LARGE TIME STEP AND OVERLAPPING GRIDS FOR CONSERVATION LAWS

A Dissertation

Presented to

the Faculty of the Department of Mathematics

University of Houston

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

By

Ilija Jegdić

May 2014

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Abstract

One focus of this dissertation is to construct a large time step Finite Volume Method for computing numerical solutions to hyperbolic systems of conservation laws. We also consider a method of overlapping spatial grids for which variants have proved to be an important consideration in large scale applications.

In practice we often run into grids which have a fairly large range of cell sizes – some cells may be relatively large compared to others which may be significantly smaller. For traditional finite volume methods, the smallest spatial cell size dictates the time step size limit when employing explicit time marching. Moreover, if a solution is obtained as a limit from a sequence of approximations which use exceedingly irregular girds, the limit solution may not even be a proper weak solution. The large time step method we propose here addresses both of these problems. We prove approximate solutions obtained are stable, and when convergent will always converge to a weak solution, regardless of relative grid cell sizes.

Overlapping grids arise often in practice in order to discretized very complicated flow domains. One problem when grids overlap is how to identify a single valued approximation. A second issue is how to interface overlapping grids in such a way to obtain a conservative scheme. The method we propose here addresses both these issues. We identify a single valued approximate solution which employs overlapping spatial grids, and we prove its limit is a weak solution. Moreover, we show the method satisfies the maximum principle and is therefore stable.

Chapter one is an introduction to the theory of hyperbolic conservation laws. In chapter two we introduce the finite volume method, approximate Riemann problem

solvers, and we establish the Lax-Wendroff Theorem for the multidimensional algorithm. In chapter three we present our large time step method and establish the theoretical results noted above. Numerical examples are also given in this chapter. In the last chapter we present our overlapping grid method. The theoretical results indicated above are proved and several numerical examples are presented.

Acknowledgements

I would like to sincerely thank my advisor, Professor Richard Sanders, for all the help and support throughout my graduate studies. Professor Sanders has taught me everything I know about the finite volume method and he helped me to expand my computational skills. He unselfishly shared his ideas with me and helped me to reach my goals.

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Also, I would like to thank my family and my friends for all their support. In particular I would like to thank my sister Katarina for all her help and support over the years. At the end I would like to dedicate this dissertation to my friends Csilla, Dara, Ljiljana, Milorad, Basrak, and Milan.

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Chapter 1

Introduction

1.1 Motivation

Conservation laws are partial differential equations that describe a variety of physical phenomena. Some examples are in gas dynamics, aerospace engineering, biological applications, etc.

The n-dimensional system of m conservation laws is given by

$$u_t^i + \nabla \cdot F_i(u) = 0, \quad i \in \{1, \dots, m\},$$

where $(x, t) \in \mathbb{R}^n \times [0, \infty)$, $u = (u^1, \dots, u^m)$ is the unknown vector valued function to be determined and $F_i = (F_1^i, \dots, F_m^i)$ is the known spatial flux defined on a domain of conservation states. Here, $\nabla \cdot F_i(u) = \sum_{j=1}^n \frac{\partial F_j^i(u)}{\partial x^j}$. The system is supplemented by an initial condition

$$u(x, 0) = u_0(x), \quad x \in \mathbb{R}^n,$$

where u_0 is a bounded and measurable function. The system is hyperbolic if any linear combination of the matrices $\left[\frac{\partial F^i}{\partial u^j}\right]$ has real eigenvalues and a complete set of eigenvectors. When the spatial flux function F is nonlinear, we have the nonlinear systems of equations. A remarkable note is that solutions of these systems may develop discontinuities even if the initial data u_0 is smooth. Therefore, the notion of weak solutions is introduced. A weak solution of the observed initial value problem is defined to be a bounded and measurable function u such that for all test functions $\varphi \in C_0^\infty(\mathbb{R}^n \times (0,\infty))$ we have

$$\int_0^\infty \int_{\mathbb{R}^n} u_t\varphi(x,t) + F(u)\cdot\nabla\varphi(x,t)\,dx\,dt + \int_{\mathbb{R}^n} u_0(x)\varphi(x,0)\,dx = 0.$$

However, a weak solution may not be unique, and to select a physically correct solution, we have to impose an extra condition which is called "entropy condition". In the case of a scalar conservation law (meaning u is a scalar function, i.e., the case $m = 1$), one such entropy condition is Kružkov Entropy Condition [8]. It states that a weak solution is the entropy solution provided

$$-\int_0^\infty \int_{\mathbb{R}^n} |u-c| + sgn\,(u-c)\,(F(u)-F(c))\cdot\nabla\varphi(x,t)\,dx\,dt \leq 0,$$

holds for all $\varphi \in C_0^\infty(\mathbb{R}^n \times (0,\infty))$, $\varphi \geq 0$ and for all $c \in \mathbb{R}$. However in case of systems (case $m > 1$), Kružkov entropy condition has not been generalized.

In the case when the flux F is nonlinear, it is very possible that for the given problem one can not find the exact solution analytically, so numerical methods have to be considered. The most-known numerical methods are finite difference, finite volume and finite element numerical methods. In finite difference methods, the system is approximated by finite differences and the solution is approximated pointwise at the grid points. In finite volume methods, the numerical solution is a picewise constant function over the given grid cells. In finite element methods, the domain is divided into subdomains and the solution is approximated most often using piecewise polynomial functions.

We recall the *method of characteristic* [2, 6] for solving initial values problems for one dimensional scalar hyperbolic conversational laws

$$u_t + f(u)_x = 0,$$

with initial condition

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}.$$

We consider a curve, $\gamma(t) = (x(t), t)$, defined by

$$x'(t) = f'\left(u\left(x(t), t\right)\right)$$

and $x(0) = x_0$. If we assume that $u(x,t)$ is a solution of a conservation law, then if differentiate u along $\gamma(t)$ we have

$$\frac{d}{dt} u\left(\gamma(t)\right) = \frac{d}{dt} u\left(x(t), t\right) = u_x x'(t) + u_t = u_x f'(u) + u_t = 0.$$

Hence, the solution u is constant along characteristic curves, i.e.,

$$u(\gamma(t)) = u(x(t), t) = u(x(0), 0) = u_0(x_0).$$

Example. The most well known-example of a nonlinear scalar conservation law is the Burgers' equation

$$u_t + f(u)_x = 0$$

where $f(u) = \frac{1}{2}u^2$. We consider the Riemann initial condition

$$u_0(x) = u(x,0) = \begin{cases} u_l, & x \leq 0 \\ u_r, & 0 < x. \end{cases}$$

Consider two cases.

1^o Let $u_l < u_r$. Following the method of characteristics, the solution is

$$u(x,t) = \begin{cases} u_l, & x \geq u_l t \\ u_r, & u_r t > x \end{cases}$$

and the question is what will be solution for $x \in (u_l t, u_r t)$. If we smooth our initial

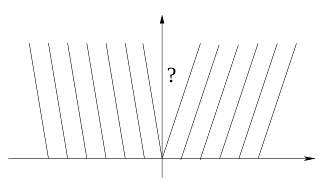
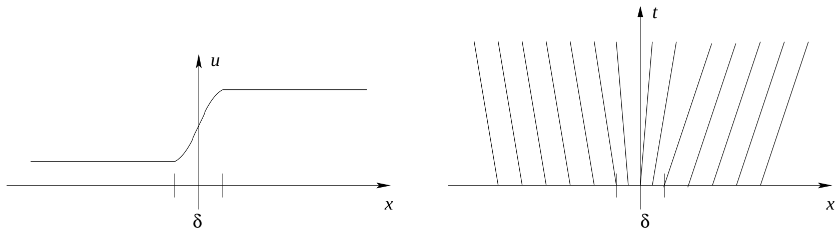


Figure 1.1: Characteristics and unknown part

data,

$$\widetilde{u}_\delta(x) = \begin{cases} u_l, & x \leq -\delta \\ w_\delta(x), & -\delta < x \leq \delta \\ u_r, & \delta < x, \end{cases}$$

where $w_\delta(x)$ is some increasing function such that $\widetilde{u}_\delta$ is a smooth function and



(a) x-u plane (b) Characteristics

Figure 1.2: Graph of $\widetilde{u}_\delta(x,t)$

$\widetilde{u}_\delta \to u_0$ as $\delta \to 0$, we get solution on all $\mathbb{R}$. Taking $\delta \to 0$ we get a solution which is constant along the rays $\frac{x}{t}$ and we can take

$$u(x,t) = \begin{cases} u_l, & x \geq u_l t \\ v\left(\frac{x}{t}\right), & u_l t < x \leq u_r t \\ u_r, & u_r t > x, \end{cases}$$

where v is some smooth function. From here we have $u_t = -\frac{x}{t^2}v'$ and $u_x = \frac{1}{t}v'$. Since $u_t + uu_x = 0$, therefore,

$$-\frac{x}{t^2}v' + \frac{1}{t}v'v = 0.$$

If we denote $\zeta = \frac{x}{t}$ we get

$$-\zeta v' + vv' = 0,$$

from where $v(\zeta) = \zeta$. Therefore,

$$u(x,t) = \frac{x}{t}$$

and a new solution for Burgers' equation is

$$u(x,t) = \begin{cases} u_l, & x \leq u_l t \\ \frac{x}{t}, & u_l t < x \leq u_r t \\ u_r, & u_r t < x. \end{cases}$$

This solution is called a rarefaction wave, and one can show that it is an entropy solution.

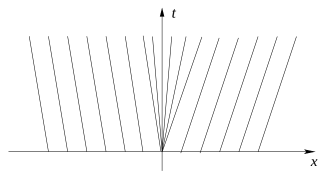


Figure 1.3: Characteristics of a rarefaction wave

If we take

$$u(x,t) = \begin{cases} u_l, & x \geq st \\ u_r, & st > x, \end{cases}$$

where s is the shock speed obtained by Rankine-Hugoniot jump condition:

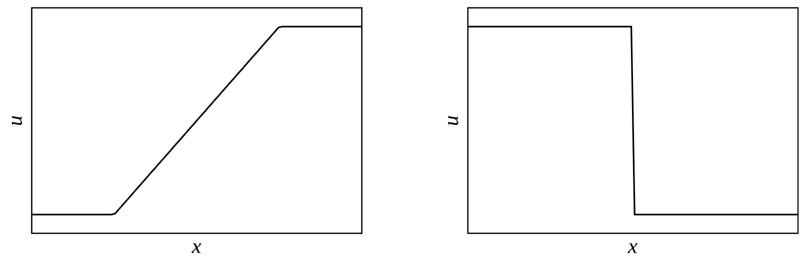
$$s(u_l - u_r) = f(u_l) - f(u_r),$$

we will get a weak solution. Notice that characteristics are pointing out from the shock. One can show that this solution is not an entropy solution.

2^o Let $u_l > u_r$. For this case we can not use characteristics because they cross for any $t > 0$. However, from the Rankine-Hugoniot jump condition we have

$$u(x,t) = \begin{cases} u_l, & x \leq st \\ u_r, & st < x, \end{cases}$$

where s is the shock speed. This solution is called the shock wave and $x = st$ is called the shock. One can show that this solution is also the entropy solution for Burgers' equation provided $u_l > u_r$. Notice that in this case characteristics are pointing into



(a) Rarefaction wave (b) Shock wave

Figure 1.4: Burgers' equation

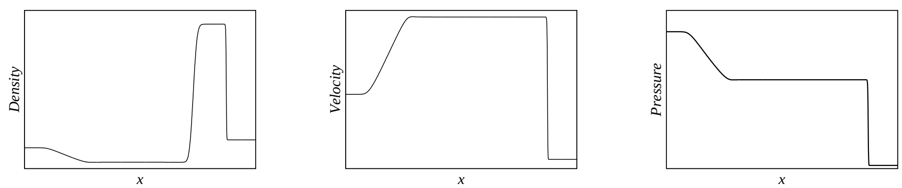
the shock.

Example. The most famous example of a system of conservation laws is the Euler system of gas dynamics equations. In one spatial dimension, the physical experiment consists of a thin tube divided in two parts by a membrane. Both parts are filled with a gas, in general, with different densities, pressures, and velocities. If the membrane is broken at time $t = 0$, the equations that describe density, pressure, and energy are given as three conservation laws (conservation of mass, linear momentum, and energy)

$$\begin{bmatrix} \rho \\ \rho v \\ \rho e \end{bmatrix}_t + \begin{bmatrix} \rho v \\ \rho v^2 + p \\ (\rho e + p) v \end{bmatrix}_x = 0$$

where ρ is density, v is velocity, e is total energy, and p pressure. The internal energy is defined to be $\varepsilon = e - \frac{1}{2}v^2$ and pressure is given by $p = (\gamma - 1)\rho\varepsilon$, where $\gamma > 1$ is constant depending on physical properties of the gas. For diatomic gases $\gamma = \frac{7}{5}$ and for monatomic gases $\gamma = \frac{5}{3}$. The speed of sound is computed as $c = \sqrt{(\gamma - 1)\gamma\varepsilon}$.

The above physical experiment corresponds to the Riemann initial value problem. Following illustrations represent solution of the given system with the initial data in so-called primitive variables, $\rho_l = 0.445$, $v_l = 0.698$, $p_l = 3.528$, $\rho_r = 0.5$, $v_r = 0$, and $p_r = 0.571$, with $\gamma = \frac{7}{5}$. Solution for this problem consists of four constant states separated by a rarefaction wave, a contact discontinuity, and a shock, and it is called the Lax shock-tube problem.



(a) Density (b) Velocity (c) Pressure

Figure 1.5: Euler equation

Chapter 2

Finite Volume Method

2.1 Introduction and Basic Ideas

In this chapter we will present finite volume method. Let us consider the d-dimensional system

$$u_t + \nabla \cdot F(u) = 0,$$

where u is unknown vector valued function defined on $\mathbb{R}^d \times [0, +\infty)$ and F is flux function defined on a domain of conversation states. Let this system be supplemented by an initial condition

$$u(x, 0) = u_0(x),$$

where u_0 is a function of bounded variation. Our goal is to find approximate solution of u at the specific time T. As we mentioned earlier, a solution obtained through finite volume method is piecewise constant function and that solution is obtained by calculating solution of local Riemann problems. In the other words, if we observe any two neighboring cells in a grid, what we have is actually a Riemann problem and by solving that Riemann problem we are getting solution in those two cells at some new time. And by repeating the procedure for all neighboring cells we are getting solution in some new time.

Let us assume that $\mathbb{R}^d$ has partition consisting of cells $\{\Omega_i \mid i \in I\}$, and with $|\Omega_i|$ we denote size of cell i.

As we know, solution obtained through finite volume method is piecewise constant function in every time step and we denote it by

$$u^n(x) = \sum_I u_i^n \chi_{\Omega_i}(x),$$

where χ represents characteristic function and u_i^n represents cell average of cell i at time step n. Clearly we have $u^0(x) = \sum_I u_i^0 \chi_{\Omega_i}(x)$, where u_i^0 is calculated using the initial data by $u_i^0 = \frac{1}{\Omega_i} \int_{\Omega_i} u_0(x)\, dx$.

Beside space, time is also discretized, meaning that if we are looking for solution at time T, then we are going to be required to do a series of time steps of size Δt until we reach the desired time. Times steps are defined by

$$\Delta t M \leq \min_i \frac{|\Omega_i|}{|\partial\Omega_i|} CFL,$$

where CFL constant is given by Courant - Friedrichs - Lewy condition, $|\partial\Omega_i|$ is the size of Ω_i's boundary, and M is constant proportional to fastest wave speed.

If we go back to our problem, $u_t + \nabla \cdot F(u) = 0$, and if we integrate it over the cell $\Omega_i \times [t_n, t_{n+1}]$ and use The Divergence Theorem, we get

$$\begin{aligned} 0 &= \int_{t_n}^{t_{n+1}} \int_{\Omega_i} u_t + \nabla \cdot F(u)\, dx\, dt \\ &= \int_{\Omega_i} u\left(x, t_{n+1}\right) - u\left(x, t_n\right)\, dx + \int_{t_n}^{t_{n+1}} \sum_k \int_{S_{i,k}} n_k \cdot F(u)\, ds\, dt, \end{aligned}$$

where $S_{i,k}$ is edge between cells i and k and n_k is corresponding outward normal. Since u is unknown function we approximate $\int_{\Omega_i} u\left(x, t_n\right)\, dx$ using the average of the approximate solution in cell i at time step n. Also $n_k \cdot F(u)$ we approximate using a numerical flux function $h_{n_k \cdot F}$, we get

$$0 = |\Omega_i| \left(u_i^{n+1} - u_i^n\right) + \Delta t \sum_k \int_{S_{i,k}} h_{n_k \cdot F}\, ds,$$

where $\Delta t = t_{n+1} - t_n$ represents time step from time t_n to time t_{n+1}.

For the numerical flux function, $h_{n_k \cdot F}$, we can take any approximation of $n_k \cdot F(u)$. The number of states on which $h_{n_k \cdot F}$ depends is not fixed, and usually it is taken to depend on two states, u_i^n and u_k^n, where Ω_i and Ω_k are two neighboring cells.

For numerical flux, we will say that it is *consistent* with $n_k \cdot F(u)$ if for all u

$$h_{n_k \cdot F}(u, u) = n_k \cdot F(u).$$

We also require for numerical flux to be *locally Lipschitz continuous*, i.e.

$$\left|h_{n_k \cdot F}\left(u_1, v_1\right) - h_{n_k \cdot F}\left(u_2, v_2\right)\right| \leq Lip\left(|u_1 - u_2| + |v_1 - v_2|\right),$$

where *Lip* is some positive constant. Lastly, we require that

$$h_{n_k \cdot F}(u_i, u_k) = -h_{-n_k \cdot F}(u_k, u_i),$$

meaning that flux that enters the cell i through edge $S_{i,k}$ is equal to the flux that exits cell k through edge $S_{i,k}$. This last condition will ensure that finite volume scheme is conservative.

As we mentioned earlier, numerical fluxes are approximation of $n_k \cdot F(u)$, and one of the methods for designing two-point numerical fluxes lies in Riemann solvers and it will be presented in following section.

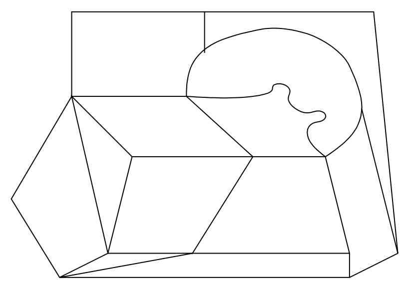


Figure 2.1: Grid in finite volume method

2.2 One-dimensional Riemann Solvers

In this section we discuss notion of one-dimensional Riemann Solvers and how they can be used for construction of numerical fluxes. Riemann Solvers were introduced by Harten [5].

Let us consider the Riemann problem for the scalar conservation law

$$u_t + f(u)_x = 0 \tag{CL}$$

with the initial condition

$$u(x,0) = u_0(x) = \begin{cases} u_l, & x \le 0 \\ u_r, & 0 < x. \end{cases}$$

From the theory we know that solution is constant along the rays $\frac{x}{t} = \mathit{const}$. We define function $R\left(u_l, u_r, \frac{x}{t}\right)$ to be exact solution of the above Riemann problem, i.e.,

$$u(x,t) = R\left(u_l, u_r, \frac{x}{t}\right).$$

This function R is called the *exact Riemann solver*. Let a be fastest wave speed, then for $x < -|a|t$ we have $R\left(u_l, u_r, \frac{x}{t}\right) = u_l$ and for $x > |a|t$ we have $R\left(u_l, u_r, \frac{x}{t}\right) = u_r$ and for $x \in (-|a|t, |a|t)$, $R\left(u_l, u_r, \frac{x}{t}\right)$ will give us exact value of solution at (x,t).

Now let us turn our attention to approximate Riemann solvers. We will say that function $R^A\left(u_l, u_r, \frac{x}{t}\right)$ is *approximate Riemann solver* if it satisfies the next two conditions

1. $R^A\left(u, u, \frac{x}{t}\right) = u$,
2. $\int_0^\Gamma \left(R^A\left(u_l, u_r, \frac{x}{t}\right) - u_r\right) - \left(R^A\left(u_l, u_r, -\frac{x}{t}\right) - u_l\right)\, dx + tf(u_r) - tf(u_l) = 0$

   for all $t > 0$ and $\Gamma > |a|t$.

If we integrate (CL) over the set $[-\Gamma, \Gamma] \times [0,t]$ and use fact that $R\left(u_l, u_r, \frac{x}{t}\right)$ solves this problem exactly, we will get that exact Riemann solver satisfies the second condition. Hence, for every $t > 0$ and $\Gamma > |a|t$

$$\int_{-\Gamma}^{\Gamma} R^A\left(u_l, u_r, \frac{x}{t}\right) - R\left(u_l, u_r, \frac{x}{t}\right)\, dx = 0.$$

In general, an approximate Riemann solver does not have to have finite speed of propagation. An example of that can be seen in the work of Perthame [10].

We finally define two-point numerical flux function $h_f(u_l, u_r)$ by

$$h_f(u_l, u_r) = \frac{1}{t}\int_0^{\Gamma} \left(R^A\left(u_l, u_r, \frac{x}{t}\right) - u_r\right)\, dx + f(u_r).$$

From the second condition we also have

$$h_f(u_l, u_r) = -\frac{1}{t}\int_{-\Gamma}^{0} \left(R^A\left(u_l, u_r, \frac{x}{t}\right) - u_l\right)\, dx + f(u_l).$$

We will say that two-point numerical flux is *monotone* if numerical flux function $h_f(u_l, u_r)$ is monotonically increasing in first argument, u_l, and monotonically decreasing in its second argument, u_r.

We will say that two-point monotone numerical flux can be *split* if we can separate monotonically increasing and monotonically decreasing parts of numerical flux. For two-point numerical flux it means $h_f(u_l, u_r) = h^+(u_l) + h^-(u_r)$, where h^+ is increasing function and h^- is decreasing function.

Next we list several well known two-point numerical fluxes.

2.2.1 Godunov Numerical Flux

Godunov numerical flux [3] is obtained by taking $R^A(u_l, u_r, 0) = R(u_l, u_r, 0)$. In the other words, Godunov numerical flux gives the exact solution for the Riemann problem. For one-dimensional problems it is given by

$$h_f(u_l, u_r) = \begin{cases} \min_{u \in [u_l, u_r]} f(u), & u_l \leq u_r \\ \max_{u \in [u_r, u_l]} f(u), & u_r \leq u_l. \end{cases}$$

Notice that Godunov flux is monotone, but it can not be split.

We remark that in the case of systems Godunov numerical flux is quite complicated to calculate.

2.2.2 Lax - Friedrichs Numerical Flux

Lax - Friedrichs numerical flux is obtained by taking

$$R^A\left(u_l, u_r, \frac{x}{t}\right) = \begin{cases} u_l, & x \le -|a|t \\ u_m, & -|a|t \le x \le |a|t \\ u_r, & |a|t \le x, \end{cases}$$

where

$$u_m = \frac{1}{2|a|t} \int_{-|a|t}^{|a|t} R\left(u_l, u_r, \frac{x}{t}\right) dx.$$

Integrating (CL) we get

$$u_m = \frac{1}{2}\left(u_l + u_r\right) - \frac{1}{2|a|}\left(f(u_r) - f(u_l)\right).$$

And, now we have

$$\begin{aligned} h_f\left(u_l, u_r\right) &= \frac{1}{t}\int_0^{|a|t} R^A\left(u_l, u_r, \frac{x}{t}\right) - u_r\, dx + f(u_r) \\ &= \frac{1}{t}\int_0^{|a|t} u_m - u_r\, dx + f(u_r) \\ &= \frac{|a|}{2}\left(u_l - u_r\right) + \frac{1}{2}\left(f(u_r) + f(u_l)\right). \end{aligned}$$

Notice that Lax - Friedrichs flux is both monotone and it can be split using

$$h^+(u_l) = \frac{1}{2}\left(|a|u_l + f(u_l)\right)$$

and

$$h^{-}(u_r) = \frac{1}{2}\left(-|a|u_r + f(u_r)\right).$$

2.2.3 Roe Numerical Flux

In 1981. Roe [11] suggested a new scheme for solving the Riemann problem. Instead of solving the exact Riemann problem he suggested solving the approximate problem

$$u_t + Au_x = 0 \tag{AppP}$$

with initial condition

$$u_0(x) = \begin{cases} u_l, & x \leq 0 \\ u_r, & x > 0, \end{cases}$$

where matrix A satisfies following conditions:

1. For any u_l and u_r, $A(u_l - u_r) = f(u_l) - f(u_r)$,

2. All eigenvalues of A are real and matrix A is diagonalizable,

3. $A(u_l, u_r) \to \frac{\partial f}{\partial u}(u)$ as $u_l \to u$ and $u_r \to u$.

The first condition is also known as the "Roe Condition" and matrix A as a Roe Matrix. In the case of scalar conservation laws A reduces to the Rankine-Hugonit shock speed. In general a Roe Matrix is not unique and there are many ways it can be computed, but we will not go into these details at this time.

Let matrices L and R be normalized matrices whose rows and columns are left and right eigenvectors of A, respectively, such that $LR = I$. Let $\Lambda = diag(\lambda_i)$ where

λ_i are eigenvalues of matrix A. One can show that solution of (AppP) can be written as

$$u(x,t) = u_l + RH\left(\frac{x}{t}I - \Lambda\right)L(u_r - u_l),$$

where $H\left(\frac{x}{t}I - \Lambda\right) = diag\left(h\left(\frac{x}{t} - \lambda_i\right)\right)$ and

$$h\left(\frac{x}{t} - \lambda_i\right) = \begin{cases} 0, & \frac{x}{t} - \lambda_i \leq 0 \\ 1, & \frac{x}{t} - \lambda_i > 0. \end{cases}$$

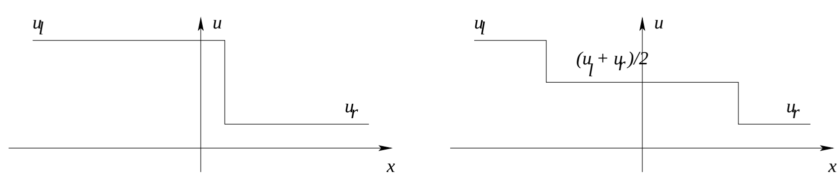
Now, taking

$$R^A\left(u_l, u_r, \frac{x}{t}\right) = u(x,t) = u_l + RH\left(\frac{x}{t}I - \Lambda\right)L(u_r - u_l),$$

we get

$$h_f(u_l, u_r) = \frac{1}{2}\left(f(u_r) + f(u_l) - |A|\left(u_r - u_l\right)\right),$$

where $|A| = R\,|\Lambda|\,L$ and $|\Lambda| = diag\left(|\lambda_i|\right)$.



(a) Regular solver (b) Fixed solver

Figure 2.2: Roe Riemann solvers

Unfortunately, solutions obtained with Roe numerical flux do not have necessarily to be entropy solutions. A fix for the scheme was derived by Harten and Hyman [5] the same year. Let $F\left(\frac{x}{t}, \varepsilon, \Lambda\right) = diag\left(F_i\left(\frac{x}{t}, \varepsilon_i, \lambda_i\right)\right)$, where

$$F_i\left(\frac{x}{t}, \varepsilon_i, \lambda_i\right) = \begin{cases} 0, & \lambda_i + \varepsilon_i \leq \frac{x}{t} \text{ or } \frac{x}{t} < \lambda_i - \varepsilon_i \\ -\frac{1}{2}, & \lambda_i - \varepsilon_i < \frac{x}{t} < \lambda_i \\ \frac{1}{2}, & \lambda_i < \frac{x}{t} < \lambda_i + \varepsilon_i \end{cases}$$

and where for example

$$\varepsilon_i \geq \max\left\{0, \lambda(u_m) - \lambda(u_l), \lambda(u_r) - \lambda(u_m)\right\}.$$

State u_m is such that Jacobian of $f(u_m)$ is A. Let us take

$$R^A\left(u_l, u_r, \frac{x}{t}\right) = u_l + R\left(H\left(\frac{x}{t}I - \Lambda\right) + F\left(\frac{x}{t}, \varepsilon, \Lambda\right)\right) L(u_r - u_l)$$

which gives

$$h_f(u_l, u_r) = \frac{1}{2}\left(f(u_r) + f(u_l) + R\, diag\left(\max\left\{\varepsilon_i, |\lambda_i|\right\}\right) L(u_r - u_l)\right).$$

We note that solutions obtained with this flux are entropy solutions, but, unfortunately Roe numerical flux and fix of Roe numerical flux can not be split.

2.2.4 Engquist-Osher Numerical Flux

The last numerical flux that we will present in this section is the Engquist-Osher [4] numerical flux. Here we generate the approximate Riemann solver as follows. Allow the solution to evolve according to characteristics. Of course this may result in a multivalued solution. Applying the Brenier pointwise projection operator [1] we arrive at

$$\begin{aligned} R^A\left(u_l, u_r, \frac{x}{t}\right) = & \sum_{\substack{u \in I(u_l, u_r) \\ u \in (f')^{-1}\left(\frac{x}{t}\right)}} u\, sgn\left(f''(u)(u_r - u_l)\right) \\ & + u_l\left(1 - h\left(\frac{x}{t} - f'(u_l)\right)\right) + u_r h\left(\frac{x}{t} - f'(u_r)\right) \end{aligned}$$

where $u \in I(u_l, u_r) = \{u|\min\{u_l, u_r\} \leq u \leq \max\{u_l, u_r\}\}$. Instead of integrating $R^A - u_r$ with respect to dx we can calculate the value by integrating with respect to

du. This yields

$$\frac{1}{t}\int_0^{\Gamma}\left(R^A\left(u_l,u_r,\frac{x}{t}\right)-u_r\right)dx=\int_{u_r}^{u_l}\max\left\{0,f'(u)\right\}\,du,$$

and so

$$h_f(u_l,u_r)=f(u_r)-\int_{u_l}^{u_r}\max\left\{0,f'(u)\right\}\,du.$$

Usually the Engquist-Osher numerical flux is written as $f^+(u_l)+f^-(u_r)$ where $f^+(u)=\int_0^u\max\left\{f'(u),0\right\}\,du$ and $f^-(u)=\int_0^u\min\left\{f'(u),0\right\}\,du$. With this notation we calculate that

$$h_f(u_l,u_r)=f^+(u_l)+f^-(u_r)+f(0).$$

We note that the Engquist-Osher numerical flux can be split. The Engquist-Osher numerical flux can be generalized for systems, Osher-Solomon [9].

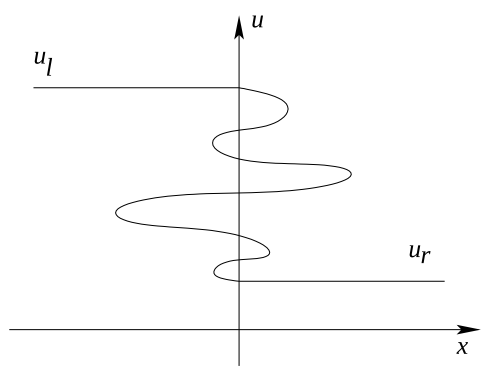


Figure 2.3: Engquisit-Osher Riemann Solver

2.3 Properties of Finite Volume Method

In the previous two sections we derived the finite volume method and we showed how several well known numerical fluxes are constructed. Now we recall finite volume method scheme as

$$0 = |\Omega_i| \left(u_i^{n+1} - u_i^n\right) + \Delta t \sum_k \int_{S_{i,k}} h_{n_k \cdot F}\, ds.$$

In one-dimension the finite volume method reads

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x_i} \left(h_f\left(u_i^n, u_{i+1}^n\right) - h_f\left(u_{i-1}^n, u_i^n\right)\right).$$

In this case Lax and Wendroff [7] have shown that if the approximate solution converges boundedly almost everywhere to a function $u(x,t)$ then $u(x,t)$ is a weak solution to the Conservation Law. Moreover, for the scalar one-dimensional equation employing monotone numerical flux functions it is known that the limit function also satisfies the Kružkov entropy condition. In case of cartesian product of one-dimensional grids this is easily extended to multiple dimensions. In the following section we will prove Lax Wendroff theorem in the n-dimensional case where grid does not have to be so simple.

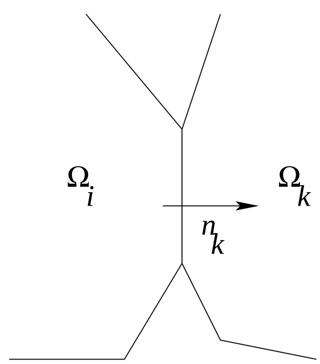


Figure 2.4: Cells and normal between them

The traditional way of designing a numerical flux for the n-dimensional case would be to use $f(u) = n_k \cdot F(u)$ and then to use ones favorite one-dimensional numerical flux to approximate $f(u)$. For example for the Lax - Friedrichs numerical flux we have

$$f(u) = \frac{1}{2}\left(F(u_r) + F(u_l)\right) \cdot n_k - a(u_r - u_l),$$

where a is a sufficiently large parameter. However, in our latter work we will use a different approach.

2.4 The Lax–Wendroff Theorem

Consider an n-dimensional system of hyperbolic conservation laws

$$u_t + \nabla \cdot F(u) = 0 \;\text{ in }\; \mathbb{R}^d \times (0, \infty), \quad u(x,0) = u_0(x), \quad x \in \mathbb{R}^d.$$

A bounded and measurable function, $u(x,t)$, is said to be a weak solution to the conservation law if for every smooth test function $\phi(x,t)$ having compact support we have

$$\int_0^\infty \int_{\mathbb{R}^d} \left(u\,\phi_t + F(u) \cdot \nabla\phi\right)\,dxdt + \int_{\mathbb{R}^d} u_0(x)\phi(x,0)\,dx = 0.$$

Now consider a partition of $\mathbb{R}^d \times [0, \infty)$

$$\Delta = \left\{\Omega_i \times [t_n, t_{n+1}) \;\mid\; i \in I,\; n \geq 0\right\}, \quad \mathbb{R}^d = \bigcup_{i \in I} \Omega_i.$$

The finite volume scheme considered here is written as follows: For each $i \in I$ update from time level t_n to t_{n+1} according to

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{|\Omega_i|} \sum_{k \in K_i} \int_{S_{i,k}} h_{F \cdot n_{i,k}}\left(u_i^n, u_k^n\right)\,ds,$$

where K_i denotes the set of cell indices k with Ω_k adjacent to Ω_i, $|\Omega_i|$ is the volume of cell Ω_i, $S_{i,k}$ is the surface between cells i and k, $n_{i,k}$ is the outward unit normal to Ω_i along $S_{i,k}$, and $\Delta t = t_{n+1} - t_n$. For ease of presentation below, we assume each normal $n_{i,k}$ is constant along side $S_{i,k}$ (with area $|S_{i,k}|$), and so

$$\int_{S_{i,k}} h_{F\cdot n_{i,k}}(u_i, u_k)\, ds = |S_{i,k}|\, h_{F\cdot n_{i,k}}(u_i, u_k).$$

Recall the numerical flux is required to be Lipschitz in both arguments as well as to satisfy

Consistency: $h_{F\cdot n}(u, u) = F(u) \cdot n.$

Conservative: $h_{F\cdot(-n)}(u_1, u_2) = -h_{F\cdot n}(u_2, u_1).$

From the computed values u_i^n, we write a piecewise constant approximate solution to the conservation law as

$$u_\Delta(x,t) = \sum_{\substack{i\in I,\\ n\geq 0}} u_i^n\, \chi_{\Omega_i}(x)\, \chi_{[t_n,t_{n+1})}(t),$$

where $\chi_{\Omega_i}(x)$ is the characteristic function of the spatial cell Ω_i and $\chi_{[t_n,t_{n+1})}(t)$ is the characteristic function of the time interval $[t_n, t_{n+1})$.

The Lax–Wendroff Theorem is often stated as follows. If $u_\Delta(x,t) \to u(x,t)$ boundedly almost everywhere on $\mathbb{R}^d \times [0,\infty)$ as $|\Delta| \to 0$ then the limit function, $u(x,t)$, is a weak solution to the conservation law. On multidimensional grids which are given by a Cartesian product of one-dimensional grids the proof is usually accomplished by employing *summation by parts.* Here however, due to the general nature of the finite volume grid, we employ *integration by parts*. With this goal in mind, consider the following.

For each $i \in I$ and $k \in K_i$ solve (up to an additive constant)

$$\nabla^2 \theta_{i,k} = \frac{|S_{i,k}|}{|\Omega_i|} \text{ in } \Omega_i, \text{ with } \frac{\partial \theta_{i,k}}{\partial n} = \begin{cases} 1 & \text{when } x \in S_{i,k} \\ 0 & \text{when } x \in \partial\Omega_i \backslash S_{i,k}, \end{cases}$$

and for $x \in \Omega_i$ define

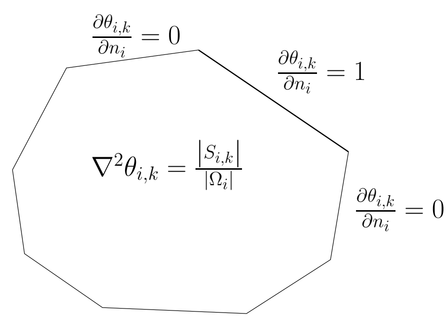


Figure 2.5: Cell, normal and boundary conditions

$$V_{i,k}(x) = \nabla \theta_{i,k}(x) \quad \Rightarrow \quad \nabla \cdot V_{i,k} = \frac{|S_{i,k}|}{|\Omega_i|}.$$

We assume the finite volume grid is sufficiently regular such that the following is valid

$$||V_{i,k}||_{\infty,\Omega_i} = C_{i,k} \leq C \text{ for all } i \in I,\ k \in K_i,$$

and this remains uniformly valid as the partition Δ tends to zero.

Remark: This assumption is obviously valid, with $C = 1$, when the spatial partition is rectangular. More generally, if the assumption is valid for a given partition with constant C it is valid for any positive scaling of the given partition with the same constant.

We require one additional constraint on our finite volume spatial partition. Let $B_\Delta(x)$ denote the ball centered at $x \in \mathbb{R}^d$ with radius $r(x)$ such that

$$x \in \Omega_i \quad \Rightarrow \quad B_\Delta(x) = B(x, r(x)) \text{ with } r(x) = \inf \left\{ r \mid \cup_{k \in K_i} \Omega_k \subseteq B(x,r) \right\}.$$

We assume there is a positive constant D such that

$$\text{for every } i \in I \sup_{x \in \Omega_i,\, k \in K_i} \frac{|B_\Delta(x)|}{|\Omega_k|} \leq D, \tag{C2}$$

which is also uniformly valid as the partition $|\Delta|$ tends to zero.

With vector valued functions $V_{i,k}$ given above observe that if for $x \in \Omega_i$ and $t \in [t_n, t_{n+1})$ we define

$$\mathcal{F}_i^n(x,t) = \sum_{k \in K_i} \left(\frac{1}{|S_{i,k}|} \int_{S_{i,k}} h_{F \cdot n_{i,k}} \left(u_i^n, u_k^n\right) \, ds \right) V_{i,k}(x),$$

$$\text{then} \quad \nabla \cdot \mathcal{F}_i^n(x,t) = \frac{1}{|\Omega_i|} \sum_{k \in K_i} \int_{S_{i,k}} h_{F \cdot n_{i,k}} \left(u_i^n, u_k^n\right) \, ds.$$

Similarly, but easier to see, if for $x \in \Omega_i$ and $t \in [t_n, t_{n+1})$ we define

$$\mathcal{U}_i^n(x,t) = \frac{t - t_n}{\Delta t} \, u_i^{n+1} + \frac{t_{n+1} - t}{\Delta t} \, u_i^n,$$

$$\text{then} \quad \frac{\partial}{\partial t} \mathcal{U}_i^n(x,t) = \frac{1}{\Delta t} \left(u_i^{n+1} - u_i^n \right).$$

So if we define $\mathcal{F}(x,t) = \sum_{n \geq 0} \sum_{i \in I} \mathcal{F}_i^n(x,t) \, \chi_{\Omega_i}(x) \chi_{[t_n, t_{n+1})}(t)$ and $\mathcal{U}(x,t)$ similarly, the finite volume scheme can be used to conclude for almost every $x \in \mathbb{R}^d$ and $t \geq 0$

$$\frac{\partial}{\partial t} \mathcal{U} + \nabla \cdot \mathcal{F} = 0.$$

Moreover, the fact that $h_{F\cdot n_{i,k}}(u_i, u_k) = -h_{F\cdot n_{k,i}}(u_k, u_i)$ implies $\mathcal{F}\cdot n$ is continuous across Ω_i cell boundaries. Also, clearly $\mathcal{U}$ is continuous in t. Therefore, for any smooth and compactly supported test function ϕ, integration by parts can be applied allowing us to write

$$\int_0^\infty \int_{\mathbb{R}^d} (\mathcal{U}\,\phi_t + \mathcal{F}\cdot\nabla\phi)\ dxdt + \int_{\mathbb{R}^d} \mathcal{U}(x,0)\phi(x,0)\,dx = 0.$$

From this we add and subtract terms to finally write

$$\int_0^\infty \int_{\mathbb{R}^d} (u_\Delta\,\phi_t + F(u_\Delta)\cdot\nabla\phi)\ dxdt + \int_{\mathbb{R}^d} u_\Delta(x,0)\,\phi(x,0)\,dx = I + II + III,$$

where terms I, II and III are given by

$$\begin{aligned} I &= \int_0^\infty \int_{\mathbb{R}^d} (u_\Delta - \mathcal{U})\,\phi_t\,dxdt, \\ II &= \int_{\mathbb{R}^d} (u_\Delta(x,0) - \mathcal{U}(x,0))\,\phi(x,0)\,dx, \\ III &= \int_0^\infty \int_{\mathbb{R}^d} (F(u_\Delta) - \mathcal{F})\cdot\nabla\phi\,dxdt. \end{aligned}$$

Now, to prove the Lax-Wendroff Theorem, suppose $u_\Delta \to u$ boundedly almost everywhere as the grid size $\Delta \to 0$. It is not difficult to show terms I and II above tend to zero as $|\Delta|$ tends to zero, and so these details are omitted. Due to the general nature of the spatial partition however, the treatment of term III is somewhat more delicate. The remainder of this section is devoted to establishing the fact that term III also tends to zero with $|\Delta|$. Once done the bounded convergence theorem can then be applied to conclude

$$\begin{aligned} &\int_0^\infty \int_{\mathbb{R}^d} (u\,\phi_t + F(u)\cdot\nabla\phi)\ dxdt + \int_{\mathbb{R}^d} u(x,0)\,\phi(x,0)\,dx \\ &\qquad = \lim_{|\Delta|\to 0} \left(\int_0^\infty \int_{\mathbb{R}^d} (u_\Delta\,\phi_t + F(u_\Delta)\cdot\nabla\phi)\ dxdt + \int_{\mathbb{R}^d} u_\Delta(x,0)\,\phi(x,0)\,dx \right) \\ &\qquad = \lim_{|\Delta|\to 0} (I + II + III) = 0, \end{aligned}$$

which is the main result of this section.

In order to estimate term III, break it up as follows

$$III = \sum_{n=0}^{\infty} \sum_{i \in I} \int_{t_n}^{t_{n+1}} \int_{\Omega_i} \left(F(u_i^n) - \mathcal{F}_i^n(x,t)\right) \cdot \nabla \phi \, dx dt,$$

and recall

$$\begin{aligned}\mathcal{F}_i^n(x,t) &= \sum_{k \in K_i} \left(\frac{1}{|S_{i,k}|} \int_{S_{i,k}} h_{F \cdot n_{i,k}}(u_i^n, u_k^n) \, ds \right) V_{i,k}(x) \\ &= \sum_{k \in K_i} h_{F \cdot n_{i,k}}(u_i^n, u_k^n) \, V_{i,k}(x).\end{aligned}$$

We now claim the following identity is valid.

$$F(u_i) = \sum_{k \in K_i} h_{F \cdot n_{i,k}}(u_i, u_i) \, V_{i,k}(x).$$

To see this is true, first observe by consistency that $h_{F \cdot n_{i,k}}(u_i, u_i) = F(u_i) \cdot n_{i,k}$. Next observe that by appealing to the usual uniqueness theorem for Laplace's equation with compatible Neumann boundary data every solution to

$$\nabla^2 \theta_i = 0 \text{ in } \Omega_i, \quad \text{with} \quad \left.\frac{\partial \theta_i}{\partial n}\right|_{x \in S_{i,k}} = F(u_i) \cdot n_{i,k},$$

must be of the form $\theta_i(x) = F(u_i) \cdot x + const$. Recall from earlier how the functions $\theta_{i,k}(x)$ were defined and that $V_{i,k}(x) = \nabla \theta_{i,k}(x)$. With these, superposition allows us to write

$$\theta_i(x) = \sum_{k \in K_i} (F(u_i) \cdot n_{i,k}) \; \theta_{i,k}(x) \quad \Rightarrow \quad \nabla \theta_i(x) = \sum_{k \in K_i} (F(u_i) \cdot n_{i,k}) \; V_{i,k}(x),$$

and since $\theta_i(x) = F(u_i) \cdot x + const$ implies $\nabla \theta_i(x) = F(u_i)$ we get

$$F(u_i) = \sum_{k \in K_i} (F(u_i) \cdot n_{i,k}) \; V_{i,k}(x) = \sum_{k \in K_i} h_{F \cdot n_{i,k}}(u_i, u_i) \, V_{i,k}(x).$$

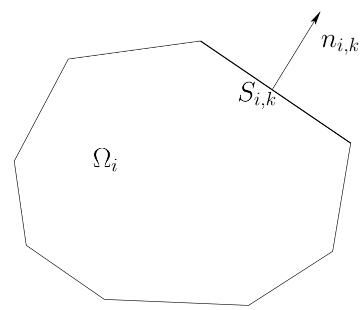


Figure 2.6: Cell, edge and normal

This fact allows us to write term III as

$$\sum_{n=0}^{\infty}\sum_{i\in I}\int_{t_n}^{t_{n+1}}\int_{\Omega_i}\sum_{k\in K_i}\left(h_{F\cdot n_{i,k}}(u_i,u_i)-h_{F\cdot n_{i,k}}(u_i,u_k)\right)V_{i,k}(x)\cdot\nabla\phi\,dxdt,$$

and since $h_{F\cdot n_{i,k}}$ is Lipschitz continuous, with say constant Lip, and our assumption that $||V_{i,k}||_{\infty,\Omega_i}\leq C$, we may estimate

$$|III|\leq Lip\,C\sum_{n=0}^{\infty}\sum_{i\in I}\int_{t_n}^{t_{n+1}}\int_{\Omega_i}\left(\sum_{k\in K_i}|u_i^n-u_k^n|\right)||\nabla\phi||\,dxdt.$$

The bracketed term above is treated as follows. Let $B(x,r)$ denote a ball centered at $x\in\Omega_i$ with radius r large enough so that it contains every adjacent cell Ω_k with $k\in K_i$. One easily sees that

$$\sum_{k\in K_i}|u_i^n-u_k^n||\Omega_k|\leq\int_{B(x,r)}|u_\Delta(x+y,t_n)-u_\Delta(x,t_n)|\,dy.$$

This and our assumption that for every $x\in\Omega_i$ we have $B_\Delta(x)$ satisfies

$$\max_{k\in K_i}\left(|B_\Delta(x)|/|\Omega_k|\right)\leq D$$

allows us to conclude

$$\sum_{k\in K_i}|u_i^n-u_k^n|\leq\frac{D}{|B_\Delta(x)|}\int_{B_\Delta(x)}|u_\Delta(x+y,t_n)-u_\Delta(x,t_n)|\,dy.$$

Therefore we have

$$\begin{aligned} III &\leq Lip\, CD \int_0^\infty \int_{\mathbb{R}^d} \frac{1}{|B_\Delta(x)|} \int_{B_\Delta(x)} |u_\Delta(x+y,t) - u_\Delta(x,t)|\, dy\, ||\nabla\phi||\, dxdt \\ &\leq Lip\, CDM \iint_{\mathrm{supp}(\phi)} \frac{1}{|B_\Delta(x)|} \int_{B_\Delta(x)} |u_\Delta(x+y,t) - u_\Delta(x,t)|\, dy\, dxdt, \end{aligned}$$

since ϕ has compact support in $\mathbb{R}^d \times [0,\infty)$ and where $M = \sup ||\nabla\phi||$. Since $u_\Delta \to u$ we can add and subtract and apply the triangle inequality to see

$$III \leq Lip\, CDM\, (A + B + C)\,,$$

where

$$\begin{aligned} A &= \iint_{\mathrm{supp}(\phi)} \frac{1}{|B_\Delta(x)|} \int_{B_\Delta(x)} |u_\Delta(x+y,t) - u(x+y,t)|\, dy\, dxdt \\ &\leq \iint_{\mathrm{supp}(\widetilde{\phi})} |u_\Delta(x,t) - u(x,t)|\, dxdt, \\ B &= \iint_{\mathrm{supp}(\phi)} \frac{1}{|B_\Delta(x)|} \int_{B_\Delta(x)} |u_\Delta(x,t) - u(x,t)|\, dy\, dxdt \\ &= \iint_{\mathrm{supp}(\phi)} |u_\Delta(x,t) - u(x,t)|\, dxdt, \\ C &= \iint_{\mathrm{supp}(\phi)} \frac{1}{|B_\Delta(x)|} \int_{B_\Delta(x)} |u(x+y,t) - u(x,t)|\, dy\, dxdt. \end{aligned}$$

Above we have used

$$\frac{1}{B_\Delta(x)} \int_{B_\Delta(x)} dy = 1,$$

and in A the notation $\mathrm{supp}(\widetilde{\phi})$ is used to take into account the change of variables $x + y \to x$. Clearly A and B tend to zero as $|\Delta| \to 0$. Proving term C tends to zero with $|\Delta|$ is done in the same way as one would show a mollified L^1 function converges in L^1.

Chapter 3

A Large Time Step Method in 1D

3.1 Introduction and Basic Ideas

Consider a system of conservation laws in one space dimension

$$u_t + f(u)_x = 0 \text{ in } R \times (0, \infty),$$

where $u = (u^1, ..., u^n)$ is the unknown vector field and $f = (f^1, ..., f^n)$ is the spatial flux density field defined on a domain of conservation states. The system is supplemented by the initial condition $u(x, 0) = u_0(x)$, where $u_0(x)$ is a bounded and measurable function.

Let $\Delta = \{x_i \mid i \in I\}$ be partition of R, with $\Delta x_i = x_i - x_{i-1}$. In the previous Chapter we gave short preview over the finite volume method and the scheme was given as

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x_i} (h_i - h_{i-1})$$

where h_i denotes the numerical flux between cells i and $i+1$. The time step Δt is taken to be such that

$$\Delta t \max_u |f'(u)| \leq \min_i \Delta x_i CFL.$$

As we can see the time step depends directly on the size of the smallest cell. In the other words, if we have at least one very small cell, our time step will be very small, regardless of the size of the other cells.

Now, let us turn our attention to numerical flux functions. Let h be an arbitrary numerical flux function, and let us consider the example in the following figures.

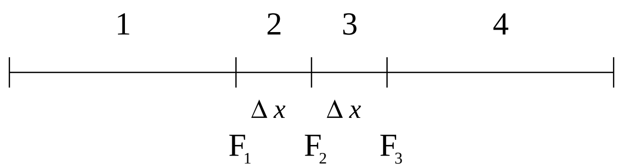


Figure 3.1: Cells and Fluxes

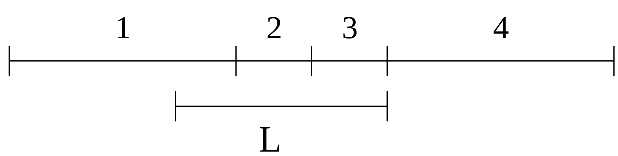


Figure 3.2: Cells and L

We have that cells 2 and 3 are much more smaller than cells 1 and 4. Now, if we assume that flow is from left to right in order to compute fluxes we are going to use information from the left. Now, let say we use information from more then one cell on the left. For the point of discussion at this time let us use information from all cells which are at distance L from the location of the flux we want to compute, where L is larger than Δx and is smaller then size of cells 1. and 4. Since the flux depends only on cells one the left we can make sort of convex combination of the fluxes one the left, and what is on the right is not important since that information will not affect the value of the flux. Similarly we do this for all fluxes, and then we have

$$\begin{aligned}
F_1 &= \frac{L}{L} h_f(u_1, u_2) \\
F_2 &= \frac{L - \Delta x}{L} h_f(u_1, u_3) + \frac{\Delta x}{L} h_f(u_2, u_3) \\
F_3 &= \frac{L - 2\Delta x}{L} h_f(u_1, u_4) + \frac{\Delta x}{L} h_f(u_2, u_4) + \frac{\Delta x}{L} h_f(u_3, u_4).
\end{aligned}$$

Now, if we turn the flow to be from the right to the left, then the information that we need is from the right side. We get

$$\begin{aligned}
F_1 &= \frac{\Delta x}{L} h_f(u_1, u_2) + \frac{\Delta x}{L} h_f(u_1, u_3) + \frac{L - 2\Delta x}{L} h_f(u_1, u_4) \\
F_2 &= \frac{\Delta x}{L} h_f(u_2, u_3) + \frac{L - \Delta x}{L} h_f(u_2, u_4) \\
F_3 &= \frac{L}{L} h_f(u_3, u_4).
\end{aligned}$$

The question is how to combine these fluxes into one formula that can work regardless the direction of the flow. The answer to this question lies in splitting fluxes. As we mentioned earlier, in splitting fluxes increasing and decreasing part of the flux h_f are separated into two different function, h^+ and h^- respectively. Recall, some examples of splitting fluxes for scalar equation are Lax-Friedrich and Engquist-Osher numerical flux. For Lax-Friedrich numerical flux we have

$$h^+(u_l) = \frac{1}{2}\left(|a|u_l + f(u_l)\right), \qquad h^-(u_r) = \frac{1}{2}\left(-|a|u_r + f(u_r)\right),$$

where $|a|$ is fastest wave speed, and for Engquist-Osher numerical flux we have

$$h^+(u_l) = \int_0^{u_l} \max\left\{f'(u), 0\right\} du, \qquad h^-(u_r) = \int_0^{u_r} \min\left\{f'(u), 0\right\} du.$$

Hence, we can now write fluxes as

$$F_1 = \frac{L}{L}h^+(u_1) + \frac{\Delta x}{L}h^-(u_2) + \frac{\Delta x}{L}h^-(u_3) + \frac{L-2\Delta x}{L}h^-(u_4)$$
$$F_2 = \frac{L-\Delta x}{L}h^+(u_1) + \frac{\Delta x}{L}h^+(u_2) + \frac{\Delta x}{L}h^-(u_3) + \frac{L-\Delta x}{L}h^-(u_4)$$
$$F_3 = \frac{L-2\Delta x}{L}h^+(u_1) + \frac{\Delta x}{L}h^+(u_2) + \frac{\Delta x}{L}h^+(u_3) + \frac{L}{L}h^-(u_4)$$

Using integral notation we can write these fluxes as

$$F_i = \frac{1}{L}\left(\int_0^L h^+\left(u(x_i - x)\right)\,dx + \int_0^L h^-\left(u(x_i + x)\right)\,dx\right),$$

for $i \in \{1, 2, 3\}$.

In the general case, let x_i be a point where we want to compute the flux and flux function is given as

$$h_i = \frac{1}{L}\left(\int_0^L h^+\left(u^n(x_i - x)\right)\,dx + \int_0^L h^-\left(u^n(x_i + x)\right)\,dx\right).$$

If we choose L such that $L > \min_i \Delta x_i$ then we have

$$\Delta t \max_u |f'(u)| \leq L\,CFL,$$

and as we can see, we can make larger time step which allows us faster marching through time.

Let the cell i be such cell that $\Delta x_i < L$. Then we have

$$\begin{aligned}
u_i^{n+1} &= u_i^n - \frac{\Delta t}{\Delta x_i}\left(h_i - h_{i-1}\right) \\
&= u_i^n - \frac{\Delta t}{\Delta x_i}\left(\frac{1}{L}\left(\int_0^L h^+\left(u^n(x_i - x)\right)\,dx + \int_0^L h^-\left(u^n(x_i + x)\right)\,dx\right)\right. \\
&\left.-\frac{1}{L}\left(\int_0^L h^+\left(u^n(x_{i-1} - x)\right)\,dx + \int_0^L h^-\left(u^n(x_{i-1} + x)\right)\,dx\right)\right) \\
&= u_i^n - \frac{\Delta t}{\Delta x_i}\frac{1}{L}\left(\int_0^L h^+\left(u^n(x_i - x)\right)\,dx - \int_0^L h^+\left(u^n(x_{i-1} - x)\right)\,dx\right. \\
&\left.+\int_0^L h^-\left(u^n(x_i + x)\right)\,dx - \int_0^L h^-\left(u^n(x_{i-1} + x)\right)\,dx\right).
\end{aligned}$$

Now let us observe the integral $\int_0^L h^-\left(u^n(x_{i-1} + x)\right)\,dx$. We have

$$\begin{aligned}
&\int_0^L h^-\left(u^n(x_{i-1} + x)\right)\,dx \\
&\qquad = \int_0^{\Delta x_i} h^-\left(u^n(x_{i-1} + x)\right)\,dx + \int_{\Delta x_i}^L h^-\left(u^n(x_{i-1} + x)\right)\,dx \\
&\qquad = \int_0^{\Delta x_i} h^-\left(u^n(x_{i-1} + x)\right)\,dx + \int_0^{L-\Delta x_i} h^-\left(u^n(x_{i-1} + y + \Delta x_i)\right)\,dy \\
&\qquad = \int_0^{\Delta x_i} h^-\left(u^n(x_{i-1} + x)\right)\,dx + \int_0^{L-\Delta x_i} h^-\left(u^n(x_i + y)\right)\,dy \\
&\qquad = \int_0^{\Delta x_i} h^-\left(u^n(x_{i-1} + x)\right)\,dx + \int_0^{L-\Delta x_i} h^-\left(u^n(x_i + x)\right)\,dx.
\end{aligned}$$

Similarly we show that

$$\int_0^L h^+\left(u^n(x_i - x)\right)\,dx = \int_0^{\Delta x_i} h^+\left(u^n(x_i - x)\right)\,dx + \int_0^{L-\Delta x_i} h^+\left(u^n(x_{i-1} - x)\right)\,dx.$$

And now for u_i^{n+1} we have

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{\Delta x_i}\frac{1}{L}\left(\int_0^{\Delta x_i} h^+\left(u^n(x_i - x)\right)\,dx - \int_{L-\Delta x_i}^{L} h^+\left(u^n(x_{i-1} - x)\right)\,dx\right.$$
$$\left. + \int_{L-\Delta x_i}^{L} h^-\left(u^n(x_i + x)\right)\,dx - \int_0^{\Delta x_i} h^-\left(u^n(x_{i-1} + x)\right)\,dx\right)$$
$$= u_i^n - \frac{\Delta t}{L}\left(\frac{1}{\Delta x_i}\int_0^{\Delta x_i} h^+\left(u_i^n\right)\,dx - \frac{1}{\Delta x_i}\int_{L-\Delta x_i}^{L} h^+\left(u^n(x_{i-1} - x)\right)\,dx\right.$$
$$\left. + \frac{1}{\Delta x_i}\int_{L-\Delta x_i}^{L} h^-\left(u^n(x_i + x)\right)\,dx - \frac{1}{\Delta x_i}\int_0^{\Delta x_i} h^-\left(u_i^n\right)\,dx\right).$$

Having u_i^{n+1} written this way it is easy to show that our method is *monotone*. Let us denote

$$G\left(u_{i-k_L}^n,\ \dots, u_i^n, \dots, u_{i+k_R}^n\right) =$$
$$= u_i^n - \frac{\Delta t}{L}\left(\frac{1}{\Delta x_i}\int_0^{\Delta x_i} h^+\left(u_i^n\right)\,dx - \frac{1}{\Delta x_i}\int_{L-\Delta x_i}^{L} h^+\left(u^n(x_{i-1} - x)\right)\,dx\right.$$
$$\left. + \frac{1}{\Delta x_i}\int_{L-\Delta x_i}^{L} h^-\left(u^n(x_i + x)\right)\,dx - \frac{1}{\Delta x_i}\int_0^{\Delta x_i} h^-\left(u_i^n\right)\,dx\right),$$

where $u_{i-k_L}^n, \dots, u_i^n, \dots, u_{i+k_R}^n$ are cells on which u_i^{n+1} depends. Let us show G is increasing function in each argument. If we differentiate G with respect to u_i^n we get

$$\frac{\partial}{\partial u_i^n} G\left(u_{i-k_L}^n, \dots, u_i^n, \dots, u_{i+k_R}^n\right) = 1 - \frac{\Delta t}{L}\left(\left(h^+\right)'\left(u_i^n\right) - \left(h^-\right)'\left(u_i^n\right)\right).$$

Now, using the way we have defined Δt and choose the CFL number we get

$$\frac{\partial}{\partial u_i^n} G\left(u_{i-k_L}^n, \dots, u_i^n, \dots, u_{i+k_R}^n\right) \geq 0,$$

therefore G is increasing function with respect to u_i^n. If we compute the derivative of G with respect to u_j^n, where $j \neq i$, we have

$$\frac{\partial}{\partial u_j^n} G\left(u_{i-k_L}^n, \dots, u_i^n, \dots, u_{i+k_R}^n\right) = \begin{cases} \gamma_j \frac{\partial}{\partial u_j^n} h^+(u_j), & j < i, \\ -\gamma_j \frac{\partial}{\partial u_j^n} h^-(u_j), & j > i, \end{cases}$$

where γ_j are some non-negative constants, giving us

$$\frac{\partial}{\partial u_j^n} G\left(u_{i-k_L}^n, \ldots, u_i^n, \ldots, u_{i+k_R}^n\right) \geq 0.$$

From here we have now that G is increasing function in all of its arguments. With this result it is easy to show *maximum principle*. Let

$$u^* = \max\left\{u_{i-k_L}^n, \ldots, u_i^n, \ldots, u_{i+k_R}^n\right\},$$

$$u_* = \min\left\{u_{i-k_L}^n, \ldots, u_i^n, \ldots, u_{i+k_R}^n\right\},$$

then we have

$$u_i^{n+1} = G\left(u_{i-k_L}^n, \ldots, u_i^n, \ldots, u_{i+k_R}^n\right) \leq G\left(u^*, \ldots, u^*, \ldots, u^*\right) = u^*,$$

$$u_i^{n+1} = G\left(u_{i-k_L}^n, \ldots, u_i^n, \ldots, u_{i+k_R}^n\right) \geq G\left(u_*, \ldots, u_*, \ldots, u_*\right) = u_*,$$

where the last equality follows from fact that method is conservative.

If $f(u) = \lambda u$ the scheme is linear, therefore, maximum principle implies linear stability.

3.2 Lax - Wendroff Theorem for the Large Time Step Method

In this section we will show that if our numerical solution converges boundedly almost everywhere to a function u as $|\Delta|$ and Δt converges to 0, then u is a weak solution of the observed hyperbolic conservation law.

Let us consider scalar conversational law

$$u_t + f(u)_x = 0 \text{ in } \mathbb{R} \times (0, \infty),$$

with initial condition

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}.$$

We assume that u_0 is a function of bounded total variation. Let $\Delta = \{x_i \mid i \in I\}$ be partition of $\mathbb{R}$, $\Delta x_i = x_i - x_{i-1}$, $\Delta X = \max_i \Delta x_i$ and $\Delta x = \min_i \Delta x_i$. Also let S be such that $\frac{\Delta X}{\Delta x} \leq S$ and let L be comparable to ΔX, i.e., $L = O(\Delta X)$.

Let u_Δ^0 be given by $u_\Delta^0(x) = \sum_I u_i^0 \chi_{[x_{i-1},x_i]}(x)$, where $u_i^0 = \frac{1}{\Delta x_i}\int_{x_i-1}^{x_i} u_0(x)\,dx$, and $u^n(x) = \sum_I u_i^n \chi_{[x_{i-1},x_i]}(x)$ where

$$u_\Delta^{n+1} = u_\Delta^n - \Delta t \frac{d}{dx} F_u.$$

Here $\Delta t \max_u |f'(u)| \leq L\,CFL$ and the CFL is given by the Courant - Friedrichs - Lewy condition. Here F_u denotes piecewise linear continuous functions obtained by interpolating numerical flux over the grid and numerical fluxes are given as

$$h_i = \frac{1}{L}\left(\int_0^L h^+\left(u^n(x_i - x)\right)\,dx + \int_0^L h^-\left(u^n(x_i + x)\right)\,dx\right).$$

Then, if $u_\Delta^n(x,t)$ converges boundedly almost everywhere to a function $u(x,t)$ as $|\Delta|$ and Δt converges to 0 then $u(x,t)$ is a weak solution to the hyperbolic conservation law. It is interesting to note condition (C2) from Section 2.4 is not required in the proof below.

Proof

We show that u_Δ^n converges to a weak solution, u, as $|\Delta|, \Delta t \to 0$.

Let $\varphi \in C_0^\infty(\mathbb{R} \times [0,\infty))$ be a smooth test function with compact support. Then

$$u_\Delta^{n+1}\varphi(x,t_n) = u_\Delta^n \varphi(x,t_n) - \Delta t \varphi(x,t_n)\frac{d}{dx}F_u.$$

We subtract $u^n_\Delta \varphi(x, t_n)$ from both sides, integrate over $\mathbb{R}$ and sum over n to get

$$\sum_{n=0}^{\infty} \int_{\mathbb{R}} \left(u^{n+1}_\Delta - u^n_\Delta\right) \varphi(x, t_n) = -\Delta t \sum_{n=0}^{\infty} \int_{\mathbb{R}} \varphi(x, t_n) \frac{d}{dx} F_u.$$

We show that u^n_Δ converges to a weak solution in two parts by analyzing the terms in above expression. First, we show that

$$\sum_{n=0}^{\infty} \Delta t \int_{\mathbb{R}} \varphi(x, t_n) \frac{d}{dx} F_u \, dx \to -\int_0^\infty \int_{\mathbb{R}} f(u) \varphi_x(x, t) \, dx \, dt,$$

as $|\Delta|, \Delta t \to 0$. And in the second part we will show that

$$\sum_{n=0}^{\infty} \int_{\mathbb{R}} \left(u^{n+1}_\Delta - u^n_\Delta\right) \varphi(x, t_n) \, dx \to -\int_0^\infty \int_{\mathbb{R}} u \varphi_t(x, t) \, dx \, dt - \int u^0 \varphi(x, 0) \, dx,$$

as $|\Delta|, \Delta t \to 0$. By putting these two parts together we get that u is a weak solution, i.e.,

$$\int_0^\infty \int_{\mathbb{R}} u \varphi_t(x, t) \, dx \, dt + \int_0^\infty \int_{\mathbb{R}} f(u) \varphi_x(x, t) \, dx \, dt + \int_{\mathbb{R}} u^0 \varphi(x, 0) \, dx = 0$$

holds for every test function $\varphi \in C_0^\infty$.

Let M be such that $|\varphi_x(x, t)| \leq M$, for $x \in \mathbb{R}$ and $t \geq 0$. Also let Lip be the Lipshitz constant for the numerical flux h.

Part I

Using integration by parts we have that

$$\sum_{n=0}^{\infty} \Delta t \int_{\mathbb{R}} \varphi(x, t_n) \frac{d}{dx} F_u \, dx = \sum_{n=0}^{\infty} \Delta t \left(\lim_{\zeta \to +\infty} F_u \varphi(x, t_n)|_{-\zeta}^{\zeta} - \int_{\mathbb{R}} F_u \varphi_x(x, t_n) \, dx \right).$$

The first term in the above sum is zero because φ has compact support. Hence,

$$\begin{aligned} \sum_{n=0}^{\infty} \Delta t \int_{\mathbb{R}} \varphi(x, t_n) \frac{d}{dx} F_u \, dx = &- \sum_{n=0}^{\infty} \Delta t \int_{\mathbb{R}} \left(F_u - f(u)\right) \varphi_x(x, t_n) \, dx \\ &- \sum_{n=0}^{\infty} \Delta t \int_{\mathbb{R}} f(u) \varphi_x(x, t_n) \, dx. \end{aligned}$$

Next, we analyze the sums in the above expression. We will show that the first sum converges to 0 and it is clear that the last sum converges to $\int_0^\infty \int_{\mathbb{R}} f(u)\varphi_x(x,t)\,dx\,dt$, as $|\Delta|, \Delta t \to 0$.

We show that $\sum_{n=0}^{\infty} \Delta t \int_{\mathbb{R}} (F_u - f(u))\,\varphi_x(x,t_n)\,dx \to 0$ as $|\Delta|, \Delta t \to 0$.

We use the triangular inequality, the fact $|\varphi_x(x,t_n)| \leq M$, and we split the integral to get

$$\left| \sum_{n=0}^{\infty} \Delta t \int_{\mathbb{R}} (F_u - f(u))\,\varphi_x(x,t_n)\,dx \right| \leq \sum_{n=0}^{\infty} \sum_{i=1}^{I} M \Delta t \int_{x_{i-1}}^{x_i} |F_u - f(u)|\,dx.$$

Let us now observe the above integral on interval $[x_{i-1}, x_i]$. We have

$$\begin{aligned}
\int_{x_{i-1}}^{x_i} |F_u - f(u)|\,dx &= \int_{x_{i-1}}^{x_i} \left| \left(1 - \frac{x - x_{i-1}}{\Delta x_i}\right) h_{i-1} + \frac{x - x_{i-1}}{\Delta x_i} h_i - h_f(u_i^n, u_i^n) \right| dx \\
&\leq \int_{x_{i-1}}^{x_i} \left(1 - \frac{x - x_{i-1}}{\Delta x_i}\right) |h_{i-1} - h_f(u_i^n, u_i^n)| + \frac{x - x_{i-1}}{\Delta x_i} |h_i - h_f(u_i^n, u_i^n)|\,dx \\
&= \frac{\Delta x_i}{2} \left(|h_{i-1} - h_f(u_i^n, u_i^n)| + |h_i - h_f(u_i^n, u_i^n)| \right).
\end{aligned}$$

Now, let us observe $|h_i - h_f(u_i^n, u_i^n)|$. We have

$$\begin{aligned}
|h_i - {}& h_f(u_i^n, u_i^n)| \\
&= \left| \frac{1}{L} \int_0^L h^+ \left(u_\Delta^n(x_i - \zeta)\right) d\zeta + \frac{1}{L} \int_0^L h^- \left(u_\Delta^n(x_i + \zeta)\right) d\zeta - h_f(u_i^n, u_i^n) \right| \\
&= \left| \frac{1}{L} \int_0^L h^+ \left(u_\Delta^n(x_i - \zeta)\right) - h^+(u_i^n)\,d\zeta + \frac{1}{L} \int_0^L h^- \left(u_\Delta^n(x_i + \zeta)\right) - h^-(u_i^n)\,d\zeta \right| \\
&\leq \frac{1}{L} \int_0^L \left| h^+ \left(u_\Delta^n(x_i - \zeta)\right) - h^+(u_i^n) \right| d\zeta + \frac{1}{L} \int_0^L \left| h^- \left(u_\Delta^n(x_i + \zeta)\right) - h^-(u_i^n) \right| d\zeta.
\end{aligned}$$

We use the fact that numerical flux is Lipschitz continuous with constant Lip to get

$$|h_i - h_f(u_i^n, u_i^n)| \leq \frac{1}{L} Lip \int_0^L |u_\Delta^n(x_i - \zeta) - u_i^n|\,d\zeta + \frac{1}{L} Lip \int_0^L |u_\Delta^n(x_i + \zeta) - u_i^n|\,d\zeta.$$

Similarly for $|h_{i-1} - h_f(u_i^n, u_i^n)|$, we get

$$\begin{aligned} |h_{i-1} - h_f\ & (u_i^n, u_i^n)| \\ & \leq \frac{1}{L} Lip \int_0^L |u_\Delta^n(x_{i-1} - \zeta) - u_i^n|\ d\zeta + \frac{1}{L} Lip \int_0^L |u_\Delta^n(x_{i-1} + \zeta) - u_i^n|\ d\zeta. \end{aligned}$$

Hence,

$$\begin{aligned} \int_{x_{i-1}}^{x_i} |F_u\ & - f(u)|\ dx \\ & \leq \frac{\Delta x_i Lip}{2L} \left(\int_0^L |u_\Delta^n(x_i - \zeta) - u_i^n|\ d\zeta + \int_0^L |u_\Delta^n(x_i + \zeta) - u_i^n|\ d\zeta \right. \\ & \left. + \int_0^L |u_\Delta^n(x_{i-1} - \zeta) - u_i^n|\ d\zeta + \int_0^L |u_\Delta^n(x_{i-1} + \zeta) - u_i^n|\ d\zeta \right). \end{aligned}$$

Now let us consider two cases:

1^o $L \leq \Delta x_i$

In this case we have

$$\int_0^L |u_\Delta^n(x_i - \zeta) - u_i^n|\ d\zeta = \int_0^L |u_i^n - u_i^n|\ d\zeta = 0$$

and

$$\int_0^L |u_\Delta^n(x_{i-1} + \zeta) - u_i^n|\ d\zeta = \int_0^L |u_i^n - u_i^n|\ d\zeta = 0.$$

Therefore,

$$\int_{x_{i-1}}^{x_i} |F_u - f(u)|\,dx$$

$$\begin{aligned}
&\le \frac{\Delta x_i Lip}{2L}\left(\int_0^L |u_\Delta^n(x_i+\zeta)-u_i^n|\,d\zeta + \int_0^L |u_\Delta^n(x_{i-1}-\zeta)-u_i^n|\,d\zeta\right)\\
&= \frac{Lip}{2L}\int_{x_{i-1}}^{x_i}\int_0^L |u_\Delta^n(x_i+\zeta)-u_\Delta^n(x_i)|\,d\zeta + \int_0^L |u_\Delta^n(x_{i-1}-\zeta)-u_\Delta^n(x_i)|\,d\zeta\,dx\\
&\le \frac{Lip}{2L}\int_{x_{i-1}}^{x_i}\int_0^L |u_\Delta^n(x+\zeta)-u_\Delta^n(x)|\,d\zeta + \int_0^L |u_\Delta^n(x-\zeta)-u_\Delta^n(x)|\,d\zeta\,dx.
\end{aligned}$$

2^o $L \ge \Delta x_i$

In this case we have

$$\int_0^L |u_\Delta^n(x_i-\zeta)- u_i^n|\,d\zeta$$

$$\begin{aligned}
&= \int_0^{\Delta x_i} |u_i^n - u_i^n|\,d\zeta + \int_{\Delta x_i}^L |u_\Delta^n(x_i-\zeta)-u_i^n|\,d\zeta\\
&= \int_{\Delta x_i}^L |u_\Delta^n(x_i-\zeta)-u_i^n|\,d\zeta = \int_0^{L-\Delta x_i} |u_\Delta^n(x_{i-1}-\zeta)-u_i^n|\,d\zeta\\
&\le \int_0^L |u_\Delta^n(x_{i-1}-\zeta)-u_i^n|\,d\zeta
\end{aligned}$$

and

$$\int_0^L |u_\Delta^n(x_{i-1}+\zeta)- u_i^n|\,d\zeta$$

$$\begin{aligned}
&= \int_0^{\Delta x_i} |u_i^n - u_i^n|\,d\zeta + \int_{\Delta x_i}^L |u_\Delta^n(x_{i-1}+\zeta)-u_i^n|\,d\zeta\\
&= \int_{\Delta x_i}^L |u_\Delta^n(x_{i-1}+\zeta)-u_i^n|\,d\zeta = \int_0^{L-\Delta x_i} |u_\Delta^n(x_i+\zeta)-u_i^n|\,d\zeta\\
&\le \int_0^L |u_\Delta^n(x_i+\zeta)-u_i^n|\,d\zeta.
\end{aligned}$$

Note, condition (C2) from Section 2.4 is not required here due to fortuitous cancellation. Therefore,

$$\begin{aligned}
&\int_{x_{i-1}}^{x_i} |F_u - f(u)| \, dx \\
&\qquad \leq \frac{\Delta x_i Lip}{2L} \left(2 \int_0^L |u_\Delta^n(x_i + \zeta) - u_i^n| \, d\zeta + 2 \int_0^L |u_\Delta^n(x_{i-1} - \zeta) - u_i^n| \, d\zeta \right) \\
&\qquad = \frac{Lip}{L} \int_{x_{i-1}}^{x_i} \int_0^L |u_\Delta^n(x_i + \zeta) - u_\Delta^n(x_i)| \, d\zeta + \int_0^L |u_\Delta^n(x_{i-1} - \zeta) - u_\Delta^n(x_i)| \, d\zeta \, dx \\
&\qquad \leq \frac{Lip}{L} \int_{x_{i-1}}^{x_i} \int_0^L |u_\Delta^n(x + \zeta) - u_\Delta^n(x)| \, d\zeta + \int_0^L |u_\Delta^n(x - \zeta) - u_\Delta^n(x)| \, d\zeta \, dx.
\end{aligned}$$

Finally, using all the above cases and estimates we have

$$\begin{aligned}
&\left| \sum_{n=0}^{\infty} \Delta t \int_{\mathbb{R}} (F_u - f(u)) \, \varphi_x(x, t_n) \, dx \right| \\
&\qquad \leq \sum_{n=0}^{\infty} \sum_{i=1}^{I} M \Delta t \frac{Lip}{L} \int_{x_{i-1}}^{x_i} \int_0^L |u_\Delta^n(x + \zeta) - u_\Delta^n(x)| \, d\zeta \\
&\qquad + \int_0^L |u_\Delta^n(x - \zeta) - u_\Delta^n(x)| \, d\zeta \, dx \\
&\qquad = \sum_{n=0}^{\infty} M \Delta t \frac{Lip}{L} \int_{\mathrm{supp}\varphi} \int_0^L |u_\Delta^n(x + \zeta) - u_\Delta^n(x)| \, d\zeta \, dx \\
&\qquad + \sum_{n=0}^{\infty} M \Delta t \frac{Lip}{L} \int_{\mathrm{supp}\varphi} \int_0^L |u_\Delta^n(x - \zeta) - u_\Delta^n(x)| \, d\zeta \, dx,
\end{aligned}$$

where $\mathrm{supp}\varphi$ denotes compact support of φ.

Since $u_\Delta^n \in L^1$ and since C_0^∞ is dense in L^1 we have that for every ε there exists a function $g \in C_0^\infty$ such that $||u_\Delta^n - g||_1 < \varepsilon$. Then

$$
\begin{aligned}
|u_\Delta^n(x-\zeta) - u_\Delta^n(x)| &\le |u_\Delta^n(x-\zeta) - g(x-\zeta)| + |g(x-\zeta) - g(x)| + |g(x) - u(x)| \\
&\le \varepsilon + |g(x-\zeta) - g(x)| + \varepsilon.
\end{aligned}
$$

Similarly

$$
\begin{aligned}
|u_\Delta^n(x+\zeta) - u_\Delta^n(x)| &\le |u_\Delta^n(x+\zeta) - g(x+\zeta)| + |g(x+\zeta) - g(x)| + |g(x) - u(x)| \\
&\le \varepsilon + |g(x+\zeta) - g(x)| + \varepsilon.
\end{aligned}
$$

Since g is continuous we have that for every ε there exists some δ_0 such that for every $L < \delta_0$ we have $|g(x \pm \zeta) - g(x)| < \varepsilon$, hence,

$$|u_\Delta^n(x-\zeta) - u_\Delta^n(x)| \le 3\varepsilon$$

and

$$|u_\Delta^n(x+\zeta) - u_\Delta^n(x)| \le 3\varepsilon.$$

Therefore,

$$
\begin{aligned}
&\left| \sum_{n=0}^{\infty} \Delta t \int_{\mathbb{R}} \left(F_u - f(u)\right) \varphi_x(x, t_n)\, dx \right| \\
&\qquad \le \sum_{n=0}^{\infty} M \Delta t \frac{Lip}{L} \int_{\mathrm{supp}\varphi} \int_0^L 3\varepsilon \, d\zeta \, dx + \sum_{n=0}^{\infty} M \Delta t \frac{Lip}{L} \int_{\mathrm{supp}\varphi} \int_0^L 3\varepsilon \, d\zeta \, dx \\
&\qquad = \sum_{n=0}^{\infty} 6 M \Delta t \varepsilon \frac{Lip}{L} \int_{\mathrm{supp}\varphi} \int_0^L \, d\zeta \, dx \\
&\qquad = \sum_{n=0}^{\infty} 6 M \Delta t \varepsilon Lip \left| \mathrm{supp}\varphi \right|.
\end{aligned}
$$

When we let $|\Delta|, \Delta t \to 0$ we get

$$\left| \sum_{n=0}^{\infty} \Delta t \int_{\mathbb{R}} (F_u - f(u))\, \varphi_x(x, t_n)\, dx \right| \to 6M\varepsilon Lip\, |\text{supp}\varphi| \int_0^{\infty} dt.$$

Since we work in finite time, we can take the time integral to be from 0 to T. Then

$$6M\varepsilon Lip\, |\text{supp}\varphi| \int_0^{T} dt = 6MTLip|\text{supp}\varphi|\varepsilon$$

implying

$$\left| \sum_{n=0}^{\infty} \Delta t \int_{\mathbb{R}} (F_u - f(u))\, \varphi_x(x, t_n)\, dx \right| \to 0,$$

as $|\Delta|, \Delta t \to 0$.

Part II

We consider

$$\sum_{n=0}^{\infty} \int_{\mathbb{R}} \left(u_{\Delta}^{n+1} - u_{\Delta}^{n}\right) \varphi(x, t_n)\, dx.$$

Using summation by parts we get

$$\begin{aligned} \sum_{n=0}^{\infty} \int_{\mathbb{R}} \left(u_{\Delta}^{n+1} - u_{\Delta}^{n}\right) \varphi(x, t_n)\, dx = &- \sum_{n=1}^{\infty} \Delta t \int_{\mathbb{R}} u_{\Delta}^{n} \frac{\varphi(x, t_n) - \varphi(x, t_{n-1})}{\Delta t}\, dx \\ &- \int_{\mathbb{R}} u_{\Delta}^{0} \varphi(x, 0)\, dx. \end{aligned}$$

If we let $\Delta t \to 0$ we will get

$$\sum_{n=0}^{\infty} \int_{\mathbb{R}} \left(u_{\Delta}^{n+1} - u_{\Delta}^{n}\right) \varphi(x, t_n)\, dx \to - \int_0^{\infty} \int_{\mathbb{R}} u\varphi_t(x, t)\, dx - \int_{\mathbb{R}} u^0 \varphi(x, 0)\, dx.$$

Combining the above two parts we showed that u_{Δ}^{n} converges to a weak solution u of hyperbolic conservation law.

3.3 Entropy Solution

In this section we show that in the scalar case the solutions to our numerical method converge to the entropy solution. This proof is very similar to the proof of convergence to a weak solution, presented in the previous section, so we will show only the parts which are different.

In the proof we rely on Kružkov entropy condition. More precisely, with u_Δ^n being defined as in the previous section, we show that if $u_\Delta^n \to u$, as $|\Delta|, \Delta t \to 0$, then for every non-negative test function $\varphi \in C_0^\infty(\mathbb{R} \times (0,\infty))$ we have

$$\int_0^\infty \int_{\mathbb{R}} \mid u - k \mid \varphi_t + sgn(u-k)\,(f(u) - f(k))\,\varphi_x\,dx\,dt \geq 0.$$

Recall that

$$u_\Delta^{n+1} = u_\Delta^n - \Delta t \frac{d}{dx} F_u$$

If we denote $u_\Delta^n - \Delta t \frac{d}{dx} F_u = G(u_\Delta^n)$, we have

$$u_\Delta^{n+1} \leq G\left(u_\Delta^n \vee k\right) \quad \text{ and } \quad u_\Delta^{n+1} \geq G\left(u_\Delta^n \wedge k\right),$$

for every $k \in R$. From here

$$u_\Delta^{n+1} - k \leq G\left(u_\Delta^n \vee k\right) - G(k) \quad \text{ and } \quad u_\Delta^{n+1} - k \geq G\left(u_\Delta^n \wedge k\right) - G(k)$$

implying

$$\begin{aligned} \mid u_\Delta^{n+1} - k \mid &\leq \max\left\{G\left(u_\Delta^n \vee k\right) - G(k), G(k) - G\left(u_\Delta^n \wedge k\right)\right\} \\ &= G\left(u_\Delta^n \vee k\right) - G\left(u_\Delta^n \wedge k\right). \end{aligned}$$

We multiply obtained expression by positive test function φ with compact support, integrate over x and sum over n, to get

$$\begin{aligned}\sum_{n=0}^{\infty}\int_{\mathbb{R}}\left|u_{\Delta}^{n+1}-k\right|\varphi\,dx \leq \sum_{n=0}^{\infty}\int_{\mathbb{R}}\varphi\left(u_{\Delta}^{n}\vee k-u_{\Delta}^{n}\wedge k\right)\,dx \\ -\sum_{n=0}^{\infty}\Delta t\int_{\mathbb{R}}\varphi\frac{d}{dx}\left(F_u\left(u_{\Delta}^{n}\vee k\right)-F_u\left(u_{\Delta}^{n}\wedge k\right)\right)\,dx.\end{aligned}$$

As in the proof for the weak convergence, we divide the proof into two parts.

First, we show

$$\begin{aligned}\sum_{n=0}^{\infty}\int_{\mathbb{R}}\big(\mid u_{\Delta}^{n+1}-k\mid-\mid u_{\Delta}^{n}-k\mid\big)\,\varphi\,dx \to \\ -\int_{0}^{\infty}\int_{\mathbb{R}}\mid u-k\mid\varphi_t\,dx\,dt-\int_{\mathbb{R}}\mid u_0-k\mid\varphi\,dx,\end{aligned}$$

as $|\Delta|, \Delta t \to 0$. This proof is similar to the Part II of the proof of convergence to the weak solution, so we omit it.

Secondly, we show that

$$\begin{aligned}\sum_{n=0}^{\infty}\Delta t\int_{\mathbb{R}}\varphi\frac{d}{dx}\left(F_u\left(u_{\Delta}^{n}\vee k\right)-F_u\left(u_{\Delta}^{n}\wedge k\right)\right)\,dx \to \\ -\int_{0}^{\infty}\int_{\mathbb{R}}\varphi_x\left(f(u\vee k)-f(u\wedge k)\right)\,dx\,dt,\end{aligned}$$

as $|\Delta|, \Delta t \to 0$. To prove this, we use integration by parts and fact that φ has compact support. We get

$$\sum_{n=0}^{\infty} \Delta t \int_{\mathbb{R}} \varphi \frac{d}{dx} \left(F_u \left(u_\Delta^n \vee k\right) - F_u \left(u_\Delta^n \wedge k\right)\right) dx$$

$$= - \sum_{n=0}^{\infty} \Delta t \int_{\mathbb{R}} \varphi_x \left(F_u \left(u_\Delta^n \vee k\right) - F_u \left(u_\Delta^n \wedge k\right)\right) dx$$

$$= - \sum_{n=0}^{\infty} \Delta t \int_{\mathbb{R}} \varphi_x \left(F_u \left(u_\Delta^n \vee k\right) - f \left(u_\Delta^n \vee k\right) - F_u \left(u_\Delta^n \wedge k\right) + f \left(u_\Delta^n \wedge k\right)\right) dx$$

$$- \sum_{n=0}^{\infty} \Delta t \int_{\mathbb{R}} \varphi_x \left(f \left(u_\Delta^n \vee k\right) - f \left(u_\Delta^n \wedge k\right)\right) dx.$$

As in Part I of the previous proof, we show that the first sum converges to 0 as $|\Delta|, \Delta t \to 0$. It is also easy to see that if $|\Delta|, \Delta t \to 0$ then

$$\sum_{n=0}^{\infty} \Delta t \int_{\mathbb{R}} \varphi_x \left(f \left(u_\Delta^n \vee k\right) - f \left(u_\Delta^n \wedge k\right)\right) dx \to \int_0^{\infty} \int_{\mathbb{R}} \varphi_x \left(f \left(u^n \vee k\right) - f \left(u^n \wedge k\right)\right) dx\, dt$$

and with this the proof is completed.

3.4 Numerical Results

In this section we will present numerical examples using our method. We consider a grid of 125 points over the interval $[0, 1]$. The grid is constructed in such a way that it contains blocks of small cells of different sizes, $\frac{1}{200}$ and $\frac{1}{300}$, and those blocks are divided by blocks of cell sizes $\frac{1}{100}$. In all cases we take the CFL number to be 0.8 relative to L which here we are taking to be the size of the largest cell, $\frac{1}{100}$.

This is equivalent to having a regular time step with CFL number 2.4. We use the Engquist-Osher numerical flux.

Example 1

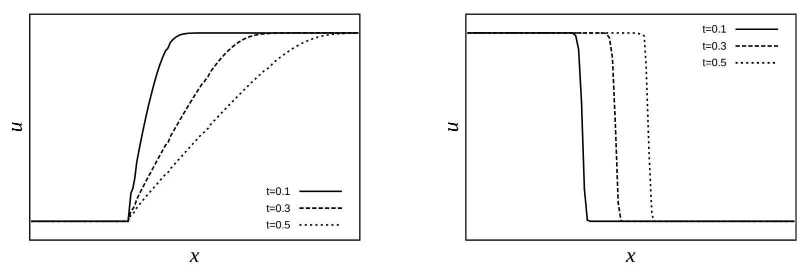
We consider the Burgers' equation,

$$u_t + uu_x = 0$$

with Riemann data conditions $u_l = 0$ and $u_r = 1$ and discontinuity is set at $x = 0.3$. Recall from Chapter 1 that the solution is a rarefaction wave. In Figure 3.3 (a) we show the solution at times $t = 0.1$, $t = 0.3$ and $t = 0.5$.

Example 2

We consider Burgers' equation when the initial data is reversed so that $u_l = 1$ and $u_r = 0$, and the discontinuity is still at $x = 0.3$. From the theory, Chapter 1, we have that the solution is a shock with speed 0.5. Figure 3.3 (b) shows the solution at times $t = 0.1$, $t = 0.3$ and $t = 0.5$.



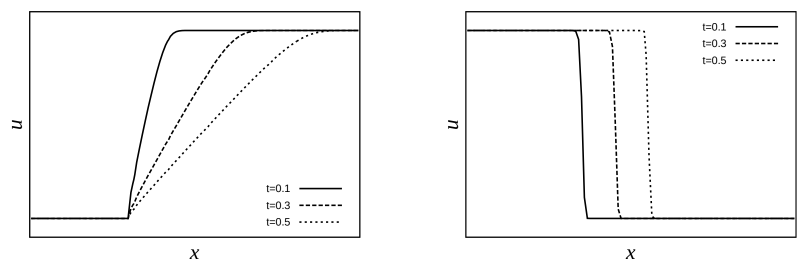
(a) Example 1 (b) Example 2

Figure 3.3: The large time step method

We remark that if L is large enough then the numerical solutions could show irregularities consisting of small steps. This problem can be avoided by choosing appropriate weights in the numerical fluxes such as

$$h_i = \frac{2}{L}\int_0^L \frac{L-x}{L} h^+\left(u^n(x_i - x)\right)\, dx + \frac{2}{L}\int_0^L \frac{L-x}{L} h^-\left(u^n(x_i + x)\right)\, dx.$$

This flux is maximum norm stable. In the following two examples, Figure 3.4 (a) and 3.4 (b), we show results using this weighted numerical flux. In both examples we used the same grid, same initial conditions and the same CFL number as in previous two examples.

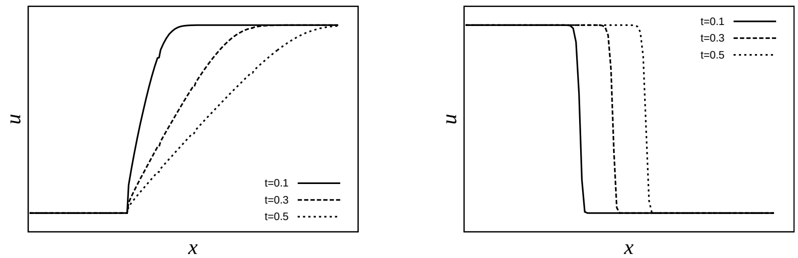


(a) Rarefaction wave (b) Shock

Figure 3.4: Modified numerical flux

The following two examples, Figures 3.5 (a) and 3.5 (b), we show results using regular large time step numerical flux. In both examples we consider the same problems as we did earlier and we use the Engquist-Osher numerical flux. This grid has the same number of points, 125, over the interval $[0, 1]$. The grid is constructed in such a way that it contains blocks of very small cells of different sizes, $\frac{1}{2000}$ and $\frac{1}{3000}$, and these blocks are divided by blocks of cells with sizes $\frac{1}{100}$. In all cases we take the CFL number to be 0.8 relative to L which here we are taking to be the size of the largest cell, $\frac{1}{100}$. This is equivalent to having a regular time step with CFL number 24.

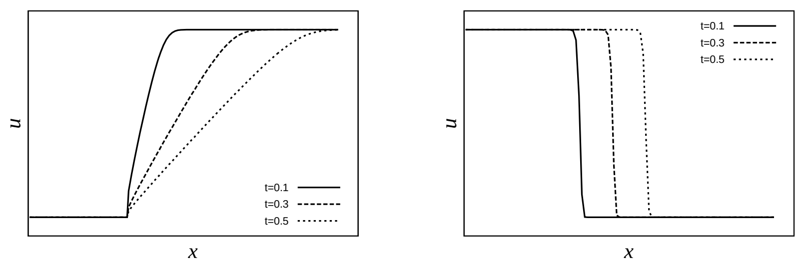
In the following two examples, Figures 3.6 (a) and 3.6 (b), we show results using



(a) Rarefaction wave (b) Shock

Figure 3.5: Modified numerical flux

the weighted numerical flux. In both examples we used the same grid, same initial conditions and the same CFL number as in previous two examples.



(a) Rarefaction wave (b) Shock

Figure 3.6: Modified numerical flux

In all examples we notice the presents of a “dog leg”. This is due the upwind numerical flux used and not to the large time step method.

3.5 Large Time Step in n Dimensions

In this section we present the idea for the large time step method in the n-dimensional case. Let us consider the system

$$u_t + \nabla \cdot F(u) = 0$$

where u is the unknown function defined on $R^n \times [0, +\infty)$ and F is an n space dimensional conservative flux function defined on the domain of conversation states. Let us assume that R^n has partition $\Delta = \{\Omega_i \mid i \in I\}$, and with $|\Omega_i|$ we are going to denote the size of cell Ω_i. Recall from Chapter 2 that the two point finite volume method is given by

$$u_i^{n+1} = u_i^n - \frac{\Delta t}{|\Omega_i|} \sum_k \int_{S_{i,k}} h_{n_k \cdot F}(u_i, u_k)\, ds.$$

With $S_{i,k}$ we have denoted the edge between neighboring cells Ω_i and Ω_k, n_k is the outward normal from the cell i to the cell k. Loosely speaking the time step size is restricted to be proportional to the smallest $diam(\Omega)/a$, where a is fastest wave speed. However for the method we present below the time step size restriction is proportional to L/a where the parameter L is independent of the grid.

We showed earlier in this chapter how a large time step method can be obtained in the one space dimensional case. Recall we use numerical fluxes which split into h^+ and h^-, and we then integrated these over a length L in order to get the large time step flux.

Let Ω_i be an arbitrary grid cell and d an arbitrary unit direction vector in R^n. Now, let us compute u_i^{n+1}. If we integrate flux h in the direction d over the length L,

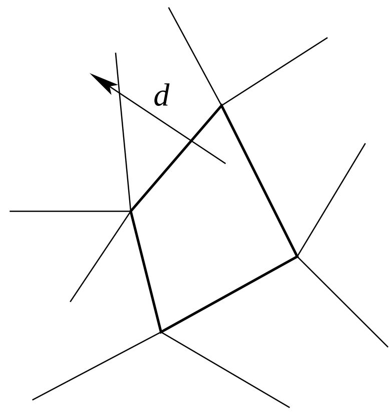


Figure 3.7: Cells and Fluxes

as in scalar case, we get that the only cells that are involved in the computation of u_i^{n+1} are those cells that are distance L from Ω_i. The question in n dimensions is in which directions should we integrate? We suggest that the flux should be integrated over L, as done earlier, and also averaged over all directions d. In other words, the formula for the numerical flux we propose is

$$h_{n_k \cdot F} = \frac{1}{|B(0,1)|} \int_{\partial B(0,1)} \left(\frac{1}{L} \int_0^L h^+_{(F\cdot d)(d\cdot n_k)} \left(u_\Delta^n \left(x - \alpha d\right)\right) d\alpha \right.$$
$$\left. + \frac{1}{L} \int_0^L h^-_{(F\cdot d)(d\cdot n_k)} \left(u_\Delta^n \left(x + \alpha d\right)\right) d\alpha \right) dS_d,$$

where $u_\Delta^n(x) = \sum_i u_i^n \chi_{\Omega_i}(x)$, x is a point on the edge $S_{i,k}$ and S_d is the surface of the unit ball centered at 0. Also, by $h_{(F\cdot d)(d\cdot n_k)}$, we think of the numerical flux h as consistent with $(F \cdot d)(d \cdot n_k)$ in the same way we did in Chapter 2. For example, let us consider a $2D$ problem with $F(u) = (f_1(u), f_2(u))$, $n_k = (n_1, n_2)$ and $d = (d_1, d_2)$. The numerical fluxes h^+ and h^- on edge $S_{i,k}$ are determined by and consistent with the one-dimensional function

$$(F(u) \cdot d)(d \cdot n_k) = (f_1(u)d_1 + f_2(u)d_2)(n_1 d_1 + n_2 d_2).$$

As one can see, the n-dimensional large time step numerical flux we proposed above is not so easy to implement. We integrated over all directions d in order to minimize possible directional bias. We have experimented with simpler variants and obtained satisfactory results.

Let us now show that our numerical flux is consistent with $F \cdot n_k$. We have

$$\begin{aligned}
\int_{\partial B(0,1)} (F \cdot d)\,(d \cdot n_k)\; dS_d &= \int_{\partial B(0,1)} ((F \cdot d)\, n_k) \cdot d\, dS_d \\
&= \int_{B(0,1)} \nabla_d \cdot ((F \cdot d)\, n_k)\; dV_d \\
&= \left(\int_{B(0,1)} \nabla_d\,(F \cdot d)\; dV_d \right) \cdot n_k \\
&= \left(\int_{B(0,1)} F\, dV_d \right) \cdot n_k \\
&= \left(\int_{B(0,1)} dV_d \right) (F \cdot n_k) \\
&= |B(0,1)|\,(F \cdot n_k)
\end{aligned}$$

where we have the fact that for constant vector n we have $\nabla \cdot (\rho n) = \nabla \rho \cdot n$. Therefore, for constant u

$$\begin{aligned}
h_{F \cdot n_k}(u) &= \frac{1}{|B(0,1)|} \int_{\partial B(0,1)} \left(\frac{1}{L} \int_0^L h^+_{(F \cdot d)(d \cdot n_k)}(u)\; d\alpha \right. \\
&\qquad \left. + \frac{1}{L} \int_0^L h^-_{(F \cdot d)(d \cdot n_k)}(u)\; d\alpha \right) dS_d \\
&= \frac{1}{|B(0,1)|} \int_{\partial B(0,1)} \left(h^+_{(F \cdot d)(d \cdot n_k)}(u) + h^-_{(F \cdot d)(d \cdot n_k)}(u) \right) dS_d \\
&= \frac{1}{|B(0,1)|} \int_{\partial B(0,1)} (F(u) \cdot d)(d \cdot n_k)\, dS_d \\
&= F(u) \cdot n_k.
\end{aligned}$$

Similar to the one-dimensional case, max-norm stability can be shown for this proposed large time step finite volume method for the scalar equation.

Chapter 4

Overlapping Grids in 1D

4.1 Introduction and Basic Ideas

Let us consider the Cauchy problem

$$w_t + f(w)_x = 0$$

with the initial condition $w(x, 0) = w_0(x)$, where $w_0(x)$ is a bounded and measurable function.

We consider this problem on two overlapping grids. Let a and b be such that $a < b$. Let the first grid, called the "bottom" grid, be the grid in the interval $(-\infty, b]$ partitioned using the partition Δ_B consisting of cells of the fixed size Δx_B. Similarly, let the second grid, called the "top" grid, be the grid in the interval $[a, \infty)$, partitioned using the partition Δ_T consisting of cells of fixed size Δx_T. For the simplicity, we assume that the cells in the bottom grid are indexed using non-positive integers ending with 0, and the cells in the top grid are indexed using non-negative integers starting with 0.

Let us denote with k_1 cell on the bottom grid such that

$$b + (k_1 - 1)\,\Delta x_B \le a \le b + k_1 \Delta x_B,$$

i.e., the cell k_1 contains the point a. Similarly let k_2 be the cell on the top grid such that

$$a + k_2 \Delta x_T \le b \le a + (k_2 + 1)\,\Delta x_T,$$

i.e., the cell k_2 contains the point b.

At this moment we introduce some more notation that will be used latter. If Δx denotes the minimum of Δx_B and Δx_T, we define

$$\Delta z_1 = a - (b + (k_1 - 1)\Delta x_B)\,, \qquad \Delta z_2 = a + k_2 \Delta x_T - b$$

and

$$\theta_1 = \min\left\{\frac{\Delta z_1}{\Delta x}, 1\right\}, \qquad \theta_2 = \min\left\{\frac{\Delta z_2}{\Delta x}, 1\right\}.$$

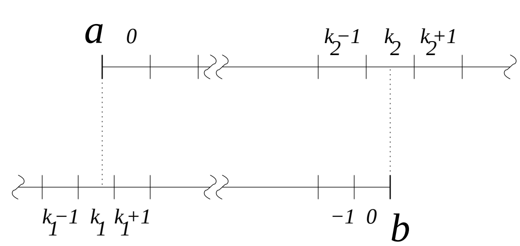


Figure 4.1: The bottom and the top grids

We present several straight forward ideas for solving the above Cauchy problem.

The first idea is that we make the union of these two grids and we *solve this problem on the union grid.* In this chapter we assume that the union of the bottom and the top grids is the grid on $\mathbb{R}$, where the partition on $\mathbb{R}$ is the union of the partitions Δ_B and Δ_T. In essence this idea is not bad, but it could happen that in the union grid we have at least one very small cell and as a consequence, the progress in time could be very slow. This issue could be resolved using some methods for large time steps such as the method we presented in the previous chapter.

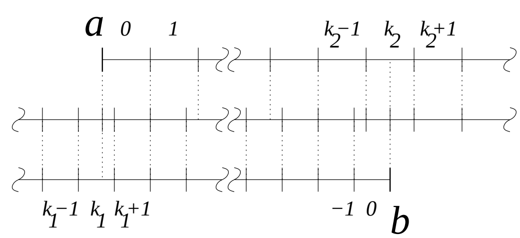


Figure 4.2: The union grid

The second way of solving the above Cauchy problem using large time step is by cutting one of the grids and *"gluing" the two grids together as one.* For example we choose the bottom grid and cut it to be in the interval $(-\infty, a]$ and we glue the top grid to the right of it. In this case, we can get one tiny cell, and it will be the cell k_1. As suggested earlier, we can use some methods for large time steps or some other methods in order to resolve this issue. Unfortunately, this idea of solving overlapping grid problem will not give the solution on the whole bottom grid and the method depends on which grid we choose to cut. Also, as we pointed out earlier our large time step method can be used only if the flux function could be separated.

Another, third method that we could use to solve the above Cauchy problem is to *solve given problem on each grid and then to average the approximate solutions on the overlap.* More precisely, let u and v be the approximate solutions on the bottom and the top grids, respectively. We define w, solution on the union of these two grids, by

$$w(x,t)=\begin{cases} u(x,t), & x\in(-\infty,a) \\ \frac{u(x,t)+v(x,t)}{2}, & x\in[a,b] \\ v(x,t), & x\in(b,\infty). \end{cases}$$

We note that this method is not conservative, and with that in mind we discard this method as well.

The method that we propose will be using the union grid and it requires the next two properties

- Conservation
- Maximum principle

Conservation is not a very challenging issue, because to achieve it we just need to "adjust" the fluxes at the end of the bottom grid and fluxes related to the cell k_2 on top grid, and the fluxes at the beginning of the top grid and those related to the cell k_1 on bottom grid. One naive idea for this could be interpolation of fluxes, i.e., $\tilde{f}\left(v_0^n\right)=\theta h\left(u_{k_1-1}^n,u_{k_1}^n\right)+(1-\theta)\,h\left(u_{k_1}^n,u_{k_1+1}^n\right)$ on the top grid and, similarly, for the end flux on the bottom grid. However, in this case, the maximum principle is not satisfied, so this solution has to be rejected. What we propose is similar to what we did in large time step discussion.

First, we extend both grids so that they cover $\mathbb{R}$ in such a way that the extension is just the projection of the other grid. In more detail, the cells left from a in the top grid are those from the bottom grid, and the cells right from b in the bottom are those cells from the top grid (see figure 3, where the middle grid is the union of the bottom and the top grid).

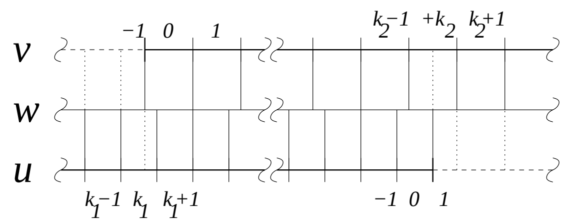


Figure 4.3: Indices and solutions

Next, in each time step we have three substeps. First, functions $\widetilde{u}$ and $\widetilde{v}$, on extended bottom and top grids, respectively, are computed using long time step discussed earlier with $L = \Delta x = \min\{\Delta x_B, \Delta x_T\}$. Second, we average these functions in order to get solution w, and finally this solution is projected back onto extended grids in order to get updated solutions, u and v for both extended grids.

One immediate question is why we are using the large time step method introduced earlier, which uses split fluxes, when we explicitly stated that we want to create the method that could be used with any numerical flux function. The reason why this is possible is because $L = \Delta x$ and when we are calculating fluxes we are using at most three cells. Hence, the split fluxes are not necessary.

We present the above three substeps in more detail. Let u^n_Δ, v^n_Δ, and w^n_Δ denote solutions at the n-th time step on the extended bottom, the extended top and the union grids, respectively. These three functions are step functions defined by

$$u_\Delta^n(x) = \sum_i u_i^n \chi_{B_i}(x), \qquad v_\Delta^n(x) = \sum_j v_j^n \chi_{T_j}(x), \qquad w_\Delta^n(x) = \sum_l w_l^n \chi_{U_l}(x),$$

where B_i is the i-th cell on the extended bottom grid and u_i^n is the value of the approximate solution in B_i, T_j is j-th cell on the extended top grid and v_j^n is the value of the approximate solution in T_j, and U_l is l-th cell on the union grid and w_l^n is the value of the approximate solution in U_l. Moreover, function χ is the characteristic function.

Now, having u_Δ^n, v_Δ^n and w_Δ^n in the n-th time step, let us compute them in time step $n+1$. We define

$$\widetilde{u}_\Delta^{n+1}(x) = \sum_i \widetilde{u}_i^{n+1} \chi_{B_i}(x), \qquad \widetilde{v}_\Delta^{n+1}(x) = \sum_j \widetilde{v}_j^{n+1} \chi_{T_j}(x),$$

where

$$\widetilde{u}_i^{n+1} = \begin{cases} u_i^n - \frac{\Delta t}{\Delta x_B}\left(h_f(u_i^n, u_{i+1}^n) - h_f(u_{i-1}^n, u_i^n)\right), & -\infty < i < 0, \\ u_i^n - \frac{\Delta t}{\Delta x_B}\left(F_l - h_f(u_{i-1}^n, u_i^n)\right), & i = 0, \\ u_i^n - \frac{\Delta t}{\Delta z_2}\left(F_r - F_l\right), & i = 1, \\ u_i^n - \frac{\Delta t}{\Delta x_T}\left(h_f(u_i^n, u_{i+1}^n) - F_r\right), & i = 2, \\ u_i^n - \frac{\Delta t}{\Delta x_T}\left(h_f(u_i^n, u_{i+1}^n) - h_f(u_{i-1}^n, u_i^n)\right), & 2 < i < \infty, \end{cases}$$

and

$$
\widetilde{v}_j^{n+1} = \begin{cases} v_j^n - \frac{\Delta t}{\Delta x_B}\left(h_f(v_j^n, v_{j+1}^n) - h_f(v_{j-1}^n, v_j^n)\right), & -\infty < j < -2, \\ v_j^n - \frac{\Delta t}{\Delta x_B}\left(f_l - h_f(v_j^n, v_{j-1}^n)\right), & j = -2, \\ v_j^n - \frac{\Delta t}{\Delta z_1}\left(f_r - f_l\right)), & j = -1, \\ v_j^n - \frac{\Delta t}{\Delta x_T}\left(h_f(v_j^n, v_{j+1}^n) - f_r\right), & j = 0, \\ v_j^n - \frac{\Delta t}{\Delta x_T}\left(h_f(v_j^n, v_{j+1}^n) - h_f(v_{j-1}^n, v_j^n)\right), & 1 \leq j < \infty. \end{cases}
$$

Here, the unknown end fluxes are computed by

$$
\begin{aligned}
f_l &= \theta_1 h_f(v_{-2}^n, v_{-1}^n) + (1-\theta_1) h_f(v_{-2}^n, v_0^n), \\
f_r &= \theta_1 h_f(v_{-1}^n, v_0^n) + (1-\theta_1) h_f(v_{-2}^n, v_0^n), \\
F_l &= \theta_2 h_f(u_0^n, u_1^n) + (1-\theta_2) h_f(u_0^n, u_2^n), \\
F_r &= \theta_2 h_f(u_1^n, u_2^n) + (1-\theta_2) h_f(u_0^n, u_2^n).
\end{aligned}
$$

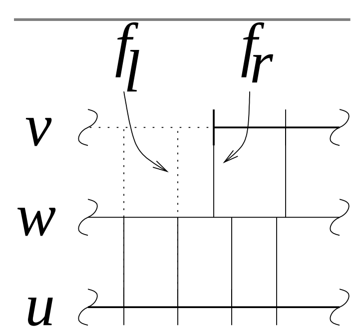


Figure 4.4: Unknown fluxes

Next, as mentioned earlier, we compute

$$
w_l^{n+1} = \frac{1}{2} \int_{U_l} \widetilde{u}_\Delta^{n+1}(x) + \widetilde{v}_\Delta^{n+1}(x)\, dx.
$$

Finally, we compute u_i^{n+1} and v_j^{n+1} by

$$u_i^{n+1} = \frac{1}{|B_i|} \int_{B_i} w_\Delta^{n+1}(x)\, dx, \qquad v_j^{n+1} = \frac{1}{|T_j|} \int_{T_j} w_\Delta^{n+1}(x)\, dx.$$

The main question is how big of a time step may we use? We claim that the time step can stay the same and that it depends only on Δx. To prove this, let us observe cell 1 on the bottom extended grid. We have

$$\begin{aligned}
\widetilde{u}_1^n &= u_1^n - \frac{\Delta t}{\Delta z_2} (F_r - F_l) \\
&= u_1^n - \frac{\Delta t}{\Delta z_2} (\theta_2 h_f(u_1^n, u_2^n) + (1-\theta_2) h_f(u_0^n, u_2^n) \\
&\qquad -\theta_2 h_f(u_0^n, u_1^n) - (1-\theta_2) h_f(u_0^n, u_2^n)) \\
&= u_1^n - \frac{\Delta t}{\Delta z_2} \theta_2 \left(h_f(u_1^n, u_2^n) - h_f(u_0^n, u_1^n) \right).
\end{aligned}$$

Now, if $\theta_2 = 1$, then we have that $\Delta z_2 \geq \Delta x$, so the time step depends on Δx, as earlier. If $\theta_2 < 1$, then $\theta_2 = \frac{\Delta z_2}{\Delta x}$, implying

$$\frac{\Delta t}{\Delta z_2} \theta_2 = \frac{\Delta t}{\Delta z_2} \frac{\Delta z_2}{\Delta x} = \frac{\Delta t}{\Delta x},$$

and again the time step depends only on Δx. We get the same result by performing similar calculations on the top extended grid.

Having the method written this way it is easy to see that both Conservation and Maximum principle hold. Beside these two properties we will show that when this method converges then the limit of the approximate solutions is the entropy solution of the above Cauchy problem.

4.2 Lax - Wendroff Theorem for the Overlapping Grids

In this section we will show that if our numerical solution converges boundedly almost everywhere to a function w as $|\Delta|$ and Δt converges to zero, then w is a weak solution of the observed hyperbolic conservation law.

Let us consider scalar conversational law

$$w_t + f(w)_x = 0 \text{ in } \mathbb{R} \times (0, \infty).$$

We assume that w_0 is a function of bounded total variation. Let Δ_B and Δ_T be partitions of half lines $(-\infty, b]$ and $[a, \infty)$ respectively with $a < b$. We assume that cells on the bottom grid are of size Δx_B and on the top grid of size Δx_T. Let $\Delta X = \max\{\Delta x_B, \Delta x_T\}$ and $\Delta x = \min\{\Delta x_B, \Delta x_T\}$, and let S be such that $\frac{\Delta X}{\Delta x} \leq S$ as $|\Delta| \to 0$.

We recall definitions of u^n_Δ, v^n_Δ and w^n_Δ, and the corresponding numerical fluxes given in previous section. For simplicity, we introduce projections on extended bottom and top grids and the union grid, P_u, P_v, and P_w, by

$$P_u(\xi)\mid_{B_i} = \frac{1}{|B_i|}\int_{B_i} \xi(x)\,dx, \qquad P_v\mid_{T_j}(\xi) = \frac{1}{|T_j|}\int_{T_j} \xi(x)\,dx,$$

$$P_w(\xi)\mid_{U_l} = \frac{1}{|U_l|}\int_{U_l} \xi(x)\,dx.$$

Using the above definition of P_u and P_v we rewrite our numerical scheme from the previous section by

$$w^{n+1}_\Delta = \frac{1}{2}\left(P_u(w^n_\Delta) + P_v(w^n_\Delta)\right) - \Delta t \frac{d}{dx}\frac{1}{2}\left(F_u + F_v\right).$$

Here F_u and F_v denote piecewise linear continuous functions obtained by interpolating numerical fluxes over the extended bottom and top grids, respectively.

Then, if w_Δ^n converges boundedly almost everywhere to a function w as $|\Delta|$ and Δt converges to 0 then w is a weak solution to the hyperbolic conservation law.

Proof

We show that w_Δ^n converges to a weak solution, w, as $|\Delta|$, $\Delta t \to 0$.

Let $\varphi \in C_0^\infty(\mathbb{R} \times [0,\infty))$ be a smooth test function with compact support. Then

$$w_\Delta^{n+1}\varphi(x,t_n) = \frac{1}{2}\left(P_u(w_\Delta^n) + P_v(w_\Delta^n)\right)\varphi(x,t_n) - \Delta t \frac{d}{dx}\frac{1}{2}\left(F_u + F_v\right)\varphi(x,t_n).$$

We subtract $w_\Delta^n\varphi(x,t_n)$ from both sides, integrate over $\mathbb{R}$ and sum over n to get

$$\begin{aligned}\sum_{n=0}^{\infty}\int_{\mathbb{R}}\left(w_\Delta^{n+1} - w_\Delta^n\right)\varphi(x,t_n)\,dx = \\ \frac{1}{2}\sum_{n=0}^{\infty}\int_{\mathbb{R}}\left(P_u(w_\Delta^n) - w_\Delta^n\right)\varphi(x,t_n)\,dx + \frac{1}{2}\sum_{n=0}^{\infty}\int_{\mathbb{R}}\left(P_v(w_\Delta^n) - w_\Delta^n\right)\varphi(x,t_n)\,dx \\ - \sum_{n=0}^{\infty}\Delta t\int_{\mathbb{R}}\frac{d}{dx}\frac{1}{2}\left(F_u + F_v\right)\varphi(x,t_n)\,dx.\end{aligned}$$

We show that w_Δ^n converges to a weak solution in three parts by analyzing the terms in above expression. First, we show that

$$\sum_{n=0}^{\infty}\Delta t\int_{\mathbb{R}}\frac{d}{dx}\frac{1}{2}\left(F_u + F_v\right)\varphi(x,t_n)\,dx \to -\int_0^\infty\int_{\mathbb{R}} f(w)\varphi_x(x,t)\,dx\,dt$$

as $|\Delta|$, $\Delta t \to 0$. Second, we show that

$$\sum_{n=0}^{\infty}\int_{\mathbb{R}}\left(P_u(w_\Delta^n) - w_\Delta^n\right)\varphi(x,t_n)\,dx \to 0$$

and

$$\sum_{n=0}^{\infty} \int_{\mathbb{R}} \left(P_v(w_\Delta^n) - w_\Delta^n\right) \varphi(x, t_n)\, dx \to 0$$

as $\Delta t \to 0$. And in the third part we will show that

$$\sum_{n=0}^{\infty} \int_{\mathbb{R}} \left(w_\Delta^{n+1} - w_\Delta^n\right) \varphi(x, t_n)\, dx \to -\int_0^{\infty} \int_{\mathbb{R}} w \varphi_t(x, t)\, dx\, dt - \int_{\mathbb{R}} w^0 \varphi(x, t_0)\, dx$$

as $|\Delta|$, $\Delta t \to 0$. By putting these three parts together we get that w is a weak solution, i.e.,

$$\int_0^{\infty} \int_{\mathbb{R}} w \varphi_t(x, t)\, dx\, dt + \int_0^{\infty} \int_{\mathbb{R}} f(w) \varphi_x(x, t)\, dx\, dt + \int_{\mathbb{R}} w^0 \varphi(x, t_0)\, dx = 0$$

holds for every test function $\varphi \in C_0^{\infty}$.

Let M be such that $|\varphi_x(x, t)| \leq M$, for $x \in \mathbb{R}$ and $t \geq 0$. Also let Lip be the Lipshitz constant for the numerical flux h.

Part I

Using integration by parts we have that

$$\sum_{n=0}^{\infty} \Delta t \int_{\mathbb{R}} \frac{d}{dx} \frac{1}{2} \left(F_u + F_v\right) \varphi(x, t_n)\, dx =$$

$$\sum_{n=0}^{\infty} \frac{1}{2} \Delta t \left(\lim_{\zeta \to +\infty} \left(F_u + F_v\right) \varphi(x, t_n)|_{-\zeta}^{\zeta} - \int_{\mathbb{R}} \left(F_u + F_v\right) \varphi_x(x, t_n)\, dx \right).$$

The first term in the above sum is zero because φ has compact support. Hence,

$$\sum_{n=0}^{\infty} \Delta t \int_{\mathbb{R}} \frac{d}{dx} \frac{1}{2} \left(F_u + F_v\right) \varphi(x, t_n)\, dx$$

$$\begin{aligned}
&= -\sum_{n=0}^{\infty} \frac{1}{2} \Delta t \int_{\mathbb{R}} \left(F_u - F_{P_u}\right) \varphi_x(x, t_n)\, dx - \sum_{n=0}^{\infty} \frac{1}{2} \Delta t \int_{\mathbb{R}} \left(F_{P_u} - f(w)\right) \varphi_x(x, t_n)\, dx \\
&\quad - \sum_{n=0}^{\infty} \frac{1}{2} \Delta t \int_{\mathbb{R}} \left(F_v - F_{P_v}\right) \varphi_x(x, t_n)\, dx - \sum_{n=0}^{\infty} \frac{1}{2} \Delta t \int_{\mathbb{R}} \left(F_{P_v} - f(w)\right) \varphi_x(x, t_n)\, dx \\
&\quad - \sum_{n=0}^{\infty} \Delta t \int_{\mathbb{R}} f(w) \varphi_x(x, t_n)\, dx
\end{aligned}$$

where $F_{P_u} = f(P_u)$ and $F_{P_v} = f(P_v)$. Next, we analyze the sums in the above expression. We will show that the first four sums converge to 0 and it is clear that the last sum converges to $\int_0^{\infty} \int_{\mathbb{R}} f(w) \varphi_x(x, t)\, dx\, dt$, as $\Delta, \Delta t \to 0$. We split the proof into two steps where we show that the first and second sums converge to zero. The proofs for the third and forth sums follow similarly.

Step I

We show that $\sum_{n=0}^{\infty} \Delta t \int_{\mathbb{R}} \left(F_u - F_{p_u}\right) \varphi_x(x, t_n)\, dx \to 0$ as $\Delta, \Delta t \to 0$.

We use the triangular inequality, the fact $|\varphi_x(x, t_n)| \leq M$, and we split the integral to get

$$
\begin{aligned}
&\left|\sum_{n=0}^{\infty}\Delta t\int_{\mathbb{R}}(F_u - F_{p_u})\,\varphi_x(x,t_n)\,dx\right| \leq \\
&\qquad \sum_{n=0}^{\infty}\sum_{i=-\infty}^{-1} M\Delta t\int_{x_{i-1}}^{x_i}|F_u - F_{p_u}|\;dx + \sum_{n=0}^{\infty} M\Delta t\int_{x_{-1}}^{x_0}|F_u - F_{p_u}|\;dx \\
&\qquad + \sum_{n=0}^{\infty} M\Delta t\int_{x_0}^{x_1}|F_u - F_{p_u}|\;dx + \sum_{n=0}^{\infty} M\Delta t\int_{x_1}^{x_2}|F_u - F_{p_u}|\;dx \\
&\qquad + \sum_{n=0}^{\infty}\sum_{i=3}^{\infty} M\Delta t\int_{x_{i-1}}^{x_i}|F_u - F_{p_u}|\;dx.
\end{aligned}
$$

1^o For $i \in \{-1,-2,\dots\}$, using the definition of the numerical fluxes, we have

$$
\begin{aligned}
&\int_{x_{i-1}}^{x_i}|F_u - F_{p_u}|\;dx \\
&\quad = \int_{x_{i-1}}^{x_i}\left|\left(1-\frac{x-x_{i-1}}{\Delta x_B}\right)h(u_{i-1}^n,u_i^n) + \frac{x-x_{i-1}}{\Delta x_B}h(u_i^n,u_{i+1}^n) - h(u_i^n,u_i^n)\right|\;dx \\
&\quad \leq \int_{x_{i-1}}^{x_i}\left(1-\frac{x-x_{i-1}}{\Delta x_B}\right)\left|h(u_{i-1}^n,u_i^n) - h(u_i^n,u_i^n)\right| \\
&\qquad + \frac{x-x_{i-1}}{\Delta x_B}\left|h(u_i^n,u_{i+1}^n) - h(u_i^n,u_i^n)\right|\;dx.
\end{aligned}
$$

We use the fact that h is Lipschitz continuous with constant Lip to get

$$
\begin{aligned}
&\int_{x_{i-1}}^{x_i} |F_u - F_{p_u}| \, dx \\
&\quad \le Lip \int_{x_{i-1}}^{x_i} \left(1 - \frac{x - x_{i-1}}{\Delta x_B}\right) \left|u_{i-1}^n - u_i^n\right| + \frac{x - x_{i-1}}{\Delta x_B} \left|u_{i+1}^n - u_i^n\right| \, dx \\
&\quad = Lip \frac{\Delta x_B}{2} \left(\left|u_{i-1}^n - u_i^n\right| + \left|u_{i+1}^n - u_i^n\right|\right) \\
&\quad = Lip \frac{\Delta x_B}{2} \frac{1}{\Delta x} \int_0^{\Delta x} d\zeta \left(\left|u_{i-1}^n - u_i^n\right| + \left|u_{i+1}^n - u_i^n\right|\right) \\
&\quad = Lip \frac{\Delta x_B}{2\Delta x} \left(\int_0^{\Delta x} |u_\Delta^n(x_{i-1} - \zeta) - u_i^n| \, d\zeta + \int_0^{\Delta x} |u_\Delta^n(x_i + \zeta) - u_i^n| \, d\zeta\right) \\
&\quad = \frac{Lip}{2\Delta x} \left(\int_{x_{i-1}}^{x_i} \int_0^{\Delta x} |u_\Delta^n(x_{i-1} - \zeta) - u_i^n| \, d\zeta \, dx \right. \\
&\qquad \left. + \int_{x_{i-1}}^{x_i} \int_0^{\Delta x} |u_\Delta^n(x_i + \zeta) - u_i^n| \, d\zeta \, dx\right) \\
&\quad = \frac{Lip}{2\Delta x} \left(\int_{x_{i-1}}^{x_i} \int_0^{\Delta x} |u_\Delta^n(x_{i-1} - \zeta) - u_\Delta^n(x)| \, d\zeta \, dx \right. \\
&\qquad \left. + \int_{x_{i-1}}^{x_i} \int_0^{\Delta x} |u_\Delta^n(x_i + \zeta) - u_\Delta^n(x)| \, d\zeta \, dx\right) \\
&\quad \le \frac{Lip}{2\Delta x} \left(\int_{x_{i-1}}^{x_i} \int_0^{\Delta x} |u_\Delta^n(x - \zeta) - u_\Delta^n(x)| \, d\zeta \, dx \right. \\
&\qquad \left. + \int_{x_{i-1}}^{x_i} \int_0^{\Delta x} |u_\Delta^n(x + \zeta) - u_\Delta^n(x)| \, d\zeta \, dx\right).
\end{aligned}
$$

We sum over all $i \in \{-1, -2, \dots\}$, we get

$$
\begin{aligned}
&\sum_{i=-\infty}^{-1} \int_{x_{i-1}}^{x_i} |F_u - F_{P_u}| \, dx \\
&\quad \le \frac{Lip}{2\Delta x} \left(\int_{-\infty}^{x_{-1}} \int_0^{\Delta x} |u_\Delta^n(x - \zeta) - u_\Delta^n(x)| \, d\zeta \, dx \right. \\
&\qquad \left. + \int_{-\infty}^{x_{-1}} \int_0^{\Delta x} |u_\Delta^n(x + \zeta) - u_\Delta^n(x)| \, d\zeta \, dx\right).
\end{aligned}
$$

2^o Next, we consider the case $i = 0$. Using the definition of the numerical fluxes and that h is Lipschitz continuous, we have

$$
\begin{aligned}
&\int_{x_{i-1}}^{x_i} \left|F_u - F_{p_u}\right| \, dx \\
&\quad = \int_{x_{i-1}}^{x_i} \Bigg| \left(1 - \frac{x - x_{i-1}}{\Delta x_B}\right) h(u_{i-1}^n, u_i^n) \\
&\qquad + \frac{x - x_{i-1}}{\Delta x_B} \left(\theta_2 h(u_i^n, u_{i+1}^n) + (1-\theta_2) h(u_i^n, u_{i+2}^n)\right) - h(u_i^n, u_i^n) \Bigg| \, dx \\
&\quad \leq \int_{x_{i-1}}^{x_i} \left(1 - \frac{x - x_{i-1}}{\Delta x_B}\right) \left|h(u_{i-1}^n, u_i^n) - h(u_i^n, u_i^n)\right| \\
&\qquad + \frac{x - x_{i-1}}{\Delta x_B} \theta_2 \left|h(u_i^n, u_{i+1}^n) - h(u_i^n, u_i^n)\right| \\
&\qquad + \frac{x - x_{i-1}}{\Delta x_B} (1-\theta_2) \left|h(u_i^n, u_{i+2}^n) - h(u_i^n, u_i^n)\right| \, dx \\
&\quad \leq Lip \int_{x_{i-1}}^{x_i} \left(1 - \frac{x - x_{i-1}}{\Delta x_B}\right) \left|u_{i-1}^n - u_i^n\right| + \frac{x - x_{i-1}}{\Delta x_B} \theta_2 \left|u_{i+1}^n - u_i^n\right| \\
&\qquad + \frac{x - x_{i-1}}{\Delta x_B} (1-\theta_2) \left|u_{i+2}^n - u_i^n\right| \, dx \\
&\quad = Lip \frac{\Delta x_B}{2} \left(\left|u_{i-1}^n - u_i^n\right| + \theta_2 \left|u_{i+1}^n - u_i^n\right| + (1-\theta_2) \left|u_{i+2}^n - u_i^n\right|\right) \\
&\quad = Lip \frac{\Delta x_B}{2} \frac{1}{\Delta x} \int_0^{\Delta x} d\zeta \left(\left|u_{i-1}^n - u_i^n\right| + \theta_2 \left|u_{i+1}^n - u_i^n\right| + (1-\theta_2) \left|u_{i+2}^n - u_i^n\right|\right) \\
&\quad = Lip \frac{\Delta x_B}{2} \frac{1}{\Delta x} \left(\int_0^{\Delta x} \left|u_{i-1}^n - u_i^n\right| \, d\zeta \right. \\
&\qquad \left. + \int_0^{\Delta x} \theta_2 \left|u_{i+1}^n - u_i^n\right| + (1-\theta_2) \left|u_{i+2}^n - u_i^n\right| \, d\zeta \right) \\
&\quad = Lip \frac{\Delta x_B}{2} \frac{1}{\Delta x} \left(\int_0^{\Delta x} \left|u_\Delta^n(x_{i-1} - \zeta) - u_i^n\right| \, d\zeta \right. \\
&\qquad \left. + \int_0^{\Delta x} \left|u_\Delta^n(x_i + \zeta) - u_i^n\right| \, d\zeta \right) \\
&\quad \leq \frac{Lip}{2} \frac{1}{\Delta x} \left(\int_{x_{i-1}}^{x_i} \int_0^{\Delta x} \left|u_\Delta^n(x - \zeta) - u_\Delta^n(x)\right| \, d\zeta \, dx \right. \\
&\qquad \left. + \int_{x_{i-1}}^{x_i} \int_0^{\Delta x} \left|u_\Delta^n(x + \zeta) - u_\Delta^n(x)\right| \, d\zeta \, dx \right).
\end{aligned}
$$

3^o When $i = 1$, we have

$$\int_{x_{i-1}}^{x_i} |F_u - F_{p_u}|\, dx \leq$$

$$\int_{x_{i-1}}^{x_i} \left| \left(1 - \frac{x - x_{i-1}}{\Delta z_2}\right) \left(\theta_2 h(u_{i-1}^n, u_i^n) + (1-\theta_2) h(u_{i-1}^n, u_{i+1}^n)\right) \right.$$
$$\left. + \frac{x - x_{i-1}}{\Delta z_2} \left((1-\theta_2) h(u_{i-1}^n, u_{i+1}^n) + \theta_2 h(u_i^n, u_{i+1}^n)\right) - h(u_i^n, u_i^n) \right| dx$$
$$\leq \int_{x_{i-1}}^{x_i} \left(1 - \frac{x - x_{i-1}}{\Delta z_2}\right) \theta_2 \left| h(u_{i-1}^n, u_i^n) - h(u_i^n, u_i^n) \right|$$
$$+ \left(1 - \frac{x - x_{i-1}}{\Delta z_2}\right) (1-\theta_2) \left| h(u_{i-1}^n, u_{i+1}^n) - h(u_i^n, u_i^n) \right|$$
$$+ \frac{x - x_{i-1}}{\Delta z_2} (1-\theta_2) \left| h(u_{i-1}^n, u_{i+1}^n) - h(u_i^n, u_i^n) \right|$$
$$+ \frac{x - x_{i-1}}{\Delta z_2} \theta_2 \left| h(u_i^n, u_{i+1}^n) - h(u_i^n, u_i^n) \right| dx$$
$$\leq Lip \int_{x_{i-1}}^{x_i} \left(1 - \frac{x - x_{i-1}}{\Delta z_2}\right) \theta_2 \left| u_{i-1}^n - u_i^n \right|$$
$$+ \left(1 - \frac{x - x_{i-1}}{\Delta z_2}\right) (1-\theta_2) \left(\left| u_{i-1}^n - u_i^n \right| + \left| u_{i+1}^n - u_i^n \right|\right)$$
$$+ \frac{x - x_{i-1}}{\Delta z_2} (1-\theta_2) \left(\left| u_{i-1}^n - u_i^n \right| + \left| u_{i+1}^n - u_i^n \right|\right)$$
$$+ \frac{x - x_{i-1}}{\Delta z_2} \theta_2 \left| u_{i+1}^n - u_i^n \right| dx$$
$$= Lip \frac{\Delta z_2}{2} (2 - \theta_2) \left(\left| u_{i-1}^n - u_i^n \right| + \left| u_{i+1}^n - u_i^n \right|\right).$$

We emphasize that $0 \leq \theta_2 \leq 1$, therefore,

$$
\begin{aligned}
&\int_{x_{i-1}}^{x_i} |F_u - F_{p_u}| \, dx \leq \\
&\quad Lip \frac{\Delta z_2}{\Delta x} \int_0^{\Delta x} d\zeta \left(\left|u_{i-1}^n - u_i^n\right| + \left|u_{i+1}^n - u_i^n\right| \right) \\
&\quad = Lip \frac{\Delta z_2}{\Delta x} \left(\int_0^{\Delta x} \left|u_{i-1}^n - u_i^n\right| \, d\zeta + \int_0^{\Delta x} \left|u_{i+1}^n - u_i^n\right| \, d\zeta \right) \\
&\quad = \frac{Lip}{\Delta x} \int_{x_{i-1}}^{x_i} dx \left(\int_0^{\Delta x} |u_\Delta^n(x_{i-1} - \zeta) - u_i^n| \, d\zeta + \int_0^{\Delta x} |u_\Delta^n(x_i + \zeta) - u_i^n| \, d\zeta \right) \\
&\quad \leq \frac{Lip}{\Delta x} \left(\int_{x_{i-1}}^{x_i} \int_0^{\Delta x} |u_\Delta^n(x - \zeta) - u_\Delta^n(x)| \, d\zeta \, dx \right. \\
&\quad \left. + \int_{x_{i-1}}^{x_i} \int_0^{\Delta x} |u_\Delta^n(x + \zeta) - u_\Delta^n(x)| \, d\zeta \, dx \right).
\end{aligned}
$$

As seen earlier in Chapter 3.2 condition (C2) from Section 2.4 is not required here.

4^o When $i=2$, we have

$$\begin{aligned}
&\int_{x_{i-1}}^{x_i} \left|F_u - F_{p_u}\right| \, dx \\
&\quad \leq \int_{x_{i-1}}^{x_i} \Bigg| \left(1 - \frac{x - x_{i-1}}{\Delta x_T}\right) \left(\theta_2 h(u_{i-1}^n, u_i^n) + (1-\theta_2) h(u_{i-2}^n, u_i^n)\right) \\
&\qquad + \frac{x - x_{i-1}}{\Delta x_T} h(u_i^n, u_{i+1}^n) - h(u_i^n, u_i^n) \Bigg| \, dx \\
&\quad \leq \int_{x_{i-1}}^{x_i} \left(1 - \frac{x - x_{i-1}}{\Delta x_T}\right) \theta_2 \left|h(u_{i-1}^n, u_i^n) - h(u_i^n, u_i^n)\right| \\
&\qquad + \left(1 - \frac{x - x_{i-1}}{\Delta x_T}\right) (1-\theta_2) \left|h(u_{i-2}^n, u_i^n) - h(u_i^n, u_i^n)\right| \\
&\qquad + \frac{x - x_{i-1}}{\Delta x_T} \left|h(u_i^n, u_{i+1}^n) - h(u_i^n, u_i^n)\right| \, dx \\
&\quad \leq Lip \frac{\Delta x_T}{2} \left(\theta_2 \left|u_{i-1}^n - u_i^n\right| + (1-\theta_2) \left|u_{i-2}^n - u_i^n\right| + \left|u_{i+1}^n - u_i^n\right|\right) \\
&\quad = Lip \frac{\Delta x_T}{2} \frac{1}{\Delta x} \int_0^{\Delta x} d\zeta \left(\theta_2 \left|u_{i-1}^n - u_i^n\right| + (1-\theta_2) \left|u_{i-2}^n - u_i^n\right| + \left|u_{i+1}^n - u_i^n\right|\right) \\
&\quad = Lip \frac{\Delta x_T}{2} \frac{1}{\Delta x} \left(\int_0^{\Delta x} \theta_2 \left|u_{i-1}^n - u_i^n\right| + (1-\theta_2) \left|u_{i-2}^n - u_i^n\right| \, d\zeta \right. \\
&\qquad \left. + \int_0^{\Delta x} \left|u_{i+1}^n - u_i^n\right| \, d\zeta \right) \\
&\quad = Lip \frac{\Delta x_T}{2} \frac{1}{\Delta x} \left(\int_0^{\Delta x} \left|u_\Delta^n(x_{i-1} - \zeta) - u_i^n\right| \, d\zeta + \int_0^{\Delta x} \left|u_\Delta^n(x_i + \zeta) - u_i^n\right| \, d\zeta\right) \\
&\quad = \frac{Lip}{2} \frac{1}{\Delta x} \left(\int_{x_{i-1}}^{x_i} \int_0^{\Delta x} \left|u_\Delta^n(x_{i-1} - \zeta) - u_i^n\right| \, d\zeta \, dx \right. \\
&\qquad \left. + \int_{x_{i-1}}^{x_i} \int_0^{\Delta x} \left|u_\Delta^n(x_i + \zeta) - u_i^n\right| \, d\zeta \, dx \right) \\
&\quad \leq \frac{Lip}{2} \frac{1}{\Delta x} \left(\int_{x_{i-1}}^{x_i} \int_0^{\Delta x} \left|u_\Delta^n(x - \zeta) - u_\Delta^n(x)\right| \, d\zeta \, dx \right. \\
&\qquad \left. + \int_{x_{i-1}}^{x_i} \int_0^{\Delta x} \left|u_\Delta^n(x + \zeta) - u_\Delta^n(x)\right| \, d\zeta \, dx \right).
\end{aligned}$$

5^o When $i \in \{3, 4, \dots\}$, we have

$$\begin{aligned}
&\int_{x_{i-1}}^{x_i} |F_u - F_{p_u}|\, dx \\
&\quad = \int_{x_{i-1}}^{x_i} \left| \left(1 - \frac{x - x_{i-1}}{\Delta x_T}\right) h(u_{i-1}^n, u_i^n) + \frac{x - x_{i-1}}{\Delta x_T} h(u_i^n, u_{i+1}^n) - h(u_i^n, u_i^n) \right| dx \\
&\quad \le \int_{x_{i-1}}^{x_i} \left(1 - \frac{x - x_{i-1}}{\Delta x_T}\right) \left| h(u_{i-1}^n, u_i^n) - h(u_i^n, u_i^n) \right| \\
&\qquad + \frac{x - x_{i-1}}{\Delta x_T} \left| h(u_i^n, u_{i+1}^n) - h(u_i^n, u_i^n) \right| \, dx \\
&\quad \le Lip \int_{x_{i-1}}^{x_i} \left(1 - \frac{x - x_{i-1}}{\Delta x_T}\right) \left| u_{i-1}^n - u_i^n \right| + \frac{x - x_{i-1}}{\Delta x_T} \left| u_{i+1}^n - u_i^n \right| \, dx \\
&\quad = Lip \frac{\Delta x_T}{2} \left(\left| u_{i-1}^n - u_i^n \right| + \left| u_{i+1}^n - u_i^n \right| \right) \\
&\quad = Lip \frac{\Delta x_T}{2} \frac{1}{\Delta x} \int_0^{\Delta x} d\zeta \left(\left| u_{i-1}^n - u_i^n \right| + \left| u_{i+1}^n - u_i^n \right| \right) \\
&\quad = \frac{Lip}{\Delta x} \frac{\Delta x_T}{2} \left(\int_0^{\Delta x} \left| u_{i-1}^n - u_i^n \right| \, d\zeta + \int_0^{\Delta x} \left| u_{i+1}^n - u_i^n \right| \, d\zeta \right) \\
&\quad = \frac{Lip}{\Delta x} \frac{\Delta x_T}{2} \left(\int_0^{\Delta x} |u_\Delta^n(x_{i-1} - \zeta) - u_i^n| \, d\zeta + \int_0^{\Delta x} |u_\Delta^n(x_i + \zeta) - u_i^n| \, d\zeta \right) \\
&\quad = \frac{Lip}{2\Delta x} \int_{x_{i-1}}^{x_i} dx \left(\int_0^{\Delta x} |u_\Delta^n(x_{i-1} - \zeta) - u_i^n| \, d\zeta + \int_0^{\Delta x} |u_\Delta^n(x_i + \zeta) - u_i^n| \, d\zeta \right) \\
&\quad \le \frac{Lip}{2\Delta x} \left(\int_{x_{i-1}}^{x_i} \int_0^{\Delta x} |u_\Delta^n(x - \zeta) - u_\Delta^n(x)| \, d\zeta \, dx \right. \\
&\qquad \left. + \int_{x_{i-1}}^{x_i} \int_0^{\Delta x} |u_\Delta^n(x + \zeta) - u_\Delta^n(x)| \, d\zeta \, dx \right).
\end{aligned}$$

We sum over all i, $i \in \{3, 4, \dots\}$ to get

$$\begin{aligned}
\sum_{i=3}^{\infty} \int_{x_{i-1}}^{x_i} |F_u - F_{p_u}| \, dx \quad &\le \quad \frac{Lip}{2\Delta x} \int_{x_2}^{\infty} \int_0^{\Delta x} |u_\Delta^n(x - \zeta) - u_\Delta^n(x)| \, d\zeta \, dx \\
&\quad + \frac{Lip}{2\Delta x} \int_{x_2}^{\infty} \int_0^{\Delta x} |u_\Delta^n(x + \zeta) - u_\Delta^n(x)| \, d\zeta \, dx.
\end{aligned}$$

Finally, using all the above cases and estimates we have

$$\begin{aligned}
&\left|\sum_{n=0}^{\infty} \Delta t \int_{-\infty}^{\infty} (F_u - F_{p_u})\, \varphi_x(x, t_n)\, dx\right| \\
&\qquad \leq \frac{MLip}{\Delta x} \sum_{n=0}^{\infty} \Delta t \int_{\text{supp}\varphi} \int_0^{\Delta x} |u_\Delta^n(x+\zeta) - u_\Delta^n(x)|\, d\zeta\, dx \\
&\qquad\quad + \frac{MLip}{\Delta x} \sum_{n=0}^{\infty} \Delta t \int_{\text{supp}\varphi} \int_0^{\Delta x} |u_\Delta^n(x-\zeta) - u_\Delta^n(x)|\, d\zeta\, dx.
\end{aligned}$$

Where suppφ denotes the support of the test function φ.

Since $u_\Delta^n \in L^1$ and since C_0^∞ is dense in L^1 we have that for every ε there exists a function $g \in C_0^\infty$ such that $||u_\Delta^n - g||_1 < \varepsilon$. Then

$$\begin{aligned}
|u_\Delta^n(x-\zeta) - u_\Delta^n(x)| &\leq |u_\Delta^n(x-\zeta) - g(x-\zeta)| + |g(x-\zeta) - g(x)| + |g(x) - u(x)| \\
&\leq \varepsilon + |g(x-\zeta) - g(x)| + \varepsilon.
\end{aligned}$$

Similarly

$$\begin{aligned}
|u_\Delta^n(x+\zeta) - u_\Delta^n(x)| &\leq |u_\Delta^n(x+\zeta) - g(x+\zeta)| + |g(x+\zeta) - g(x)| + |g(x) - u(x)| \\
&\leq \varepsilon + |g(x+\zeta) - g(x)| + \varepsilon.
\end{aligned}$$

Since g is continuous we have that for every ε there exists some δ_0 such that for every $\Delta X < \delta_0$ we have $|g(x \pm \zeta) - g(x)| < \varepsilon$, hence,

$$|u_\Delta^n(x-\zeta) - u_\Delta^n(x)| \leq 3\varepsilon$$

and

$$|u_\Delta^n(x+\zeta) - u_\Delta^n(x)| \leq 3\varepsilon.$$

Therefore,

$$\left|\sum_{n=0}^{\infty} \Delta t \int_{-\infty}^{\infty} (F_u - F_{p_u})\, \varphi_x(x, t_n)\, dx\right| \to \frac{6MLip}{\Delta x}\varepsilon \int_0^{\infty} \int_{\mathrm{supp}\varphi} \int_0^{\Delta x} d\zeta\, dx\, dt.$$

Since we work in finite time, we can take the time integral to be from 0 to T. Then

$$\frac{6MLip}{\Delta x}\varepsilon \int_0^{T} \int_{\mathrm{supp}\varphi} \int_0^{\Delta x} d\zeta\, dx\, dt = 6MTLip|\mathrm{supp}\varphi|\varepsilon$$

implying

$$\left|\sum_{n=0}^{\infty} \Delta t \int_{\mathbb{R}} (F_u - F_{p_u})\, \varphi_x(x, t_n)\, dx\right| \to 0$$

as $|\Delta|$, $\Delta t \to 0$.

Step II

We show that $\sum_{n=0}^{\infty} \Delta t \int_{\mathbb{R}} (F_{p_u} - f(w))\, \varphi_x(x, t_n)\, dx \to 0$ as $|\Delta| \to 0$ and $\Delta t \to 0$.

Note

$$\left|\sum_{n=0}^{\infty} \Delta t \int_{\mathbb{R}} (F_{p_u} - f(w))\, \varphi_x(x, t_n)\, dx\right| \le \sum_{n=0}^{\infty} \Delta t \sum_{l} \int_{x_{l-1}}^{x_l} |F_{p_u} - f(w)|\, |\varphi_x(x, t_n)|\, dx.$$

Using that h is Lipschitz continuous with constant Lip and that $|\varphi_x(x, t_n)| \le M$, we have

$$\left|\sum_{n=0}^{\infty} \Delta t \int_{\mathbb{R}} (F_{p_u} - f(w))\, \varphi_x(x, t_n)\, dx\right| \le 2MLip \sum_{n=0}^{\infty} \Delta t \int_{\mathbb{R}} |P_u(w_\Delta^n) - w_\Delta^n|\, dx.$$

If we let $\Delta t \to 0$ we will get

$$2MLip \sum_{n=0}^{\infty} \Delta t \int_{\mathbb{R}} |P_u(w_\Delta^n) - w_\Delta^n|\, dx \to 2MLip \int_0^{\infty} \int_{\mathbb{R}} |P_u(w_\Delta^n) - w_\Delta^n|\, dx.$$

Then

$$\lim_{\Delta t \to 0} \left|\sum_{n=0}^{\infty} \Delta t \int_{\mathbb{R}} (F_{p_u} - f(w))\, \varphi_x(x, t_n)\, dx\right| \le 2MLip \int_0^{\infty} \int_{\mathbb{R}} |P_u(w_\Delta^n) - w_\Delta^n|\, dx.$$

Since we have, $\int_{-\infty}^{\infty} |P_u(w_\Delta^n) - w_\Delta^n| \; dx = 0$, it follows that

$$\lim_{\Delta t \to 0} \left| \sum_{n=0}^{\infty} \Delta t \int_{\mathbb{R}} (F_{p_u} - f(w)) \; \varphi_x(x, t_n) \, dx \right| \\ \leq 2MLip \int_0^{\infty} \int_{\mathbb{R}} |P_u(w_\Delta^n) - w_\Delta^n| \; dx = 0.$$

By proving steps I and II we have

$$\sum_{n=0}^{\infty} \Delta t \int_{\mathbb{R}} \frac{d}{dx} \frac{1}{2} (F_u + F_v) \, \varphi(x, t_n) \, dx \to -\int_0^{\infty} \int_{\mathbb{R}} f(w) \varphi_x(x, t_n) \, dx \, dt$$

as $|\Delta| \to 0$ and $\Delta t \to 0$.

II Part

We show that $\sum_{n=0}^{\infty} \int_{\mathbb{R}} (P_u(w_\Delta^n) - w_\Delta^n) \, \varphi(x, t_n) \, dx \to 0$ as $\Delta t \to 0$. Note

$$\begin{aligned} &\sum_{n=0}^{\infty} \int_{\mathbb{R}} (P_u(w_\Delta^n) - w_\Delta^n) \, \varphi(x, t_n) \, dx \\ &\quad = \sum_{n=0}^{\infty} \sum_{l} \int_{x_l}^{x_{l+1}} (P_u(w_\Delta^n) - w_\Delta^n) \, (\varphi(x, t_n) - \varphi(x_l, t_n) + \varphi(x_l, t_n)) \; dx \\ &\quad = \sum_{n=0}^{\infty} \sum_{l} \int_{x_l}^{x_{l+1}} (P_u(w_\Delta^n) - w_\Delta^n) \, (\varphi(x, t_n) - \varphi(x_l, t_n)) \; dx \\ &\quad + \sum_{n=0}^{\infty} \sum_{l} \varphi(x_l, t_n) \int_{x_l}^{x_{l+1}} P_u(w_\Delta^n) - w_\Delta^n \, dx. \end{aligned}$$

We have $\int_{x_l}^{x_{l+1}} P_u(w_\Delta^n) \, dx = \int_{x_l}^{x_{l+1}} w_\Delta^n \, dx$ for all l, implying $\int_{x_l}^{x_{l+1}} P_u(w_\Delta^n) - w_\Delta^n \, dx = 0$. Therefore,

$$\sum_{n=0}^{\infty} \int_{\mathbb{R}} (P_u(w_\Delta^n) - w_\Delta^n) \varphi(x, t_n) \, dx \\ = \sum_{n=0}^{\infty} \sum_{l} \int_{x_l}^{x_{l+1}} (P_u(w_\Delta^n) - w_\Delta^n) \, (\varphi(x, t_n) - \varphi(x_l, t_n)) \; dx.$$

Since, $|\varphi(x,t_n)-\varphi(x_l,t_n)| \le M\,|x-x_l| \le M\Delta x_B \le M\Delta t$, then

$$\begin{aligned}
&\left|\sum_{n=0}^{\infty}\int_{\mathbb{R}}\left(P_u(w_\Delta^n)-w_\Delta^n\right)\varphi(x,t_n)\,dx\right| \\
&\qquad\le \sum_{n=0}^{\infty}\sum_{l}\int_{x_l}^{x_{l+1}}\left|P_u(w_\Delta^n)-w_\Delta^n\right|\left|\varphi(x,t_n)-\varphi(x_l,t_n)\right|\,dx \\
&\qquad\le \sum_{n=0}^{\infty} M\Delta t\int_{\mathbb{R}}\left|P_u(w_\Delta^n)-w_\Delta^n\right|\,dx.
\end{aligned}$$

If we let $\Delta t \to 0$ we will get

$$\sum_{n=0}^{\infty} M\Delta t\int_{\mathbb{R}}\left|P_u(w_\Delta^n)-w_\Delta^n\right|\,dx \to M\int_0^{\infty}\int_{\mathbb{R}}\left|P_u(w_\Delta^n)-w_\Delta^n\right|\,dx\,dt.$$

Therefore,

$$\lim_{\Delta t\to 0}\left|\sum_{n=0}^{\infty}\int_{\mathbb{R}}\left(P_u(w_\Delta^n)-w_\Delta^n\right)\varphi(x,t_n)\,dx\right| \le M\int_0^{\infty}\int_{\mathbb{R}}\left|P_u(w_\Delta^n)-w_\Delta^n\right|\,dx\,dt.$$

Now, since we have that $\int \left|P_u(w_\Delta^n)-w_\Delta^n\right|\,dx = 0$, then

$$\lim_{\Delta t\to 0}\left|\sum_{n=0}^{\infty}\int_{\mathbb{R}}\left(P_u(w_\Delta^n)-w_\Delta^n\right)\varphi(x,t_n)\,dx\right| \le 0.$$

Similarly, we prove that

$$\sum_{n=0}^{\infty}\int_{\mathbb{R}}\left(P_v(w_\Delta^n)-w_\Delta^n\right)\varphi(x,t_n)\,dx \to 0$$

as $\Delta t \to 0$.

Part III

We consider

$$\sum_{n=0}^{\infty}\int_{\mathbb{R}}\left(w_\Delta^{n+1}-w_\Delta^n\right)\varphi(x,t_n)\,dx.$$

Using summation by parts we get

$$\sum_{n=0}^{\infty} \int_{\mathbb{R}} (w_\Delta^{n+1} - w_\Delta^n) \varphi(x, t_n)\, dx$$
$$= -\sum_{n=1}^{\infty} \Delta t \int_{\mathbb{R}} w_\Delta^n \frac{\varphi(x,t_n) - \varphi(x, t_{n-1})}{\Delta t}\, dx - \int_{\mathbb{R}} w_\Delta^0 \varphi(x, t_0)\, dx.$$

If we let $\Delta t \to 0$ we will get

$$\sum_{n=0}^{\infty} \int_{\mathbb{R}} \left(w_\Delta^{n+1} - w_\Delta^n \right) \varphi(x, t_n)\, dx \to -\int_0^\infty \int_{\mathbb{R}} w \varphi_t(x,t)\, dx - \int_{\mathbb{R}} w^0 \varphi(x, t_0)\, dx.$$

Combining the above three parts we showed that w_Δ^n converges to a weak solution w of hyperbolic conservation law.

4.3 Entropy Solution

In this section we show that the solutions to our numerical method converge to the entropy solution. This proof is very similar to the proof of convergence to a weak solution, presented in the previous section, so we will show only the parts which are different.

In the proof we rely on Kružkov entropy condition. More precisely, with w_Δ^n being defined as in the previous section, we show that if $w_\Delta^n \to w$, as $|\Delta|$, $\Delta t \to 0$, then for every non-negative test function $\varphi \in C_0^\infty (\mathbb{R} \times [0, T])$ we have

$$\int_0^\infty \int_{\mathbb{R}} \mid w - k \mid \varphi_t + sgn(u - k) \left(f(u) - f(k) \right) \varphi_x \, dx\, dt \geq 0.$$

Recall that

$$w_\Delta^{n+1} = \frac{1}{2} \left(P_u(w_\Delta^n) + P_v(w_\Delta^n) \right) - \Delta t \frac{d}{dx} \frac{1}{2} \left(F_u + F_v \right).$$

If we denote $\frac{1}{2}\left(P_u(w_\Delta^n)+P_v(w_\Delta^n)\right)-\Delta t\frac{d}{dx}\frac{1}{2}\left(F_u+F_v\right)=G(w_\Delta^n)$, we have

$$w_\Delta^{n+1}\leq G\left(w_\Delta^n\vee k\right)\quad\text{ and }\quad w_\Delta^{n+1}\geq G\left(w_\Delta^n\wedge k\right),$$

for every $k\in\mathbb{R}$. From here

$$w_\Delta^{n+1}-k\leq G\left(w_\Delta^n\vee k\right)-G(k)\quad\text{ and }\quad w_\Delta^{n+1}-k\geq G\left(w_\Delta^n\wedge k\right)-G(k)$$

implying

$$\begin{aligned}\mid w_\Delta^{n+1}-k\mid &\leq \max\left\{G\left(w_\Delta^n\vee k\right)-G(k),G(k)-G\left(w_\Delta^n\wedge k\right)\right\}\\ &=G\left(w_\Delta^n\vee k\right)-G\left(w_\Delta^n\wedge k\right).\end{aligned}$$

We subtract $\mid w_\Delta^n-k\mid$ from both sides, multiply obtained expression by positive test function φ with compact support, integrate over x and sum over n, to get

$$\begin{aligned}\sum_{n=0}^{\infty}&\int_{\mathbb{R}}\left(\mid w_\Delta^{n+1}-k\mid-\mid w_\Delta^n-k\mid\right)\varphi\,dx\\
&\leq\frac{1}{2}\sum_{n=0}^{\infty}\int_{\mathbb{R}}\varphi\left(P_u\left(w_\Delta^n\vee k\right)-w_\Delta^n\vee k\right)\,dx\\
&\quad-\frac{1}{2}\sum_{n=0}^{\infty}\int_{\mathbb{R}}\varphi\left(P_u\left(w_\Delta^n\wedge k\right)-w_\Delta^n\wedge k\right)\,dx\\
&\quad+\frac{1}{2}\sum_{n=0}^{\infty}\int_{\mathbb{R}}\varphi\left(P_v\left(w_\Delta^n\vee k\right)-w_\Delta^n\vee k\right)\,dx\\
&\quad-\frac{1}{2}\sum_{n=0}^{\infty}\int_{\mathbb{R}}\varphi\left(P_v\left(w_\Delta^n\wedge k\right)-w_\Delta^n\wedge k\right)\,dx\\
&\quad-\frac{1}{2}\sum_{n=0}^{\infty}\Delta t\int_{\mathbb{R}}\varphi\frac{d}{dx}\left(F_u\left(w_\Delta^n\vee k\right)-F_u\left(w_\Delta^n\wedge k\right)\right)\,dx\\
&\quad-\frac{1}{2}\sum_{n=0}^{\infty}\Delta t\int_{\mathbb{R}}\varphi\frac{d}{dx}\left(F_v\left(w_\Delta^n\vee k\right)-F_v\left(w_\Delta^n\wedge k\right)\right)\,dx.\end{aligned}$$

As in the proof for the weak convergence, we divide the proof into three parts.

First, we show

$$\sum_{n=0}^{\infty} \int_{\mathbb{R}} \left(\mid w_{\Delta}^{n+1} - k \mid - \mid w_{\Delta}^{n} - k \mid\right) \varphi \, dx$$
$$\to - \int_0^{\infty} \int_{\mathbb{R}} \mid w - k \mid \varphi_t \, dx \, dt - \int_{\mathbb{R}} \mid w_0 - k \mid \varphi \, dx$$

as $|\Delta|$, $\Delta t \to 0$. This proof is similar to Part III of the proof of convergence to the weak solution, so we omit it.

Secondly, we show that

$$\sum_{n=0}^{\infty} \int_{\mathbb{R}} \varphi \left(P_u \left(w_{\Delta}^{n} \vee k\right) - w_{\Delta}^{n} \vee k\right) \, dx \to 0,$$
$$\sum_{n=0}^{\infty} \int_{\mathbb{R}} \varphi \left(P_u \left(w_{\Delta}^{n} \wedge k\right) - w_{\Delta}^{n} \wedge k\right) \, dx \to 0$$

and

$$\sum_{n=0}^{\infty} \int_{\mathbb{R}} \varphi \left(P_v \left(w_{\Delta}^{n} \vee k\right) - w_{\Delta}^{n} \vee k\right) \, dx \to 0,$$
$$\sum_{n=0}^{\infty} \int_{\mathbb{R}} \varphi \left(P_v \left(w_{\Delta}^{n} \wedge k\right) - w_{\Delta}^{n} \wedge k\right) \, dx \to 0$$

as $\Delta t \to 0$. This proof is similar to the Part II of the proof of convergence to the weak solution, so we omit it.

Finally, we show that

$$\sum_{n=0}^{\infty} \Delta t \int_{\mathbb{R}} \varphi \frac{d}{dx} \left(F_u \left(w_{\Delta}^{n} \vee k\right) - F_u \left(w_{\Delta}^{n} \wedge k\right)\right) \, dx$$
$$\to - \int_0^{\infty} \int_{\mathbb{R}} \varphi_x \left(f(w \vee k) - f(w \wedge k)\right) \, dx \, dt$$
$$\sum_{n=0}^{\infty} \Delta t \int_{\mathbb{R}} \varphi \frac{d}{dx} \left(F_v \left(w_{\Delta}^{n} \vee k\right) - F_v \left(w_{\Delta}^{n} \wedge k\right)\right) \, dx$$
$$\to - \int_0^{\infty} \int_{\mathbb{R}} \varphi_x \left(f(w \vee k) - f(w \wedge k)\right) \, dx \, dt$$

as $|\Delta|$, $\Delta t \to 0$. To prove this, we use integration by parts and fact that φ has compact support, to get

$$\begin{aligned}
&\sum_{n=0}^{\infty}\Delta t \int_{\mathbb{R}} \varphi \frac{d}{dx}\left(F_u\left(w_\Delta^n \vee k\right) - F_u\left(w_\Delta^n \wedge k\right)\right) dx \\
&\quad = -\sum_{n=0}^{\infty}\Delta t \int_{\mathbb{R}} \varphi_x \left(F_u\left(w_\Delta^n \vee k\right) - F_u\left(w_\Delta^n \wedge k\right)\right) dx \\
&\quad = -\sum_{n=0}^{\infty}\Delta t \int_{\mathbb{R}} \varphi_x \left(F_u\left(w_\Delta^n \vee k\right) - F_{P_u}\left(w_\Delta^n \vee k\right) - F_u\left(w_\Delta^n \wedge k\right) + F_{P_u}\left(w_\Delta^n \wedge k\right)\right) dx \\
&\qquad -\sum_{n=0}^{\infty}\Delta t \int_{\mathbb{R}} \varphi_x \left(F_{P_u}\left(w_\Delta^n \vee k\right) - f\left(w_\Delta^n \vee k\right) - F_{P_u}\left(w_\Delta^n \wedge k\right) + f\left(w_\Delta^n \wedge k\right)\right) dx \\
&\qquad -\sum_{n=0}^{\infty}\Delta t \int_{\mathbb{R}} \varphi_x \left(f\left(w_\Delta^n \vee k\right) - f\left(w_\Delta^n \wedge k\right)\right) dx.
\end{aligned}$$

As in Part I (*Step II*) of the previous proof, we show that the second sum converges to 0 as $|\Delta|$, $\Delta t \to 0$. It is also easy to see that

$$\begin{aligned}
&\lim_{\Delta t \to 0} \sum_{n=0}^{\infty}\Delta t \int_{\mathbb{R}} \varphi_x \left(f\left(w_\Delta^n \vee k\right) - f\left(w_\Delta^n \wedge k\right)\right) dx \\
&\qquad\qquad = \int_0^{\infty}\int_{\mathbb{R}} \varphi_x \left(f\left(w_\Delta^n \vee k\right) - f\left(w_\Delta^n \wedge k\right)\right) dx\, dt.
\end{aligned}$$

Let us now show that

$$\left|\sum_{n=0}^{\infty}\Delta t \int_{\mathbb{R}} \varphi_x \left(F_u\left(w_\Delta^n \vee k\right) - F_{P_u}\left(w_\Delta^n \vee k\right) - F_u\left(w_\Delta^n \wedge k\right) + F_{P_u}\left(w_\Delta^n \wedge k\right)\right) dx\right| \to 0,$$

as $|\Delta|$, $\Delta t \to 0$. Using the triangular inequality and the fact that φ_x is bounded by some constant M, we have

$$\left|\sum_{n=0}^{\infty} \Delta t \int_{\mathbb{R}} \varphi_x \left(F_u\left(w_\Delta^n \vee k\right) - F_{P_u}\left(w_\Delta^n \vee k\right) - F_u\left(w_\Delta^n \wedge k\right) + F_{P_u}\left(w_\Delta^n \wedge k\right)\right) dx\right| \leq$$

$$\leq M \sum_{n=0}^{\infty} \Delta t \sum_i \int_{x_{i-1}}^{x_i} \left|\widetilde{h}\left(w_\Delta^n \vee k\right) - \widetilde{h}\left(w_\Delta^n \vee k\right)\right| + \left|\widetilde{h}\left(w_\Delta^n \wedge k\right) - \widetilde{h}\left(w_\Delta^n \wedge k\right)\right| dx$$

where $\widetilde{h}$ is the numerical flux between the cells $i-1$ and i. Let I_i be set of all indices being used to compute flux between cells $i-1$ and i. Since $\widetilde{h}$ is Lipschitz with constant L, we get

$$M \sum_{n=0}^{\infty} \Delta t \sum_i \int_{x_{i-1}}^{x_i} \left|\widetilde{h}\left(w_\Delta^n \vee k\right) - \widetilde{h}\left(w_\Delta^n \vee k\right)\right| + \left|\widetilde{h}\left(w_\Delta^n \wedge k\right) - \widetilde{h}\left(w_\Delta^n \wedge k\right)\right| dx \leq$$

$$\leq ML \sum_{n=0}^{\infty} \Delta t \sum_i \int_{x_{i-1}}^{x_i} \sum_{\zeta \in I_i} \left|u_\zeta^n \vee k - u_i^n \vee k\right| + \left|u_\zeta^n \wedge k - u_i^n \wedge k\right| dx.$$

If we carefully calculate these absolute values we will get

$$ML \sum_{n=0}^{\infty} \Delta t \sum_i \int_{x_{i-1}}^{x_i} \sum_{\zeta \in I_i} \left|u_\zeta^n \vee k - u_i^n \vee k\right| + \left|u_\zeta^n \wedge k - u_i^n \wedge k\right| dx =$$

$$= ML \sum_{n=0}^{\infty} \Delta t \sum_i \int_{x_{i-1}}^{x_i} \sum_{\zeta \in I_i} \left|u_\zeta^n - u_i^n\right| dx.$$

As in the previous proof, Part I (*Step I*), we get

$$\left|\sum_{n=0}^{\infty} \Delta t \int_{\mathbb{R}} \varphi_x \left(F_u\left(w_\Delta^n \vee k\right) - F_{P_u}\left(w_\Delta^n \vee k\right) - F_u\left(w_\Delta^n \wedge k\right) + F_{P_u}\left(w_\Delta^n \wedge k\right)\right) dx\right| \to 0,$$

as $\Delta, \Delta t \to 0$. Similarly, we show

$$\left|\sum_{n=0}^{\infty} \Delta t \int_{\mathbb{R}} \varphi_x \left(F_v\left(w_\Delta^n \vee k\right) - F_{P_v}\left(w_\Delta^n \vee k\right) - F_v\left(w_\Delta^n \wedge k\right) + F_{P_v}\left(w_\Delta^n \wedge k\right)\right) dx\right| \to 0,$$

as $|\Delta|$, $\Delta t \to 0$, and with this the proof is completed.

4.4 Numerical Results

In this section we present three numerical examples using our method. In all cases we take the CFL number to be 0.8 and we use the Godunov numerical flux.

First we consider the Burgers' equation, with initial conditions $w_l = 0$ and $w_r = 1$ and discontinuity is set to be at 0.3. From the theory of Hyperbolic Conservation Laws, we have that solution is a rarefaction wave. Top and bottom grids have 100 points each, the bottom grid is on $[0, 1]$ and the top grid is on $[0.6732, 1.6732]$. Figure 4.5 shows the solution at times $t = 0.3$, $t = 0.5$ and $t = 1$. We note that when $t = 0.3$ the rarefaction wave did not pass the overlap, when $t = 0.5$ the rarefaction wave is in the overlap, and when $t = 1$. the rarefaction wave passed the overlap and it is in the top grid.

Figure 4.6 shows the solution when the initial data is reversed so that $w_l = 1$ and $w_r = 0$, and discontinuity is still at 0.3. Figure 4.6 shows the solution at times $t = 0.1$, $t = 0.3$ and $t = 0.5$. From the theory, we have that the solution is a shock with speed 0.5.

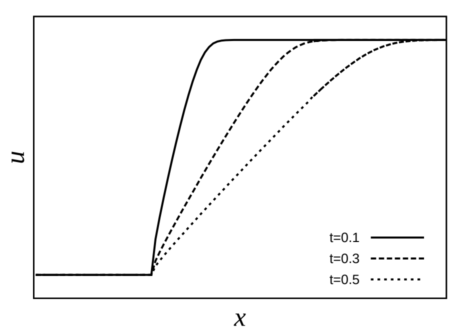


Figure 4.5: Rarefaction wave

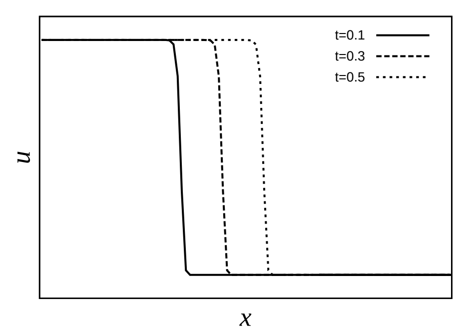


Figure 4.6: Shock wave

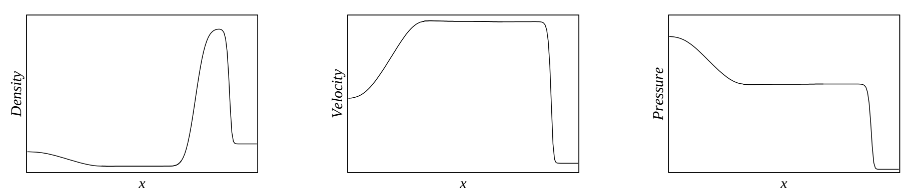
For the third example we choose the Euler gas dynamic equations

$$W_t + F(W)_x = 0$$

where $W = (\rho, \rho v, \rho e)$ and $F(W) = (\rho v, \rho v^2 + p, (\rho e + p)v)$. Also, $e = \varepsilon + \frac{v^2}{2}$, $p = (\gamma - 1)\rho\varepsilon$ and γ is taken 1.4. Further, we denote $c^2 = p_\rho + p\frac{p_\varepsilon}{\rho^2}$. The initial data is taken from the Lax problem,

$$W_0(x) = W(x,0) = \begin{cases} W_l, & x \leq 0.5 \\ W_r, & 0.5 < x \end{cases}$$

where W_l and W_r are obtained from the primitive variables $\rho_l = 0.445$, $v_l = 0.698$, $p_l = 3.528$, $\rho_r = 0.5$, $v_r = 0$, and $p_r = 0.571$. From the theory we know that the solution consists of 4 states separated by a rarefaction wave, a contact discontinuity and a shock.



(a) Density (b) Velocity (c) Pressure

Figure 4.7: Lax problem

We remark one downside of our numerical method. Consider the example

$$u_t = 0$$

with say Riemann initial data. Clearly, the solution is constant in time. However, if the discontinuity is in the overlap, our approximate solution contains some numerical dissipation. This issue is due to averaging which was necessary in order to avoid some numerical irregularities.

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