in general position. Then we have*

$$(q - (n+1))T_f(r) \leq \sum_{j=1}^{q} N_f^{(n)}(r, H_j) + O(\log T_f(r) + \log r) + \frac{n(n+1)}{2}\mathfrak{X}_\sigma(r) \ \|.$$

Theorem 4.13. *Let $f_1, f_2 : (Y, \sigma) \to \mathbb{P}^n(\mathbb{C})$ be two linearly non-degenerate holomorphic maps. Let $H_1, \ldots, H_q$ be hyperplanes in $\mathbb{P}^n(\mathbb{C})$, located in general position. Assume that*

(i) $f_1^{-1}(H_i) = f_2^{-1}(H_i)$ for $i = 1, \ldots, q$,

(ii) Let $k \leq n$ *be a positive integer such that* $f_1^{-1}(\cap_{i=1}^{k+1} H_{j_i}) = \emptyset$ $(1 \leq i_1 < \cdots < i_{k+1} \leq q)$,

(iii) $f_1 = f_2$ *on* $\cup_{j=1}^{q} f_1^{-1}(H_j)$.

We further assume that

$$\rho := \limsup_{r \to \infty} \frac{\mathfrak{X}_\sigma(r)}{T_{f_1}(r) + T_{f_2}(r)} < +\infty.$$

Assume that

$$q - (n+1) - \frac{2knq}{q - 2k + 2kn} - n(n+1)\rho > 0,$$

then $f_1 \equiv f_2$.

Proof. The argument is similar. Assume that $f_1 \not\equiv f_2$. By using Theorem 4.12 and Lemma 3.9 we get

$$\begin{aligned} & (q - (n+1))(T_{f_1}(r) + T_{f_2}(r)) \\ \leq\ & \frac{2kn}{q - 2k + 2kn} N_\chi(r, 0) + O(\max_{1 \leq i \leq 2} \log T_{f_i}(r) + \log r) + n(n+1)\mathfrak{X}_\sigma(r) \ \| \\ \leq\ & \frac{2knq}{q - 2k + 2kn} (T_{f_1}(r) + T_{f_2}(r)) + O(\max_{1 \leq i \leq 2} \log T_{f_i}(r) + \log r) + n(n+1)\mathfrak{X}_\sigma(r) \ \|. \end{aligned} \tag{4.2}$$

Hence we get

$$q - (n+1) - \frac{2knq}{q - 2k + 2kn} \leq n(n+1)\rho.$$

This proves the Theorem. □

The unit disc case. Nevanlinna theory also works for meromorphic functions, or more generally holomorphic maps, on the unit-disc, as long as they are admissible, i.e grow fast enough. However, slow growing maps on the unit disc exhibit a different behavior. For them, an essential part of the remainder in the Second Main Theorem is no longer small enough. In this section, we consider a wider range of holomorphic maps on the unit-disk. Before doing this, we first state the value sharing results for admissible mappings on the unit-disc (see, for example,[Fan99]). We first recall the Second Main Theorem.

Theorem 4.14 (**Second Main Theorem for the Unit Disc**)**.** *Let f be a meromorphic function on the unit disc $\triangle(1)$ with $\lim_{r\to 1^-} T_f(r) = \infty$ and $a_1, \ldots, a_q$ distinct points in $\mathbb{C} \cup \{\infty\}$. Then,*

$$(q-2)T_f(r) \leq \sum_{j=1}^{q} N_f^{(1)}(r, a_j) + O\left(\log \frac{1}{1-r}\right) \ \|,$$

where $\|$ means the inequality holds for all $r \in (0,1)$ except for a set E with $\int_E \frac{dr}{1-r} < \infty$.

This theorem can be easily extended to holomorphic maps from $\triangle(1)$ into $\mathbb{P}^n(\mathbb{C})$, similar to the Cartan's Second Main Theorem (see [Car33]). We'll omit the statement here.

Definition 4.15. The holomorphic map $f : \triangle(1) \to \mathbb{P}^n(\mathbb{C})$ is said to be admissible if

$$\limsup_{r\to 1^-} \frac{T_f(r)}{\log \frac{1}{1-r}} = \infty.$$

Similar to the proof of Corollary 4.9, but instead of using H. Cartan's Second Main Theorem (see [Ru01a]) for unit-disc, we can prove the following result for admissible mappings.

Theorem 4.16. *Let $f_1, f_2 : \triangle(1) \to \mathbb{P}^n(\mathbb{C})$ be admissible linearly non-degenerate holomorphic maps. Let $H_1, \ldots, H_q$ be hyperplanes in $\mathbb{P}^n(\mathbb{C})$, located in general position. Assume that*

(i) $f_1^{-1}(H_i) = f_2^{-1}(H_i)$ for $i = 1, \ldots, q$,

(ii) Let $k \le n$ be a positive integer such that $f_1^{-1}(\cap_{i=1}^{k+1} H_{j_i}) = \emptyset$ $(1 \le i_1 < \cdots < i_{k+1} \le q)$,

(iii) $f_1 = f_2$ on $\cup_{j=1}^{q} f_1^{-1}(H_j)$.

If $q > \frac{1}{2}(2k + n + 1 + \sqrt{8kn^2 + 4k^2 + 4kn - 4k + (n+1)^2})$ (in particular, if $q > (n+1)(k+1)$), then $f_1 \equiv f_2$.

Open Riemann surfaces with a complete metric. Nevanlinna theory was also extended by Fujimoto [Fuj86] to holomorphic maps on the unit-disc (or a general open Riemann surface with a complete metric) which may not be admissible, but still satisfy a (mild) growth condition. To deal with the general case, we put our set up in a more general context. We consider (instead of the unit-disc) an open Riemann surface M with the pseudo-metric ds^2. Here by a pseudo-metric, we mean it can be locally represented by $ds^2 = \lambda(z)|dz|^2$ such that $\lambda > 0$ outside a finite set of points on M, so we can define the divisor of ds^2 as $\nu_{ds^2} = \nu_\lambda$. We define the *Ricci form* by $\mathrm{Ric}[ds^2] := dd^c[\log \lambda]$ as a current (Note that in Fujimoto's book [Fuj93b], the *Ricci form* is $\mathrm{Ric}[ds^2] := -dd^c[\log \lambda]$). Let $f : M \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic map. For $\rho \ge 0$ we say that f satisfies the condition (C_ρ) if there exists a compact set K such that

$$\mathrm{Ric}[ds^2] \prec \rho \Omega_f$$

on $M - K$, where Ω_f is the pull-back by f of the Fubini-Study form $\Omega = dd^c \log \|w\|^2$ on $\mathbb{P}^n(\mathbb{C})$. Here for two currents Ω_1, Ω_2 on some open set U in M and a positive constant c, by the notation

$$\Omega_1 \prec_c \Omega_2$$

we mean that there is a divisor υ and a bounded continuous real-valued function k on M with mild singularities on U such that $\upsilon(z) > c$ for each $z \in Supp(\upsilon)$ and

$$\Omega_1 + [\upsilon] = \Omega_2 + dd^c[\log |k|^2]$$

on U. For brevity, by $\Omega_1 \prec \Omega_2$, we mean $\Omega_1 \prec_c \Omega_2$ for some constant c.

Theorem 4.17. *Let $H_1, \ldots, H_q$ be hyperplanes in $\mathbb{P}^n(\mathbb{C})$, located in general position. Let M be an open Riemann surface with a complete continuous pseudo-metric ds^2. Let $f, g : M \to \mathbb{P}^n(\mathbb{C})$ be two linearly non-degenerate holomorphic maps satisfying the condition (C_ρ). Assume that*

(i) $f^{-1}(H_i) = g^{-1}(H_i)$ for $i = 1, \ldots, q$,

(ii) Let $k \leq n$ be a positive integer such that $f^{-1}(\cap_{i=1}^{k+1} H_{j_i}) = \emptyset$ $(1 \leq i_1 < \cdots < i_{k+1} \leq q)$,

(iii) $f = g$ on $\cup_{j=1}^{q} f^{-1}(H_j)$.

If

$$q > (n+1) + \frac{2knq}{q - 2k + 2kn} + \frac{1}{2}n(n+1)\rho,$$

then $f \equiv g$.

To prove our theorem, we need the following result from Fujimoto (see [Fuj93b]).

Lemma 4.18 ([Fuj93b], Theorem 4.2.6)**.** *Let M be a open Riemann surface with a complete continuous pseudo-metric ds^2 and let $d\tau^2$ be a continuous pseudo-metric on M whose curvature is strictly negative outside a compact set K. Assume that there exists a constant p with $0 < p < 1$ such that*

$$\mathrm{Ric}[ds^2] \prec_{1-p} p(\mathrm{Ric}[d\tau^2])$$

on $M - K$. Then M is of finite type, namely, M is biholomorphic to a compact Riemann surface with finitely many points removed.

Note that the $\mathrm{Ric}[ds^2]$ defined in Fujimoto's book [Fuj93b] is different from the definition here by a negative sign.

Proof of Theorem 4.17. Let $f,\ g : M \to \mathbb{P}^n(\mathbb{C})$ be two linearly non-degenerate holomorphic maps with (local) reduced representations $F = (f_0, \cdots, f_n)$ and $G = (g_0, \cdots, g_n)$, respectively. Assume that

$$S := q - (n+1) - (n^2 + 2n - 1)\frac{2q}{N} - \frac{2knq}{q - 2k + 2kn}.$$

We can choose N big enough such that $S > 0$ and

$$\frac{q - (n+1) - \frac{n(n+1)\rho}{2} - \frac{2knq}{q-2k+2kn}}{n^2 + 2n - 1 + \rho\sum_{p=0}^{n}(n-p)^2} > \frac{2q}{N} > \frac{q - (n+1) - \frac{n(n+1)\rho}{2} - \frac{2knq}{q-2k+2kn}}{1 + n^2 + 2n - 1 + \rho\sum_{p=0}^{n}(n-p)^2}.$$

Let

$$l := \left(\frac{n(n+1)\rho}{2} + \sum_{p=0}^{n-1}(n-p)^2\frac{2q\rho}{N}\right)/S. \tag{4.3}$$

Then we have

$$1 - \frac{2q}{NS} < l < 1. \tag{4.4}$$

Denote by

$$\phi_k^F(H_j) = \frac{|F_k(H_j)|^2}{|F_k|^2}, \qquad \phi_k^G(H_j) = \frac{|G_k(H_j)|^2}{|G_k|^2}.$$

For an arbitrary holomorphic coordinate z, we set

$$h_1 := c \left[\frac{|F|^{q-(n+1)-(n^2+2n-1)\frac{2q}{N}} |F_n|^{1+\frac{2q}{N}} \prod_{p=1}^{n-1} |F_p|^{\frac{4q}{N}}}{\prod_{j=1}^{q} |F(H_j)| \prod_{p=0}^{n-1} \prod_{j=1}^{q} (N - \log \phi_p^F(H_j))} \right]^{2\beta_n},$$

$$h_2 := c \left[\frac{|G|^{q-(n+1)-(n^2+2n-1)\frac{2q}{N}} |G_n|^{1+\frac{2q}{N}} \prod_{p=1}^{n-1} |G_p|^{\frac{4q}{N}}}{\prod_{j=1}^{q} |G(H_j)| \prod_{p=0}^{n-1} \prod_{j=1}^{q} (N - \log \phi_p^G(H_j))} \right]^{2\beta_n},$$

where $\beta_n^{-1} = \frac{n(n+1)}{2} + \frac{2q}{N} \sum_{p=0}^{n-1} (n-p)^2$. Let $d\tau^2 = \eta(z)|dz|^2$ be the pseudo-metric given by

$$\eta(z) := \left(\frac{|\chi(z)|}{|F(z)|^q |G(z)|^q} \right)^{\frac{2kn\beta_n}{q-2k+2kn}} \sqrt{h_1(z) h_2(z)},$$

where χ is the auxiliary function given in (3.7). Note that if choose another local coordinate u instead of z, then each F_k (as well as G_k) is multiplied by $\left|\frac{dz}{du}\right|^{p(p+1)/2}$ so $\eta(z)$ is multiplied by $\left|\frac{dz}{du}\right|$. Hence $d\tau^2$ is well-defined on $M - K$ independently of the choice of holomorphic local coordinate z. From Proposition 1 and Lemma 4 in [PR16], we know that $d\tau^2$ is continuous pseudo-metric on M whose curvature is strictly negative outside a compact set K. Let

$$\psi_{j,p}^F = \sum_{l \neq i_1, \cdots, i_p} a_{j,l} W(f_l, f_{i_1}, \cdots, f_{i_p})$$

and

$$\psi_{j,p}^G = \sum_{l \neq i_1, \cdots, i_p} a_{j,l} W(g_l, g_{i_1}, \cdots, g_{i_p}).$$

By the assumption that f and g are linearly non-degenerate, $\psi_{j,p}^F$ and $\psi_{j,p}^G$ do not vanish identically, and thus have only isolated zeros since they are both holomorphic. Note that, from definition,

$$|\psi_{j,p}^F| < |F_p(H_j)| \qquad \text{and} \qquad |\psi_{j,p}^G| < |G_p(H_j)|.$$

Denote by $C := \sup_{0<x\leq 1} x^{2/N}(N - \log x)$. Since $0 < \phi_p^F(H_j) \leq 1$ for all p and j,

$$\frac{1}{N - \log \phi_p^F(H_j)} \geq \frac{1}{C} \phi_p^F(H_j)^{2/N} = \frac{1}{C} \frac{|F_p(H_j)|^{4/N}}{|F_p|^{4/N}} \geq \frac{1}{C} \frac{|\psi_{j,p}^F|^{4/N}}{|F_p|^{4/N}}.$$

Similarly,

$$\frac{1}{N - \log \phi_p^G(H_j)} \geq \frac{1}{C} \frac{|\psi_{j,p}^G|^{4/N}}{|G_p|^{4/N}}.$$

Hence if we let

$$\xi := \left(\frac{\prod_{j=1}^q \prod_{p=0}^{n-1} \left(|\psi_{j,p}^F \psi_{j,p}^G|\right)^{4/N} \prod_{p=0}^{n-1} \prod_{j=1}^q (N - \log \phi_p^G(H_j))}{\prod_{p=0}^{n-1} (|F_p||G_p|)^{4/N}} \right)^{\beta_n},$$

then ξ is a well-defined bounded function on $M - K$ which does not depend on a choice of a holomorphic local coordinate z. From the definition of ξ and η, we see that

$$(\xi\eta)^{1/\beta_n} = (|F||G|)^S \phi_1,$$

where

$$\phi_1 := \frac{|\chi|^{\frac{2kn}{q-2k+2kn}} |F_n G_n|^{1+\frac{2q}{N}} \prod_{p=0}^{n-1} \prod_{j=1}^q |\psi_{j,p}^F \psi_{j,p}^G|^{4/N}}{\prod_{j=1}^q |F(H_j)G(H_j)|}$$

and $S = q-(n+1)-(n^2+2n-1)\frac{2q}{N} - \frac{2knq}{q-2k+2kn} > 0$. The above gives

$$|FG|^{\rho}\phi_1^{\rho/S} = (\xi\eta)^{\frac{\rho}{S\beta_n}}.$$

Notice that ϕ_1 is holomorphic outside K, so by the Poincare-Lelong formula,

$$\frac{1}{2}\rho(\Omega_f+\Omega_g) + \frac{\rho}{S}[\phi_1 = 0] = \frac{\rho}{S\beta_n}dd^c[\log\eta] + \frac{\rho}{S\beta_n}dd^c[\log\xi]. \tag{4.5}$$

This gives $\frac{1}{2}\rho(\Omega_f+\Omega_g) \prec l(\mathrm{Ric}[d\tau^2])$ where $l = \frac{\rho}{S\beta_n}$ (which is the same as in (4.3)) and $d\tau^2 = \eta(z)|dz|^2$. On the other hand, from our assumption, $\mathrm{Ric}[ds^2] \prec \rho\Omega_f$ and $\mathrm{Ric}[ds^2] \prec \rho\Omega_g$ and so $\mathrm{Ric}[ds^2] \prec \frac{1}{2}\rho(\Omega_f+\Omega_g)$. Hence, from (4.5), $\mathrm{Ric}[ds^2] \prec l(\mathrm{Ric}[d\tau^2])$. As we can make $1-l$ small by letting N big enough (see (4.4)), this gives $\mathrm{Ric}[ds^2] \prec_{1-l} l\,\mathrm{Ric}[d\tau^2]$. Hence, by Lemma 4.18 since $l<1$, M is biholomorphic to a compact Riemann surface with finitely many points removed, and the problem is reduced to Theorem 4.8. The Theorem is thus proved by applying Theorem 4.8.

Chapter 5

Uniqueness Results for p-adic Holomorphic Mappings into $\mathbb{P}^n(\mathbb{C}_p)$

5.1 Non-Archimedean Value Distribution Theory

In this section, we consider the value-sharing problem for p-adic holomorphic maps. The reason we put this chapter is that the p-adic holomorphic curves have many similarities with the holomorphic maps defined on a compact Riemann surface of genus $g = 0$. In particular, we notice that the error term $(g(S) - 1)$ appeared in the Second Main Theorem for holomorphic maps (see Theorem 3.4) serves the same role as the term $-\frac{n(n+1)}{2} \log r$ in the Second Main Theorem for p-adic holomorphic maps (see Theorem 5.7), which are both negative when $g(S) = 0$. So we shall see that the statements of value sharing for p-adic holomorphic maps are similar to the statements of value sharing for holomorphic maps from the compact Riemann surface of genus zero. This is the main motivation that we include this chapter in this thesis.

We recall some definitions (see [Ru01b]). Let p be a prime number, and let $|\ |_p$ be the standard p-adic valuation on $\mathbb{Q}$ normalized so that $|p|_p = p^{-1}$. Let $\mathbb{Q}_p$ be the completion of $\mathbb{Q}$ with respect to the norm $|\ |_p$, and let $\mathbb{C}_p$ be the completion of the algebraic closure of $\mathbb{Q}_p$. Note that it is a theorem that $\mathbb{C}_p$ is algebraically closed. For the simplicity, we denote the p-adic norm $|\ |_p$ on $\mathbb{C}_p$ by $|\ |$. We also note that the result also works for a general complete, algebraic closed non-Archimedean field of characteristic zero.

It is known that an infinite sum converges in a non-Archimedean norm if and only if its general term approaches zero. So an expression of the form $h(z) = \sum_{n=0}^{\infty} a_n z^n, \quad a_n \in \mathbb{C}_p$ is well defined whenever $|a_n z^n| \to 0,\ as\ n \to \infty$. Such functions are called *p-adic analytic functions*. Let $h(z)$ be p-adic analytic function on $|z| < R$. Define, for each $0 < r < R$, $M_h(r) = \max_{|z|=r} |h(z)|$.

Lemma 5.1 ([AS71], Lemma)**.** *The following statements hold:*

(1) $M_h(r) = \max_{n\geq 0} |a_n| r^n$,

(2) The maximum on the right of (1) attained for a unique value of n except for a discrete sequence of values $\{r_\nu\}$ in the open interval $(0, R)$,

(3) If $r \notin \{r_\nu\}$ and $|z| = r < R$, then $|h(z)| = M_h(r)$,

(4) If h is a non-constant p-adic entire function, then $M_h(r) \to \infty$ as $r \to \infty$,

(5) $M_{h'}(r) \leq M_h(r)/r$ $(r > 0)$ *(so* $\log M_{h'}(r) \leq \log M_h(r)) - \log r$*),*

(6) For two analytic functions f, g, $M_{fg}(r) = M_f(r) M_g(r)$.

Let $B(r)$ be the open disc which is defined by $B(r) = \{z|\ |z| < r\}$, and we use $B[r]$ to denote the closed disc.

We also define $\nu(r,h) = \max_{n\geq 0}\{n \mid |a_n|r^n = |h|_r\}$, $\nu(r,h)$ is called the *central index.*

We have the following Poisson-Jensen Formula in [CY97].

Theorem 5.2 ([CY97], Theorem 3.1)**.** *The central index $\nu(r,h)$ increases as $r \longrightarrow \rho$, where ρ is defined in open subset $B(r) = \{z : |z| < r\}$ for $0 < r < \rho$, and satisfies the formula*

$$\log|h|_r = \log|a_{\nu(0,h)}| + \int_0^r \frac{\nu(t,h) - \nu(0,h)}{t} dt + \nu(0,h)\log r$$

where $\nu(0,h) = \lim_{r\to 0^+} \nu(r,h)$.

We also have the following Weierstrass preparation theorem in [CY97].

Theorem 5.3 ([CY97], Theorem 2.2)**.** *There exists a unique monic polynomial P of degree $\nu(r,h)$ and a p-adic analytic function g on $B[r]$ such that $h = gP$, where g does not have any zero inside $B[r]$, and P has exactly $\nu(r,h)$ zeros, counting multiplicities.*

Let $n_h(r,0)$ denote the number of zeros of h in $B[r]$, counting multiplicity. Define the *valence function* of h by

$$N_h(r,0) = \int_0^r \frac{n_h(t,0) - n_h(0,0)}{t} dt + n_h(0,0)\log r.$$

Weierstrass Preparation Theorem shows that

$$n_h(r,0) = \nu(r,h),$$

and the Poisson-Jensen Formula implies that

$$N_h(r, 0) = \log |h|_r - \log |a_{n_h(0,0)}|.$$

Let $f : \mathbb{C}_p \to \mathbb{P}^n(\mathbb{C}_p)$ be a p-adic holomorphic map. Let $\tilde{f} = (f_0, \ldots, f_n)$, be a reduced representative of f, where $f_0, \ldots, f_n$ are p-adic entire functions on $\mathbb{C}_p$ and have no common zeros. The Nevanlinna-Cartan characteristics function $T_f(r)$ is defined by

$$T_f(r) = \log ||f||_r$$

where

$$||f||_r = max\{|f_0|_r, \ldots, |f_n|_r\}.$$

Let Q be a homogeneous polynomial (form) in $n+1$ variables with coefficients in $\mathbb{C}_p$. We consider the p-adic entire function $Q \circ f = Q(f_0, \ldots, f_n)$ on $\mathbb{C}_p$. Let $n_f(r, Q)$ be the number of zeros of $Q \circ f$ in the disk $\{z \mid |z|_p < r\}$ counting multiplicity. Set

$$N_f(r, Q) = \int_0^r \frac{n_f(t, Q) - n_f(0, Q)}{t} dt + n_f(0, Q) \log r,$$

$$m_f(r, Q) = \log \frac{||f||_r^d}{|Q \circ f|_r}$$

if $Q \circ f \not\equiv 0$.

Forms $Q_1, \ldots, Q_q$, where $q > n$, are said to be *admissible* if no set of $n+1$ forms in this system has common zeros in $\mathbb{C}^{n+1} - \{0\}$.

We have the following First Main Theorem in [Ru01b].

Theorem 5.4 ([Ru01b], p.1267). *Let $f : \mathbb{C}_p \to \mathbb{P}^n(\mathbb{C}_p)$ be a p-adic holomorphic map, and let Q be homogeneous forms of degree d. If $Q(f) \not\equiv 0$, then for every real number r with $0 < r < \infty$*

$$m_f(r, Q) + N_f(r, Q) = dT_f(r) + O(1),$$

where $O(1)$ is a constant independent of r.

Proof. By definition, we have

$$\begin{aligned} m_f(r, Q) + N_f(r, Q) &= \log \frac{||f||_r^d}{|Q \circ f|_r} + \int_0^r \frac{n_f(t, Q) - n_f(0, Q)}{t} dt + n_f(0, Q) \log r \\ &= \log ||f||_r^d - \log |Q \circ f|_r + \int_0^r \frac{n_f(t, Q) - n_f(0, Q)}{t} dt \\ &\quad + n_f(0, Q) \log r \\ &= d \log ||f||_r - \log |Q \circ f|_r + \int_0^r \frac{n_f(t, Q) - n_f(0, Q)}{t} dt \\ &\quad + n_f(0, Q) \log r \\ &= dT_f(r) - \log |Q \circ f|_r + \int_0^r \frac{n_f(t, Q) - n_f(0, Q)}{t} dt \\ &\quad + n_f(0, Q) \log r. \end{aligned}$$

By Poisson-Jensen Formula, (Theorem 5.2),

$$\int_0^r \frac{n_f(t, Q) - n_f(0, Q)}{t} dt + n_f(0, Q) \log r - \log |Q \circ f|_r = O(1)$$

Therefore, $m_f(r, Q) + N_f(r, Q) = dT_f(r) + O(1)$, which proves the theorem. □

Definition 5.5. Let $f_1(z), \ldots, f_n(z)$ be p-adic entire functions on $\mathbb{C}_p$. Let

$$W = \begin{vmatrix} f_0 & f_1 & \ldots & f_n \\ f_0^1 & f_1^1 & \ldots & f_n^1 \\ \vdots & \vdots & \ddots & \vdots \\ f_0^{(n)} & f_1^{(n)} & \ldots & f_n^{(n)} \end{vmatrix}.$$

The determinant W is called the *Wronskian* of $f_1, f_2, \ldots, f_n$, which is normally written as $W := W(f_0, \ldots, f_n)$.

Theorem 5.6. Properties of the Wronski determinants:

(a) $W(f_0, \ldots, f_n) \not\equiv 0$ *if and only if* $(f_0, \ldots, f_n)$ *are linearly independent.*

(b) If $(g_0, \ldots, g_n) = (f_0, \ldots, f_n)B$, *where* B *is a constant* $(n+1) \times (n+1)$ *matrix, then* $W(g_0, \ldots, g_n) = \det B \times W(f_0, \ldots, f_n)$.

(c) $W(gf_0, \ldots, gf_n) = g^{n+1}W(f_0, \ldots, f_n)$ *for every function* g.

(d) $L(f_0, \ldots, f_n) := W(f_0, \ldots, f_n)/(f_0, \ldots f_n)$.

(e) $L(gf_0, \ldots, gf_n) = L(f_0, \ldots, f_n)$.

We also have p-adic Second Main Theorem as follows.

Theorem 5.7 ([TT14], p.95)**.** *Let f be a linearly non-degenerate p-adic holomorphic map from* $\mathbb{C}_p$ *into* $\mathbb{P}^n(\mathbb{C}_p)$ *and* $H_1, ..., H_q$ *be hyperplanes in* $\mathbb{P}^n(\mathbb{C}_p)$ *in general position. Then*

$$(q - n - 1)T_f(r) \leq \sum_{j=1}^{q} N_f^{(n)}(r, H_j) - \frac{n(n+1)}{2} \log r + O(1),$$

for all $r \geq 1$.

Proof. Let $L_1, \ldots, L_q$ be the linear forms defining $H_1, \ldots, H_q$. For any fixed $r > 1$, we take $\mu(r,0), \cdots, \mu(r,n) \subset \{1, \ldots, q\}$ such that

$$0 < |L_{\mu(r,0)}(f)|_r \leq \cdots \leq |L_{\mu(r,n)}(f)|_r \leq |L_j(f)|_r,$$

for $f \in \{1, \ldots, q\} \backslash \{\mu(r,0), \ldots, \mu(r,n)\}$. By the assumption that $H_1, ..., H_q$ are in general position, $L_{\mu(r,0)}, \ldots, L_{\mu(r,n)}$ are linearly independent, we can write $f_i = \sum_{j=0}^{n} \tilde{a}_{ij} L_{\mu(r,j)}$. Thus we get

$$\|f\|_r \leq C \max_{0 \leq j \leq n} |L_{\mu(r,j)}|_r.$$

Hence

$$\frac{\|f\|_r}{|Lj(f)|_r} \leq C,$$

for any $j \in \{1, \ldots, q\} \backslash \{\mu(r,0), \ldots, \mu(r,n)\}$. Thus

$$\prod_{j=1}^{q} \frac{||f\|_r \|L_j\|}{|L_j(f)|_r} \leq C \prod_{i=0}^{n} \frac{||f\|_r \|L_{\mu(r,i)}\|}{|L_{\mu(r,i)}(f)|_r}.$$

Then by the Wronskian determinant of $f_0, \ldots, f_n$, we have

$$\begin{aligned} \sum_{j=1}^{q} m_f(r, H_j) &= \sum_{i=0}^{n} \log \frac{||f\|_r \|L_{\mu(r,i)}\|}{|L_{\mu(r,i)}(f)|_r} \\ &= \log \frac{|W|_r}{|L_{\mu(r,i)}(f)|_r} + \log \frac{\|f\|_r}{|W|_r} + O(1) \\ &= (n+1)T_f(r) - N_W(r,0) + \log \frac{|W|_r}{|L_{\mu(r,i)}(f)|_r} + O(1). \end{aligned}$$

Here, by the property of the Wronskian and property (5) in the Lemma 5.1 above, we have

$$\log \frac{|W|_r}{|L_{\mu(r,i)}(f)|_r} = \log \frac{|W(L_{\mu(r,0)}(f), \ldots, L_{\mu(r,n)}(f))|_r}{|L_{\mu(r,i)}(f)|_r} + O(1) \leq -\frac{n(n+1)}{2} \log r.$$

Thus we have

$$\sum_{j=1}^{q} m_f(r, H_j) \leq (n+1)T_f(r) - N_W(r,0) - \frac{n(n+1)}{2} \log r + O(1)$$

or by the Theorem 5.4

$$(q-n-1)T_f(r) \leq \sum_{j=1}^{q} N_f(r, H_j) - N_W(r,0) - \frac{n(n+1)}{2} \log r + O(1).$$

It is easy to prove that

$$\sum_{j=1}^{q} N_f(r, H_j) - N_W(r,0) \leq \sum_{j=1}^{q} N_f^{(n)}(r, H_j).$$

This proves the Theorem. □

5.2 Preliminary Results on Uniqueness Theorem for p-adic Holomorphic Mappings into $\mathbb{P}^n(\mathbb{C}_p)$

We first discuss the result of Ru [Ru01c].

Theorem 5.8 ([Ru01c], Theorem 2.2). *Let $f, g : \mathbb{C}_p \to \mathbb{P}^n(\mathbb{C}_p)$ be two p-adic linearly non-degenerate holomorphic maps. Let $H_1, \ldots, H_{3n+1}$ be hyperplanes in $\mathbb{P}^n(\mathbb{C}_p)$ located in general position. Assume that* *(i)* $f^{-1}(H_j) = g^{-1}(H_j), 1 \leq j \leq 3n+1$,

(ii) For each $i \neq j$, $f^{-1}(H_i) \cap f^{-1}(H_j) = \emptyset$.

If $f(z) = g(z)$ *for every point* $z \in \cup_{j=1}^{q} f^{-1}(H_j)$, *then* $f \equiv g$.

Remark:

(1) This Theorem can be compared with Theorem 3.7. In particular, the role of $-\frac{n(n+1)}{2}\log r$ (being negative) in this theorem is similar to the role of $(2g-2)$ appeared in Theorem 3.7 which is negative in the case $g = 0$.

(2) Recently, Qiming Yan, in [Yan11], and Tan and Trinh, in [TT14], improved Ru's result from $3n+1$ to $2n+2$.

Proof. Similar to the proof of Theorem 3.7 in the case *genus* $= 0$, assume that $f_1 \not\equiv f_2$, then $\Phi := F_{1j_0} - F_{2j_0} \not\equiv 0$ for some $1 \leq j_0 \leq q$.

By Theorem 5.7, we get for all $r \geq 1$

$$\begin{aligned}
(q-n-1)T_f(r) &\leq \sum_{j=1}^{q} N_f^{(n)}(r, H_j) - \frac{n(n+1)}{2}\log r + O(1) \\
&\leq n\sum_{j=1}^{q} \bar{N}_f(r, H_j) - \frac{n(n+1)}{2}\log r + O(1).
\end{aligned}$$

Same for g,

$$(q-n-1)T_g(r) \leq n\sum_{j=1}^{q} \bar{N}_g(r, H_j) - \frac{n(n+1)}{2}\log r + O(1).$$

By summing up of both sides of the above inequality for all $r \geq 1$,

$$(q-(n+1))(T_f(r) + T_g(r)) \leq 2n(N_\Phi(r, 0)) - n(n+1)\log r + O(1).$$

Hence

$$(q-(n+1))(T_f(r)+T_g(r)) \leq 2n(T_f(r)+T_g(r)) - n(n+1)\log r + O(1).$$

This gives a contradiction. $\square$

Theorem 5.9 ([TT14], Theorem 1.3)**.** *Let f and g be non-Archimedean linearly non-degenerate holomorphic maps from an algebraically closed field κ of characteristic $p \geq 0$ into $\mathbb{P}^n(\kappa)$ defined by the representations $f=(f_1,\dots,f_n)$ and $g=(g_1,\dots,g_n)$, respectively. Let $H_j(\omega_0,\dots,\omega_n)$, $L_j(\omega_0,\dots,\omega_n)$ $(j=1,\dots,2n+2)$ be homogeneous linear polynomials such that $\{H_j\}_{j=1}^{2n+2}$ and $\{L_j\}_{j=1}^{2n+2}$ are in general position. Assume that*

(i) $f^{-1}(H_j)=g^{-1}(L_j)$, for all $1 \leq j \leq 2n+2$,

(ii) $f^{-1}(H_i)\cap f^{-1}(H_j)=\emptyset$ for all $1 \leq i < j \leq 2n+2$, and

(iii) $\frac{(f,H_i)}{(g,L_i)} \equiv \frac{(f,H_j)}{(g,L_j)}$ on $\cup_{k=1}^{2n+2} f^{-1}(H_k)$ for all $k \neq i,j$ and $1 \leq i < j \leq 2n+2$.

Then, $\frac{(f,H_1)}{(g,L_1)} \equiv \cdots \equiv \frac{(f,H_{2n+2})}{(g,L_{2n+2})} \equiv c$, where c is a nonzero constant in κ.

5.3 New Results

We improve the above results as follows (Compare to Theorem 3.8. above).

Theorem 5.10. *Let $f,\ g : \mathbb{C}_p \to \mathbb{P}^n(\mathbb{C}_p)$ be two p-adic linearly non-degenerated holomorphic maps. Let $H_1,\dots,H_q$ be hyperplanes in $\mathbb{P}^n(\mathbb{C}_p)$ located in general position. Assume that*

(i) $f^{-1}(H_i)=g^{-1}(H_i)$ for $i=1,\dots,q$,

(ii) Let $k \leq n$ *be a positive integer such that* $i \neq j, f^{-1}(\cap_{i=1}^{k+1} H_{j_i}) = \emptyset$ $(1 \leq i_1 < \cdots < i_{k+1} \leq q)$,

(iii) $f = g$ *on* $\cup_{j=1}^{q} f^{-1}(H_j)$.

If $q \geq \frac{1}{2}(2k + n + 1 + \sqrt{8kn^2 + 4k^2 + 4kn - 4k + (n+1)^2}$, *then* $f \equiv g$.

Proof. Assume that $f \not\equiv g$. Similar to the proof of Theorem 3.7 in the genus zero case, by applying Theorem 5.7 and Lemma 3.9, we get

$$\begin{aligned}
& (q - (n+1))(T_f(r) + T_g(r)) \\
\leq \; & \sum_{j=1}^{q} (N_f^{(n)}(r, H_j) + N_g^{(n)}(r, H_j)) - n(n+1)\log r + O(1) \\
\leq \; & \left(\frac{2kn}{q - 2k + 2kn} \right) N_\chi(r, 0) - n(n+1)\log r \\
\leq \; & \left(\frac{2knq}{q - 2k + 2kn} \right) (T_f(r) + T_g(r)) - n(n+1)\log r + O(1)
\end{aligned}$$

which gives a contradiction. □

Chapter 6

Uniqueness Results for Gauss Map of Minimal Surfaces

6.1 Theory of Minimal Surfaces and Gauss Maps

Let $x = (x_1, x_2, x_3) : M \to \mathbb{R}^3$ be an oriented surface immersed in $\mathbb{R}^3$. By definition, the classical Gauss map $g : M \to S^2$ is the map which maps each point $p \in M$ to the point in S^2 corresponding to the unit normal vector of M at p. On the other hand, S^2 is canonically identified with the extended complex plane $\mathbb{C} \cup \{\infty\}$ or $\mathbb{P}^1(\mathbb{C})$ by the stereographic projection. We may consider the Gauss map g as a map of M into $\mathbb{P}^1(\mathbb{C})$.

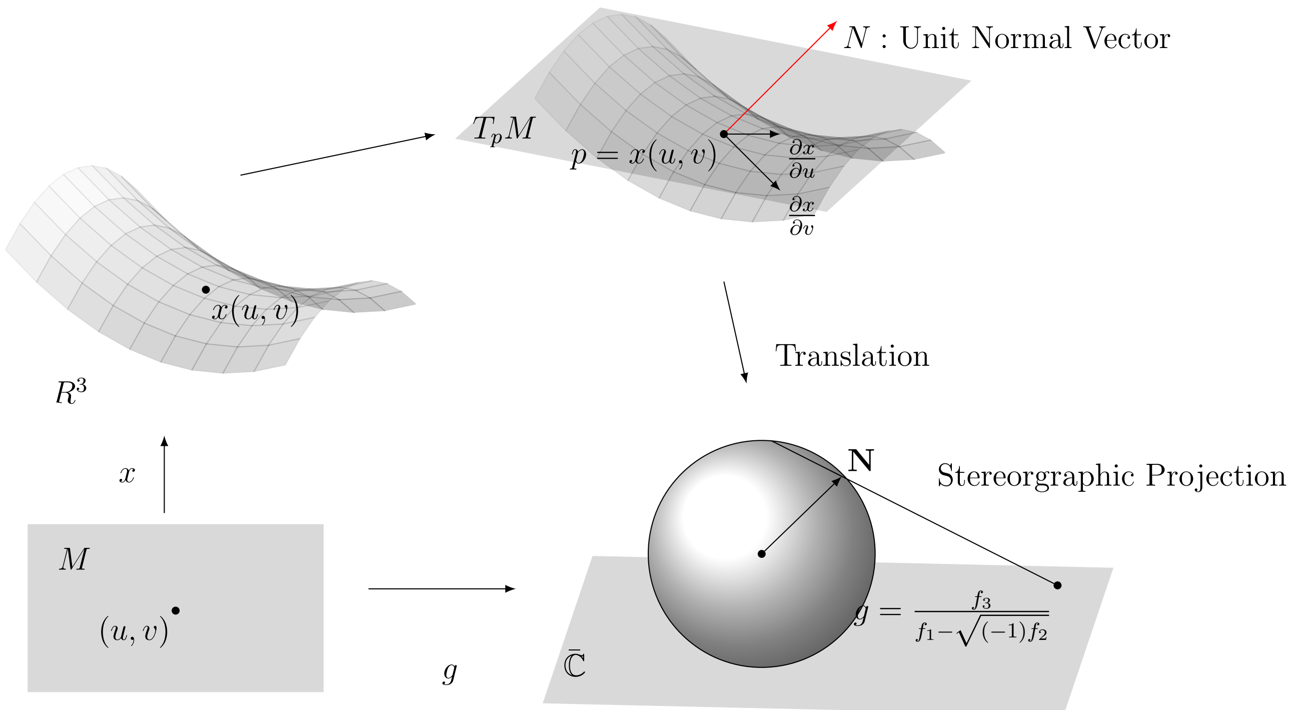

Figure 6.1: The Classical Gauss Map

Using system of isothermal coordinates (u, v) and by letting $z = u + \sqrt{-1}v$, we can regard M as a Riemann surface. Furthermore, if M is minimal then g is a holomorphic map. This gives us a great link to study surfaces between geometry and Nevanlinna Theory.

We begin by recalling some notions concerning a surface $x = (x_1, \ldots, x_m) : M \to \mathbb{R}^m$ immersed in $\mathbb{R}^m$, which means that M is a connected, oriented real 2-dimensional differentiable manifold and x is a differentiable map of M into $\mathbb{R}^m$ which has maximal rank everywhere.

For a point $p \in M$, take a system of local coordinates (u_1, u_2) around p which are positively oriented. The vectors $\frac{\partial x}{\partial u_1}\big|_p$ and $\frac{\partial x}{\partial u_2}\big|_p$ are tangent to M at p and linearly independent because x is an immersion. This shows that the tangent plane of M at p is given by

$$T_p(M) := \left\{\lambda \frac{\partial x}{\partial u_1}\bigg|_p + \mu \frac{\partial x}{\partial u_2}\bigg|_p ; \lambda, \mu \in \mathbb{R}\right\}$$

and the space of all vectors which are normal to M at p, say the normal space of M at p, is given by

$$N_p(M) := \left\{N; \left(N, \frac{\partial x}{\partial u_1}\bigg|_p\right) = \left(N, \frac{\partial x}{\partial u_2}\bigg|_p\right) = 0\right\},$$

where (X, Y) denotes inner product of vectors X and Y. The metric ds^2 on M induced from the standard metric on $\mathbb{R}^m$, called the *the first fundamental form* on M, is given by

$$\begin{aligned} ds^2 &:= |dx|^2 = (dx, dx) \\ &= \left(\frac{\partial x}{\partial u_1} du_1 + \frac{\partial x}{\partial u_2} du_2, \frac{\partial x}{\partial u_1} du_1 + \frac{\partial x}{\partial u_2} du_2\right) \\ &= g_{11} du_1^2 + 2g_{12} du_1 du_2 + g_{22} du_2^2, \end{aligned}$$

where

$$g_{ij} := \left(\frac{\partial x}{\partial u_i}, \frac{\partial x}{\partial u_j}\right) \qquad 1 \leq i, j \leq 2.$$

We now recall the notion of isothermal coordinates for a surface with a metric ds^2. A system of local coordinates (u_1, u_2) on an open set U in M is called an *isothermal* if

$$ds^2 = \lambda^2(du_1^2 + du_2^2),$$

for some positive C^∞ function λ on U, i.e. $g_{11} = g_{22},\ g_{12} = 0$.

Theorem 6.1 ([Fuj93b], Theorem 1.1.9)**.** *For every surface M there is a family of isothermal local coordinates whose domains cover the totality of M.*

Proposition 6.2 ([Fuj93b], Proposition 1.1.12)**.** *For an oriented surface M with a metric ds^2, if we take two systems of positively oriented isothermal local coordinates (u, v) and (x, y), then $w := u + \sqrt{-1}v$ is a holomorphic function in $z := x + \sqrt{-1}y$ on the common domain of definition.*

Let $x : M \to \mathbb{R}^m$ be an oriented surface with a Riemannian metric ds^2. With each positive isothermal local coordinate (u, v), we associate the complex function $z := u + \sqrt{-1}v$. By Proposition 6.2, the surface M has a complex structure so that these complex-valued functions define holomorphic local coordinates on M, and so M may be considered as a Riemann surface. From now on, we always regard M as a Riemann surface.

Theorem 6.3 ([Fuj93b], Theorem 1.1.15)**.** *Let $x = (x_1, \ldots, x_m) : M \to \mathbb{R}^m$ be a surface immersed in $\mathbb{R}^m$, which is considered as a Riemann surface as above. Then, M is minimal if and only if each x_i is a harmonic function on M, namely,*

$$\Delta_z x_i \equiv \Big(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\Big)x_i = 0 \quad (1 \leq i \leq m)$$

for every holomorphic local coordinate $z = u + \sqrt{-1}v$.

We next explain the generalized Gauss map of a surface $x = (x_1, \dots, x_m) : M \to \mathbb{R}^m$ immersed in $\mathbb{R}^m$.

Firstly, consider the set of all oriented 2-planes in $\mathbb{R}^m$ which contain the origin and denote it by $G_{2,\mathbb{R}^m}$. To clarify the set $G_{2,\mathbb{R}^m}$, we regard it as a subset of the $(m-1)$-dimensional complex projective space $\mathbb{P}^{m-1}(\mathbb{C})$ as follows. To each $P \in G_{2,\mathbb{R}^m}$, taking a positively oriented basis $\{X, Y\}$ of P such that

$$|X| = |Y|, (X, Y) = 0, \tag{6.1}$$

we assign the point $\Phi(P) = \pi(X - \sqrt{-1}Y)$, where π denotes the canonical projection of $\mathbb{C}^m - \{0\}$ onto $\mathbb{P}^{m-1}(\mathbb{C})$, namely, the map which maps each $p = (w_1, \dots, w_m) \neq (0, \dots, 0)$ to the equivalence class

$$[w_1 : \cdots : w_m] := \{(cw_1, \dots, cw_m);\ c \in \mathbb{C} - \{0\}\}.$$

On the other hand, $\Phi(P)$ is contained in the quadric

$$Q_{m-2}(\mathbb{C}) := \{[w_1 : \cdots : w_m]; w_1^2 + \cdots + w_m^2 = 0\}\ (\subset \mathbb{P}^{m-1}(\mathbb{C})).$$

In fact, for a positive basis $\{X, Y\}$ satisfying (6.1) we have

$$(X - \sqrt{-1}Y, X - \sqrt{-1}Y) = (X, X) - 2\sqrt{-1}(X, Y) - (Y, Y) = 0.$$

Conversely, take an arbitrary point $Q \in Q_{m-2}(\mathbb{C})$. If we choose some $W \in \mathbb{C}^m - \{0\}$ with $\pi(W) = Q$ and write $W = X - \sqrt{-1}Y$ with real vectors X and Y, then X and Y satisfy the condition (6.1) and Φ maps the oriented 2-plane P with positive basis

$\{X, Y\}$ to the point Q. This shows that Φ is injective.

Now, consider a surface $x = (x_1, \ldots, x_m) : M \to \mathbb{R}^m$ immersed in $\mathbb{R}^m$. For each point $p \in M$, the oriented tangent plane $T_p(M)$ is canonically identified with an element of $G_{2,\mathbb{R}^m}$ after the parallel translation which maps p to the origin.

Definition 6.4. The generalized *Gauss map* of a surface M is defined as the map of M into $Q_{m-2}(\mathbb{C})$ which maps each point $p \in M$ to $\Phi(T_p(M))$.

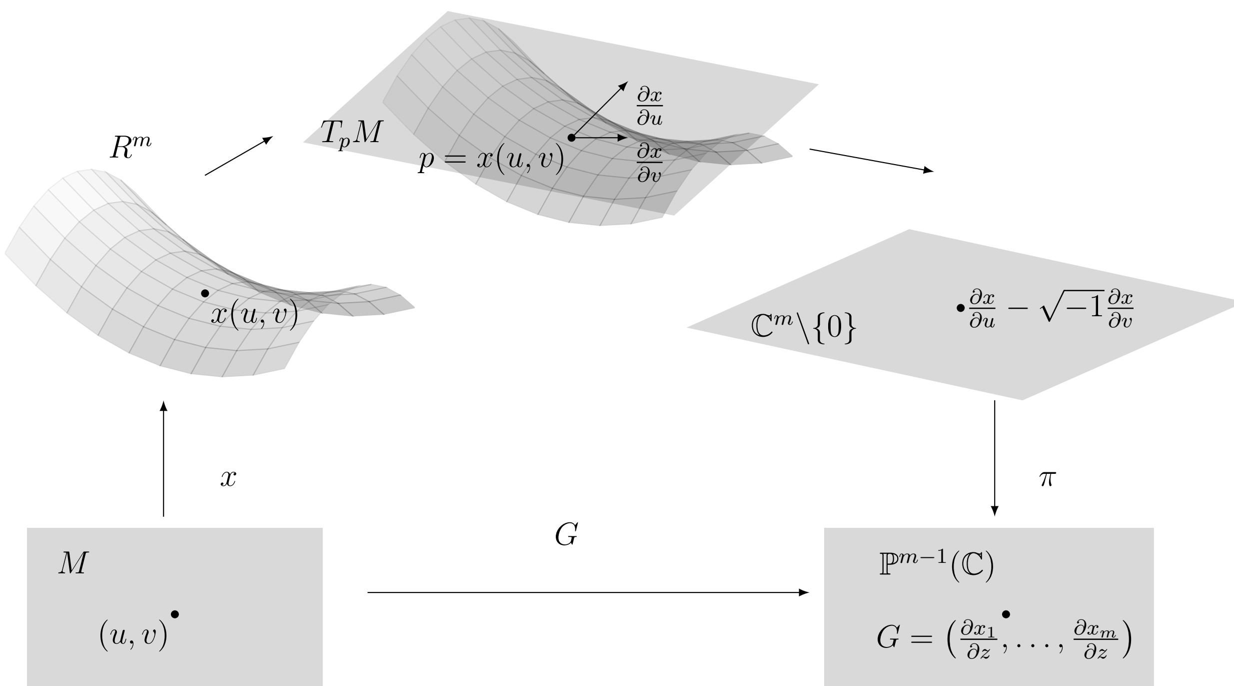

Figure 6.2: The Generalized Gauss Map

Usually, the Gauss map is defined as the conjugate of the Gauss map defined as above.

For a system of positively oriented isothermal local coordinates (u, v) the vectors

$$X = \frac{\partial x}{\partial u}, \quad Y = \frac{\partial x}{\partial v}$$

give a positive basis of $T_p(M)$ satisfying the condition (6.1) because of the isothermal condition. Therefore, the *Gauss map* of M is locally given by

$$G = \pi(X - \sqrt{-1}Y) = \left[\frac{\partial x_1}{\partial z} : \frac{\partial x_2}{\partial z} : \cdots : \frac{\partial x_m}{\partial z}\right],$$

where $z = u + \sqrt{-1}v$.

We have the following proposition for any surfaces to be minimal.

Proposition 6.5 ([HO80] Theorem 1.1)**.** *A surface $x : M \to \mathbb{R}^m$ is minimal if and only if the Gauss map $G : M \to \mathbb{P}^{m-1}(\mathbb{C})$ is holomorphic.*

Proof. Assume that M is minimal. We then have

$$\begin{aligned}
\frac{\partial}{\partial \bar{z}}\left(\frac{\partial x}{\partial z}\right) &= \frac{\partial}{\partial \bar{z}}\left(\frac{1}{2}\left(\frac{\partial x}{\partial u} - i\frac{\partial x}{\partial v}\right)\right) \\
&= \frac{1}{2}\left(\frac{1}{2}\left(\frac{\partial}{\partial u}\left(\frac{\partial x}{\partial u} - i\frac{\partial x}{\partial v}\right) + i\frac{\partial}{\partial v}\left(\frac{\partial x}{\partial u} - i\frac{\partial x}{\partial v}\right)\right)\right) \\
&= \frac{1}{4}\left(\frac{\partial^2 x}{\partial u^2} - i\frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + i\frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial^2 x}{\partial v^2}\right) \\
&= \frac{1}{4}\left(\frac{\partial^2 x}{\partial u^2} + \frac{\partial^2 x}{\partial v^2}\right) \\
&= \frac{1}{4}\Delta x \\
&= 0.
\end{aligned}$$

This shows that $\frac{\partial x}{\partial z}$ satisfies Cauchy-Riemann's equation. Hence, the Gauss map G is holomorphic.

Conversely, assume that G is holomorphic. The problem is local. For a holomorphic local coordinate z we set $f_i = \frac{\partial x_i}{\partial z}$ $(1 \leq i \leq m)$. After a suitable change of indices, we may assume that f_m has no zero. Since f_i/f_m are holomorphic, we have

$$\begin{aligned} \frac{1}{4}\Delta x_i &= \frac{\partial^2 x_i}{\partial z \partial \tilde{z}} = \frac{\partial}{\partial \tilde{z}}\left(\frac{f_i}{f_m} f_m\right) \\ &= \frac{\partial}{\partial \tilde{z}}\left(\frac{f_i}{f_m}\right) f_m + \frac{f_i}{f_m}\frac{\partial f_m}{\partial \tilde{z}} = f_i \frac{1}{f_m}\frac{\partial f_m}{\partial \tilde{z}} \end{aligned}$$

for $i = 1, 2, \ldots, m$. Write

$$\frac{1}{f_m}\frac{\partial f_m}{\partial \tilde{z}} = h_1 + \sqrt{-1}h_2$$

with real-valued functions h_1, h_2 and take the real parts of both sides of the above equation to see

$$\Delta x = 2\left(\frac{\partial x}{\partial u}h_1 + \frac{\partial x}{\partial v}h_2\right) \in T_p(M).$$

From property of Laplace operator which is $(\Delta x, X) = 0$ for each $x \in T_p(M)$, we obtain $(\Delta x, \Delta x) = 0$ and so $\Delta x = 0$. This implies that M is a minimal surface by virtue of Theorem 6.3. □

Since we will work on complete minimal surfaces, we shall explain what the "completeness" means here.

Definition 6.6. A divergent curve on a Riemannian manifold M is a differentiable map $\gamma : [0, 1) \to M$ such that for every compact subset $K \subset M$ there exist a $t_0 \in (0, 1)$

with $\gamma(t) \notin K$ for all $t > t_0$. That is, γ leaves every compact subset of M.

Definition 6.7. A Riemannian manifold M is said to be *complete* if every divergent curve $\gamma : [0, 1) \to M$ has unbounded length.

We recall the following three results obtained from Chern and Ossermann paper [CO67].

Theorem 6.8 ([CO67], Theorem 1)**.** *Let $x : M \to \mathbb{R}^m$ be a complete regular minimal surface. Then the following conditions are equivalent:*

(a) $x : M \to \mathbb{R}^m$ has finite total curvature $C(M)$,

(b) There is an integer d such that $G(M)$ intersects at most d times any hyperplane which does not contain it (the number d is called the degree of G),

(c) The Gauss map is algebraic,

(d) M is conformally equivalent to a compact surface $\bar{M}$ punctured at a finite number of points $P_1, \ldots, P_r$.

Theorem 6.9 ([CO67], Theorem 2)**.** *If $x : M \to \mathbb{R}^m$ is a complete regular surface with r boundary components, then $C(S) \leq 2\pi(\chi - r)$, where χ is the Euler characteristic.*

Theorem 6.10 ([CO67], Theorem 3)**.** *The total curvature of a complete regular surface in $\mathbb{R}^m$ is either $-\infty$ or $-2\pi d$, where d is the integer in statement (b) of Theorem 6.8*

6.2 Preliminary Results on Uniqueness Theorem for Gauss Map of Complete Minimal Surfaces in $\mathbb{R}^m$ with finite total curvature

Theorem 6.11 ([JR07], Theorem 4.1). *Consider two complete minimal surfaces $M, \tilde{M}$ immersed in $\mathbb{R}^m$ with finite total Gauss curvatures. Let $G, \tilde{G}$ be the generalized Gauss map of $M, \tilde{M}$ respectively. Suppose that there is a conformal diffeomorphism Φ between M and $\tilde{M}$ and let $G_1 := G, G_2 := \tilde{G} \circ \Phi$. Assume that G_1, G_2 are linearly non-degenerate. Let $H_1, \ldots, H_q$ be the hyperplanes in $\mathbb{P}^{m-1}(\mathbb{C})$ in general position. Assume that*

(i) $G_1^{-1}(H_j) = G_2^{-1}(H_j)$ for $j = 1, \ldots, q$,

(ii) For every $i \neq j, G_1^{-1}(H_i) \cap G_1^{-1}(H_j) = \emptyset$,

(iii) $G_1 = G_2$ on $\cup_{j=1}^{q} G_1^{-1}(H_j)$.

If $q \geq \frac{1}{2}(m^2 + 5m - 4)$ then $G_1 \equiv G_2$.

Proof. Assume that $G_1 \not\equiv G_2$. Since the minimal surface $x : M \to \mathbb{R}^m$ has finite total curvature $C(M)$, by Theorem 6.8, M is conformally equivalent to a compact surface $\bar{M}$ punctured at a finite number of points $P_1, \ldots, P_r$ and the generalized Gauss map G extends holomorphically to $G : \bar{M} \to \mathbb{P}^{m-1}(\mathbb{C})$.

By Theorem 6.9 and Theorem 6.10,

$$\begin{aligned} -2\pi \deg(G_1) &= C(S) \\ &\leq 2\pi(\chi - r) \\ &= 2\pi(2 - 2g - r - r) \end{aligned}$$

$$= \quad 2\pi(2-2g-2r).$$

As before, we can find $(c_1, \ldots, c_m) \in \mathbb{C}^m$ such that, if we let $H_c = \{c_1 x_1 + \cdots + c_m x_m = 0\}$, then $G_1^{-1}(H_c) \cap G_1^{-1}(H_j) = \emptyset$ and $G_2^{-1}(H_c) \cap G_2^{-1}(H_j) = \emptyset$ for $j = 1, \ldots, q$. We fix such H_c and let $\Phi = \frac{L_{j_0}(G_1)}{L_c(G_1)} - \frac{L_{j_0}(G_2)}{L_c(G_2)} \not\equiv 0$. Let $E_{G_1} = \cup_{j=1}^{q} G_1^{-1}(H_j)$. By the Theorem 3.5 with $E = \{P_1, \ldots, P_r\}$, we get

$$\begin{aligned}
(q-m)\deg(G_1) &\leq \sum_{j=1}^{q} \sum_{P \notin E} \min\{m-1, \upsilon_P(L_j(G_1))\} \\
&\quad + \frac{1}{2}m(m-1)\{2(g-1)+r\} \\
&\leq (m-1)|E_{G_1}| + \frac{1}{2}m(m-1)\{\deg(G_1) - r\}\} \\
&< (m-1)|E_{G_1}| + \frac{1}{2}m(m-1)\deg(G_1).
\end{aligned}$$

The above inequality also holds for G_2. Therefore, for $i = 1, 2$, $\left(q - \frac{1}{2}m(m+1)\right)\deg(G_i) < (m-1)|E_{G_i}|$. By the assumptions,

$$|E_{G_1}| = |E_{G_2}| \leq |\{\Phi = 0\}| \leq \deg(G_1) + \deg(G_2).$$

Hence

$$\left(q - \frac{1}{2}m(m+1)\right)(\deg(G_1) + \deg(G_2)) < 2(m-1)(\deg(G_1) + \deg(G_2)).$$

Therefore

$$q < \frac{1}{2}m(m+1) + 2(m-1) = \frac{1}{2}(m^2 + 5m - 4)$$

which gives a contradiction. This finishes the proof. $\square$

6.3 New Results

Theorem 6.12. *Consider two complete minimal surfaces $M, \tilde{M}$ immersed in $\mathbb{R}^m$ with finite total Gauss curvatures. Let $G, \tilde{G}$ be the generalized Gauss map of $M, \tilde{M}$ respectively. Suppose that there is a conformal diffeomorphism Φ between M and $\tilde{M}$ and let $G_1 := G, G_2 := \tilde{G} \circ \Phi$. Assume that G_1, G_2 are linearly non-degenerate. Let $H_1, \ldots, H_q$ be the hyperplanes in $\mathbb{P}^{m-1}(\mathbb{C})$ in general position. Assume that*

(i) $G_1^{-1}(H_j) = G_2^{-1}(H_j)$ for $j = 1, \ldots, q$,

(ii) Let $k \leq m-1$ be a positive integer such that $i \neq j, G_1^{-1}(\cap_{i=1}^{k+1} H_{j_i}) = \emptyset$ $(1 \leq i_1 < \cdots < i_{k+1} \leq q)$,

(iii) $G_1 = G_2$ on $\cup_{j=1}^{q} G_1^{-1}(H_j)$.

If

$$q - m - \frac{2k(m-1)q}{q - 4k + 2km} - \frac{1}{2}m(m-1) \geq 0,$$

then $G_1 \equiv G_2$.

Proof. Assume that $G_1 \not\equiv G_2$. Since the minimal surface $x : M \to \mathbb{R}^m$ has finite total curvature $C(M)$, by Theorem 6.8, M is conformally equivalent to a compact surface $\bar{M}$ punctured at a finite number of points $P_1, \ldots, P_l$ and the generalized Gauss map G_i extends holomorphically to $G_i : \bar{M} \to \mathbb{P}^{m-1}(\mathbb{C})$ for $i = 1, 2$. So we are in the similar situation in the case 2 of the proof of Theorem 4.8. Similar to (4.1) with $n = m - 1$, we get

$$\begin{aligned}
& (q-m)(\deg(G_1) + \deg(G_2)) \\
\leq \; & \frac{2k(m-1)q}{q-4k+2km}(\deg(G_1) + \deg(G_2)) + m(m-1)\{2(g-1) + l\}.
\end{aligned}$$

Now by Theorem 6.9 and Theorem 6.10, for $i = 1, 2$,

$$-2\pi \deg(G_i) = C(S) \leq 2\pi(\chi - l) = 2\pi(2 - 2g - l - l) = 2\pi(2 - 2g - 2l).$$

Hence,

$$\begin{aligned}(q-m)(\deg(G_1) + \deg(G_2)) \leq \frac{2k(m-1)q}{q-4k+2km}(\deg(G_1) + \deg(G_2)) \\ +\frac{1}{2}m(m-1)(\deg(G_1) + \deg(G_2)) - m(m-1)l.\end{aligned}$$

Hence,

$$q - m - \frac{2k(m-1)q}{q-4k+2km} - \frac{1}{2}m(m-1) < 0.$$

which gives a contradiction. This finishes the proof. □

Finally, we show that the results of Park-Ru on the general case (without the assumption that M has finite total curvature) (see [PR16]) can be derived by applying Theorem 4.17.

Theorem 6.13 ([PR16], Main Theorem)**.** *Consider two complete minimal surfaces $M, \tilde{M}$ immersed in $\mathbb{R}^m$. Let $G, \tilde{G}$ be the generalized Gauss map of $M, \tilde{M}$ respectively. Suppose that there is a conformal diffeomorphism Φ between M and $\tilde{M}$ and let $G_1 := G, G_2 := \tilde{G} \circ \Phi$. Assume that G_1, G_2 are linearly non-degenerate. Let $H_1, \ldots, H_q$ be hyperplanes in $\mathbb{P}^{m-1}(\mathbb{C})$ in general position. Assume that*

(i) $G_1^{-1}(H_j) = G_2^{-1}(H_j)$ for $j = 1, \ldots, q$,

(ii) Let $k \leq m-1$ be a positive integer such that $i \neq j, G_1^{-1}(\cap_{i=1}^{k+1} H_{j_i}) = \emptyset$ $(1 \leq i_1 < \cdots < i_{k+1} \leq q)$,

(iii) $G_1 = G_2$ *on* $\cup_{j=1}^{q} G_1^{-1}(H_j)$.

If

$$q > \frac{(m^2+m+4k)+\sqrt{(m^2+m+4)^2+16km(m+1)(m-2)}}{4},$$

then $G_1 \equiv G_2$.

Proof. By considering a universal cover, we can, without loss of generality, assume that M is either $\mathbb{C}$ or the unit-disc. In the case M is $\mathbb{C}$, then Corollary 4.9 can be applied. In the case when $M = \triangle(1)$, we notice that the metric on M is induced by $\|G(z)\|^2|dz|^2$, so we can take $\rho = 1$. Thus the theorem is derived by using Theorem 4.17 with $n = m - 1$ and $\rho = 1$. $\square$

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