STRUCTURE IN OPERATOR ALGEBRAS

A Dissertation

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In Partial Fulfillment

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Doctor of Philosophy

By

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Abstract

In this dissertation, we define two new classes of operator algebras; matricial operator algebras and scattered operator algebras. The C^*-algebras of compact operators play an important role in C^*-algebra theory, and they are widely used in mathematical physics and quantum mechanics. We define 1-matricial algebras using a sequence of invertible operators on a Hilbert space, and σ-matricial algebras are c_0-sums of 1-matricial algebras. These operator algebras, in some sense, generalize the class of C^*-algebras of compact operators to a non-selfadjoint setting. They possess many properties similar to the properties of the C^*-algebras of compact operators. We present a 'Wedderburn type' structure theorem that characterizes the σ-matricial algebras. We define scattered operator algebras using a composition series where each consecutive quotient is a 1-matricial algebra. Note that a C^*-algebra is scattered if and only if it has a composition series where each consecutive quotient is a C^*-algebra of compact operators. Hence, our definition of scattered operator algebras is quite natural. We present many results on the structure of the scattered operator algebras and show that they have some properties generalizing the properties of scattered C^*-algebras. For example, the dual of a scattered operator algebra has the Radon-Nikodým property and scattered operator algebras are Asplund spaces. Working with a composition series requires us to develop some tools for general operator algebras, and in particular, quotient operator algebras. For example we utilize frequently the isomorphism theorems and a correspondence theorem for operator algebras; as well as the results about the structure of the diagonal of a quotient operator algebra.

Contents

Chapter 1

Introduction

1.1 The Classical Theory

One of the greatest contributors to noncommutative ring theory was Wedderburn. In 1905, he proved that every simple ring that is finite dimensional over a division ring is a matrix ring. Later, he had the idea of splitting a ring into two parts; a part which is called 'radical' and the left-over part which is called 'semisimple'. He then used matrix rings to classify the semisimple part. Wedderburn's structure theorems were formulated for finite dimensional algebras over a field. In the late 1920's, the Wedderburn theory was extended to noncommutative rings satisfying chain conditions by Artin. In the 1940's Jacobson proved results of the Wedderburn-Artin type for rings without chain conditions. One of his basic tools was what is now called the 'Jacobson radical'.

In Banach algebra theory, there were attempts to find Banach algebra variants of these results. Kaplansky's work in 1947 on dual rings and dual Banach algebras is of great importance [32]. To understand the structure of noncommutative Banach algebras, he introduced CCR and GCR algebras and examined their structure theory [33]. In 1954, Bonsall and Goldie defined a new class of Banach algebras called annihilator algebras which generalize Kaplansky's dual Banach algebras [13]. In the late 1950's, Tomiuk presented the structure theory of complemented Banach algebras [51].

In C^*-algebra theory, similar endeavors gave rise to the study of a new class called Type I C^*-algebras. Glimm proved many diverse characterization of these, showing that in the separable case this class coincides with Kaplansky's GCR algebras [28]. In light of Glimm's results, the class of Type I C^*-algebras has sometimes been considered as the class of reasonable C^*-algebras; this is the class of C^*-algebras with a tractable representation theory. Some of the subclasses of this class are dual C^*-algebras and scattered C^*-algebras. We are interested in finding non-selfadjoint analogues of this theory. However, it seems too ambitious to start with generalizing Type I C^*-algebras. Hence, we start with first generalizing the dual C^*-algebras (the C^*-algebras of compact operators) and then generalizing scattered C^*-algebras into a non-selfadjoint setting.

Let H be a separable infinite dimensional Hilbert space. If we let $\mathbb{K}(H)$ denote the set of all compact operators on a Hilbert space H, then $\mathbb{K}(H)$ is a C^*-subalgebra of $B(H)$ and we call it an *elementary C^*-algebra*. If A is a C^*-subalgebra of $\mathbb{K}(H)$, then it is a c_0-direct sum of elementary C^*-algebras. These algebras admit characterizations similar to Wedderburn's theorem; for example, they are exactly the 'dual' C^*-algebras (in the sense of Kaplansky [32]) and also annihilator C^*-algebras [13]. We refer to them as 'the C^*-algebras of compact operators' or as 'annihilator C^*-algebras'. Indeed, the class of C^*-algebras of compact operators is the simplest subclass of C^*-algebras; but, it has many beautiful characterizations and a great importance in C^*-algebra theory. This class of C^*-algebras has applications in many fields such as mathematical physics, quantum mechanics, the theory of extensions, KK-theory, Atiyah-Singer index theory, Voiculescu's theory of approximate equivalence, Brown-Douglas-Fillmore theory, and so on.

In topology, a space is called *scattered* if it does not contain any perfect subsets. If K is a compact space, then K is scattered if and only if every Radon measure on K is atomic [48, Section 19]. To generalize this notion to C^*-algebras, scattered C^*-algebras were first defined in 1977 by H.E. Jensen [30], and studied by M. L. Rothwell [46], A. J. Lazar [37], and C. Chu [21] in 1980. A C^*-algebra A is said to be *scattered* if every positive functional on A is *atomic*; equivalently, if every positive functional on A is the sum of a finite or infinite sequence of pure functionals. Some recent work by M. Kusuda [36] shows that scattered C^*-algebras are quite connected to AF C^*-algebras and to the C^*-algebras which have real rank zero; scattered C^*-algebras are very rich in terms of projections. Of course, this class contains the class of the C^*-algebras of compact operators.

1.2 The Non-selfadjoint Setting

A *(concrete)* operator algebra A is a norm closed subalgebra of $B(H)$, for some Hilbert space H. Note that this subalgebra is not necessarily selfadjoint. If A is an operator space and a Banach algebra such that there exists $\pi : A \to B(H)$ which is a completely isometric isomorphism, then we say that A is an *(abstract) operator algebra*. In 1990, Blecher, Ruan, and Sinclair gave an abstract characterization of operator algebras and showed that every unital abstract operator algebra is a concrete operator algebra [11]. This result is a fundamental result in the theory of operator algebras and since then the theory has progressed enormously. One of the tasks of the researchers in this area is to find non-selfadjoint analogues of the notions and tools that are available in the C^*-algebra theory. This, together with the facts mentioned in the previous section, led us to try finding the analogues of the C^*-algebra of compact operators and scattered C^*-algebras in the theory of operator algebras.

In a joint work with D. P. Blecher and S. Sharma, we introduced matricial algebras [3]. We define 1-matricial algebras in terms of matrix units, and σ-matricial algebras are c_0-sums of 1-matricial algebras. The class of 1-matricial algebras possess many similar properties to the class of C^*-algebras of compact operators. To name a few, 1-matricial algebras are simple and semisimple, they are generated by 'matrix units', they have dense socle, and the spectrum of every element has no nonzero limit points. For semiprime approximately unital operator algebras, we obtained a 'Wedderburn type' structure theorem which gives many characterizations of σ-matricial algebras.

This work is presented in Chapter 4.

One of the many nice characterizations of the scattered C^*-algebras is that they contain a composition series where the consecutive quotients are isometric to $\mathbb{K}(H)$ for some Hilbert space H. This led us to consider generalizing the scattered C^*-algebras to a non-selfadjoint setting by using a composition series where each consecutive quotient is completely isometrically isomorphic to a 1-matricial algebra. In Chapter 5, we define a composition series for an operator algebra and then define scattered operator algebras using a composition series where the building blocks are 1-matricial algebras. We show that scattered operator algebras possess some properties that are similar to the properties of scattered C^*-algebras. For example, the dual of a scattered operator algebra has the Radon-Nikodým property, and hence a scattered operator algebra is an Asplund space.

In our attempt at defining scattered operator algebras using a composition series, we needed to understand the structure of quotient operator algebras, by which we mean A/J where A is an operator algebra and J is an ideal in A. We needed to develop some tools to work with general operator algebras, and in particular, with quotient operator algebras. For example, in Chapter 3, we present the operator algebra versions of the First, Second, and Third Isomorphism Theorems and the Correspondence Theorem, which are very useful tools in algebra. We also study the structure of the diagonal of a quotient operator algebra.

Chapter 3 is joint work with our advisor D. P. Blecher and some of these results will appear in [2], which also has many other results not in this dissertation. The initial idea in Chapter 5 was proposed by D. P. Blecher. Most of the results in

Chapter 5 are due to the author, although Blecher helped the author with many corrections and some of the more difficult and technical points when we were stuck. The rest of Chapter 5 is joint work with him, as are the parts of Chapter 4 which are not in [3]. I thank D. P. Blecher for his continuous guidance and support during the preparation of this dissertation.

Chapter 2

Background and Preliminary Results

2.1 Algebra

For algebraic terms and notations, we follow [41].

An element x in an algebra A is *quasi-invertible* if $1 - x$ is invertible in A^1. We let $qi(A) = \{x \in A : 1 - x \text{ is invertible in } A^1\}$. The *spectrum* of an element $x \in A$ is defined as $Sp(x) = \{\lambda \in \mathbb{C} : \lambda 1 - x \text{ has no inverse in } A\}$ if A is unital. If A is not unital, we can define the spectrum of an element as the spectrum of its image in the unitization A^1. Palmer gives a definition of the spectrum in terms of the quasi-invertible elements of A (see [41, Definition 2.1.5]). The *spectral radius* of x is $r(x) = \sup\{|\lambda| : \lambda \in Sp(x)\}$.

The *radical* (or *Jacobson radical*) of an algebra A is the intersection of kernels of the irreducible representations of A, and it is denoted by $\mathrm{Rad}(A)$. There are many characterizations of the Jacobson radical; for example, $\mathrm{Rad}(A) = \{a \in A : A^1 a \subset qi(A)\}$. We refer the reader to Chapter 4 in [41] for other characterizations. The algebra A is said to be *semisimple* if $\mathrm{Rad}(A) = (0)$, and is said to be *Jacobson-radical* if $\mathrm{Rad}(A) = A$.

An algebra A is *simple* if it does not contain any nontrivial ideals. We say that the algebra is *semiprime* if for any ideal J in A, $J^2 = (0)$ implies that $J = (0)$. A semisimple algebra is semiprime since any nil ideal is contained in the Jacobson radical.

An idempotent $e \in A$ is *minimal* if eAe is a division algebra. An ideal (resp. left ideal, right ideal) is called *minimal* if it is minimal among the set of nonzero ideals (resp. left ideals, right ideals) ordered by inclusion. In a semiprime algebra, every minimal left (resp. right) ideal of A has the form Ae (resp. eA) for a minimal idempotent $e \in A$. Conversely, if $e \in A$ is a minimal idempotent, then Ae (resp. eA is a minimal left (resp. right) ideal. Note that in a semiprime Banach algebra the minimal left (resp. right) ideals are all closed.

A very fundamental structure theorem, which can be attributed to Cartan, Wedderburn or Artin, characterizes unital finite dimensional algebras that are semiprime.

Theorem 2.1.1 (Cartan - Wedderburn - Artin Theorem)**.** *If A is a unital finite dimensional algebra over $\mathbb{C}$, then the following are equivalent.*

(i) A is semisimple,

(ii) A is semiprime,

(iii) $A \cong \oplus_{k=1}^{m} M_{n_k}$.

Proof. As mentioned earlier, semisimple implies semiprime. Suppose A semiprime. If A has no nontrivial right ideals then every element is right invertible, so $A = \mathbb{C}\,1$ by Gelfand-Mazur Theorem. So, WLOG assume that A has a nontrivial right ideal, hence a minimal right ideal, hence a minimal idempotent. If (e_k) is a maximal set of mutually orthogonal minimal idempotents in A, set $e = \sum_k e_k$. If $e \neq 1$ then the right ideal $(1-e)A$ contains a minimal idempotent. Indeed, in a semiprime algebra, the minimal right ideals are of the form fA for an algebraically minimal idempotent $f \in A$. This is a contradiction to the maximality of (e_k), so $\sum_k e_k = 1$. Hence, $B = \oplus_{k=1}^{n} A_k$ as right B-modules, where $A_k = e_k B$ are minimal right ideals, and e_k are algebraically minimal idempotents. The elementary Schur lemma implies that $Hom_A(A_i, A_j) = (0)$ if A_i is not A-isomorphic to A_j, and $Hom_A(A_i) = D_i$ is a division algebra, so it is congruent to $\mathbb{C}$. Hence, we have:

$$B \cong Hom_B(\oplus_{k=1}^{n} A_k) \cong \oplus_{k=1}^{m} M_{n_k}(D_k) \cong \oplus_{k=1}^{m} M_{n_k}$$

for integers n_k. $\square$

The *left socle* (resp. *right socle*) of A is the sum of all minimal left (resp. right) ideals of A. If the left and right socles coincide, then they form the *socle* of A, denoted by A_F. Note that if A is semiprime, the left and right socle of A coincide and A has socle. In this case, A_F equals the left ideal, right ideal and the ideal generated by the set of minimal idempotents of A.

Let S be a subset of an algebra A. Then, the *left annihilator* of S in A is $LA(S) = \{a \in A : ab = 0 \text{ for } b \in A\}$. The *right annihilator* of S in A is $RA(S) = \{a \in A : ba = 0 \text{ for } b \in A\}$. Sometimes we use the notations $R(S)$ or $L(S)$ instead of $RA(S)$ or $LA(S)$.

A *modular annihilator algebra* is a semiprime algebra where each maximal modular left (resp. right) ideal M satisfies $RA(M) \neq (0)$ (resp. $LA(M) \neq (0)$). This is also equivalent to A/A_F being a radical algebra. For other (several) equivalent definitions, we refer the reader to [41, Theorem 8.4.5].

An algebra A is called a *Duncan modular annihilator algebra* (DMA) if it is semiprime and satisfies $LA(A_F) = RA(A_F) = A_J$. Observe that if A is semisimple, then A is DMA if and only if $LA(A_F) = RA(A_F) = (0)$. Note that a modular annihilator algebra is a DMA algebra. However, the converse is not true; for example, $B(H)$, where H is an infinite dimensional Hilbert space, is DMA but not a modular annihilator algebra [41, Section 8.6]. The class of semisimple Duncan modular annihilator algebras contains the class of Banach algebras where every element has finite or countable spectrum [41, Theorem 8.6.2]. The converse of this statement is not true. For example, if $A = \ell^\infty([0,1])$, then A is a commutative semisimple DMA algebra and the function $f(t) = t$ has uncountable spectrum.

A Banach algebra A is a *right* (resp. *left*) *annihilator algebra* if $R(I) \neq (0)$ (resp. $L(J) \neq (0)$) for any proper closed left ideal I (resp. right ideal J) of A. An *annihilator algebra* is both a left and a right annihilator algebra. Kaplansky studied a class of algebras which he called *dual* [32], these satisfy $R(L(J)) = J$ and $L(R(I)) = I$ for J, I as above. Note that dual algebras are annihilator algebras.

2.2 The C^*-algebras of Compact Operators

A C^*-algebra is an involutive Banach algebra A satisfying the C^*-identity: $\|a^*a\| = \|a\|^2$, for all $a \in A$, where $a \mapsto a^*$ denotes the adjoint on A. By Gelfand's theorem, a commutative unital C^*-algebra is $*$-isomorphic to $C(K)$; the algebra of continuous functions on a compact Hausdorff space K. That is, any commutative unital C^*-algebra can be regarded as a $C(K)$ algebra for an appropriate K.

An element p in a C^*-algebra is a projection if $p = p^2 = p^*$; or, equivalently, if $p = p^2$ and $\|p\| \leq 1$. The partial ordering on projections is $p \leq q \Leftrightarrow pq = p \Leftrightarrow qp = p$. A projection $p \in A^{**}$ is said to be *open* if it is a weak*-limit of an increasing net (b_t) of elements in A with $0 \leq b_t \leq 1$. A projection $q \in A^{**}$ is said to be *closed* if $1 - q$ is open. A closed projection $p \in A^{**}$ is said to be *compact* if there exists a positive element $b \in A$ with $\|b\| \leq 1$ such that $p \leq b$.

A projection $p \in A^{**}$ is open if and only if it is the support projection of a left (resp. right) ideal J in A. Here, the left (resp. right) ideal is $J = A^{**}p \cap A$ (resp. $J = pA^{**} \cap A$), and $J^{\perp\perp} = A^{**}p$ (resp. $J^{\perp\perp} = pA^{**}$). The support projection p of J is the weak*-limit of any increasing right cai (resp. left cai) for J. That is, the support projection is unique. For a two-sided ideal J in A, the support projection $p \in A^{**}$ is the weak*-limit of any cai in J and we have $J = pA^{**} \cap A = A^{**}p \cap A$. If we quotient the C^*-algebra by the ideal, we get the following embedding:

$$A/J \subset (A/J)^{**} \cong A^{**}/J^{\perp\perp} \cong A^{**}(1 - p).$$

Let H be a Hilbert space. A bounded operator T on H is called *compact* if the image of the unit ball of H under T has compact closure in the norm topology of H. The set $\mathbb{K}(H)$ of all compact operators on H is a closed two-sided ideal in $B(H)$ and is therefore a C^*-algebra itself; we sometimes refer to $\mathbb{K}(H)$ as an *elementary C^*-algebra.* It is easy to see that $\mathbb{K}(H)$ is not unital; but it does contain several finite rank projections. In fact, $\mathbb{K}(H)$ is the norm closure of the set of finite rank operators on H. Every nonzero projection in $\mathbb{K}(H)$ is finite dimensional and is a finite sum of orthogonal minimal projections [4, Lemma 1.4.1]. Note that $\mathbb{K}(H)$ is simple; it does not contain any nontrivial ideals. A notion that characterizes $\mathbb{K}(H)$ is that every irreducible representation on $\mathbb{K}(H)$ is equivalent to the identity representation [4, Section 1.4].

If A is a C^*-subalgebra of $\mathbb{K}(H)$, then there exist Hilbert spaces $\{H_i\}_{i\in I}$ such that A is isomorphic to the direct sum $\oplus_{i\in I}\mathbb{K}(H_i)$, where the direct sum consists of elements $(T_i) \in \Pi\,\mathbb{K}(H_i)$ with $\|T_i\| \to 0$ (a 'c_0-direct sum'). We say that A is a *C^*-algebra of compact operators.* The C^*-algebras of compact operators admit characterizations similar to Wedderburn's theorem. It is known that a C^*-algebra is an annihilator algebra if and only if it is 'dual' in the sense of Kaplansky [32], and these are precisely the c_0-sums of elementary C^*-algebras. Hence, the class of C^*-algebras of compact operators coincides with the class of annihilator C^*-algebras and we sometimes refer to them as 'annihilator C^*-algebras'. This class of C^*-algebras has several other characterizations; these can be found in Kaplansky's or Dixmier's works (for example [24, Exercise 4.7.20]). We state some of these characterizations. If A is a C^*-algebra, then the following are equivalent:

(i) There is a faithful $*$-representation $\pi : A \to \mathbb{K}(H)$ as compact operators on some Hilbert space H.

(ii) A is $*$-isomorphic to $\oplus_i^0 \mathbb{K}(H_i)$ (a 'c_0-sum') for Hilbert spaces H_i.

(iii) For every closed left ideal J of A, $R(J) \neq (0)$.

(iv) The sum of minimal left (resp. minimal right) ideals of A is dense in A.

(v) Every closed left (resp. right) ideal J in A is of the form Ae (resp. eA) for a projection in $M(A)$.

(vi) For every selfadjoint $x \in A$, $Sp_A(x) \setminus \{0\}$ is discrete.

(vii) For fixed $a, b \in A$, the map $x \mapsto axb$ is compact.

(viii) A is an ideal in its bidual.

2.3 Scattered C^*-algebras

A topological space X is *scattered* (or *dispersed*) if for every closed subset C of X, the set of isolated points of C is dense in C. Equivalently, X is a scattered space if no nonempty closed subset of X is dense in itself; for every closed subset C of X, the closure of the interior of C is not C. Another characterization is that X does not contain any perfect subsets. For example, $X = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{1 - \frac{1}{n} : n \in \mathbb{N}\}$ is scattered. Every discrete space is scattered. Note that 2^ω (Cantor's set) is not scattered.

In [47], W. Rudin studied the properties of $C(K)$ where K is a compact Hausdorff space which is scattered. He showed that the linear functionals on $C(K)$ have a very simple structure. A. Pelczynski and Z. Semadeni gave several necessary and sufficient conditions for a compact Hausdorff space K to be scattered in terms of $C(K)$ [44]. To name a few, a compact Hausdorff space K is scattered if and only if

- every continuous image of K is scattered,
- the space K is zero dimensional and 2^ω is not a continuous image of K,
- for every separable subspace X of $C(K)$, the space X^* is separable,
- every linear functional over $C(K)$ is of the form $T(x) = \sum_{n=1}^{\infty} a_n x(t_n)$, where (t_n) is a fixed sequence of points of K and $\sum_{n=1}^{\infty} |a_n| < \infty$.

To generalize this notion to C^*-algebras, H. E. Jensen defined scattered C^*-algebras in terms of the positive functionals on the C^*-algebra. A C^*-algebra A is said to be *scattered* if every positive functional on A is atomic; or equivalently, any positive functional on A is the sum of a finite or infinite sequence of pure functionals [30]. Combining some results from [30], [37] and [21], we can state the following theorem to list some of the characterizations of scattered C^*-algebras.

Theorem 2.3.1. *Let A be a C^*-algebra. The following are equivalent.*

(i) A is scattered.

(ii) Each nondegenerate representation of A is unitarily equivalent to a subrepresentation of a sum of irreducible representations.

(iii) Each projection in the enveloping von Neumann algebra B majorizes a minimal projection in B.

(iv) *A has a composition series* $\{I_\alpha\}$ *such that each* $I_{\alpha+1}/I_\alpha$ *is an elementary* C^**-algebra.*

(v) *Every selfadjoint element in A has countable spectrum.*

(vi) *The dual of A has the Radon-Nikodým property.*

Note that a Banach space X is said to have the *Radon-Nikodým property* if for any finite measure space (Ω, Σ, μ), and any μ-continuous vector measure $L : \Sigma \to X$ of bounded total variation, there exists a Bochner integrable function $g : \Omega \to X$ such that $L(E) = \int_E g \,\mathrm{d}\mu$ for all E in Σ. If X has the Radon-Nikodým property, then it has the Krein-Milman property. That is, every (norm) closed bounded convex subset of X is the (norm) closed convex hull of its extreme points. The converse is also true if X is a dual space. A Banach space X has RNP if and only if X is an Asplund space [50, Theorem 1]. Also, a Banach space X is an Asplund space if and only if every separable subspace has a separable dual [50].

An element $x \in A$ is abelian if the hereditary C^*-subalgebra $[xAx]^-$ is commutative. A C^*-algebra is *Type I* if every quotient of A contains a nonzero abelian element. A characterization is that A is Type I if and only if it is GCR [5, Section IV.1]. By [30, Theorem 2.3], a scattered C^*-algebra is Type I.

The existence of projections in C^*-algebras is important for understanding their structure. In [17], L. G. Brown and G. K. Pedersen showed that several conditions on the abundance of projections are equivalent. A C^*-algebra is said to have *real rank zero* if every selfadjoint element in A^1 can be approximated by an invertible selfadjoint element in A^1. Having real rank zero is equivalent to having the following properties [15]:

- (FS) The selfadjoint elements with finite spectrum are dense in A_{sa}.
- (HP) Every nonzero HSA has an approximate identity consisting of projections.
- (IP) if $p, q \in A^{**}$ are mutually orthogonal projections where p is compact and q is closed, then there is a projection $r \in A$ such that $p \leq r \leq 1 - q$.

Note that having real rank zero is the noncommutative analogue of being totally disconnected. Indeed, for the commutative case; for $C_0(X)$ where X is locally compact and Hausdorff, $C_0(X)$ has real rank zero if and only if X is totally disconnected. If $C(K)$ is scattered where K is a compact Hausdorff space, we know that it has real rank zero. However, the converse is not true in general. For example, $C(2^\omega)$ has real rank zero but it is not scattered.

Recently, M. Kusuda studied scattered C^*-algebras ([35] and [36]) in connection to being AF, being Type I or having real rank zero. He showed that a C^*-algebra is scattered if and only if every C^*-subalgebra of A is AF, and if and only if every C^*-subalgebra of A has real rank zero [36, Theorem 2.3].

2.4 Operator Algebras

An *operator space* is a norm closed subspace X of $B(H)$, the bounded operators on a Hilbert space H. Besides a vector space structure, an operator space has a norm structure. The space of $n \times n$ matrices over X, $M_n(X)$, inherits a norm $\|.\|_n$

via the identification $M_n(X) \subset M_n(B(H)) \cong B(H^{(n)})$ isometrically, where $H^{(n)}$ denotes the Hilbert space direct sum of n copies of H. If $T : X \to Y$ is a linear map between operator spaces, for $n \in \mathbb{N}$, we define $T_n : M_n(X) \to M_n(Y)$ by $T_n([x_{ij}]) = [T(x_{ij})]$, for $[x_{ij}] \in M_n(X)$. We say that the map T is *completely bounded* if $\|T\|_{cb} \overset{def}{=} \sup_n \|T_n\|$ is finite and T is *completely contractive* if $\|T\|_{cb} \leq 1$. The map T is said to be a *complete isometry* if each T_n is an isometry, and a *complete quotient map* if each T_n is a quotient map (that is, each T_n takes the open ball of $M_n(X)$ onto the open ball of $M_n(Y)$).

A *(concrete) operator algebra* A is a norm closed subalgebra of $B(H)$, for some Hilbert space H. Note that this subalgebra is not necessarily selfadjoint. Observe that any operator algebra is an operator space and a Banach algebra. Conversely, if A is both an operator space and a Banach algebra, then A is an *(abstract) operator algebra* if there exists a Hilbert space H and a completely isometric homomorphism $\pi : A \to B(H)$. We identify any two operator algebras A and B as the "same" if there exists an algebra isomorphism $\pi : A \to B$ which is a complete isometry, in which case we write '$A \cong B$ completely isometrically isomorphically'.

We say that the operator algebra is *unital* if A contains the identity I_H of $B(H)$. However, we will mostly focus on operator algebras that are approximately unital.

A *left* (resp. *right*) *approximate identity* for A is a net $\{e_t\}$ in A such that $e_t a \to a$ (resp. $ae_t \to a$), for all $a \in A$. A *bounded approximate identity* (bai) is a bounded two-sided approximate identity. A contractive approximate identity (cai) is a two-sided approximate identity $\{e_t\}$ with $\|e_t\| \leq 1$. We say that A is an *approximately unital operator algebra* if it contains a cai. Since every C^*-algebra has a cai, the class of approximately unital operator algebras contains the class of C^*-algebras.

If A is an operator algebra which does not contain an identity, then there is a natural way to adjoin a unit to the algebra. We call this *the unitization* A^1 of A, and A^1 is a unital operator algebra containing A as an ideal with codimension one. To construct A^1, we regard A as a subalgebra of $B(H)$ for some Hilbert space H, and then take $A^1 = \text{span}\{A, I_H\}$. Note that, up to completely isometric isomorphisms, this unitization does not depend on the embedding $A \subset B(H)$ [7, Corollary 2.1.15]. Meyer gives a very useful extension principle. If $A \subset B(H)$ is not unital and $A^1 = \text{span}\{A, I_H\}$, then every contractive (resp. completely contractive) homomorphism $\pi : A \to B(K)$ extends to a contractive (resp. completely contractive) homomorphism $\pi^0 : A^1 \to B(K)$, where π^0 is defined by $\pi^0(a+\lambda I_H) = \pi(a)+\lambda I_K$ [7, Theorem 2.1.13]. Note that if A is already unital, then there is essentially a unique unital operator algebra containing A completely isometrically as a codimension 1 ideal. In fact, this unitization is the ∞-direct sum $A \oplus^\infty \mathbb{C}$.

In C^*-algebra theory, one often passes to the second dual, as A^{**} is a von Neumann algebra and one has more tools to study von Neumann algebras. For similar purposes, it is important to study the second dual of a nonselfadjoint operator algebra. If A is an operator algebra, then A^{**} is also an operator algebra (for details about the Arens product, we refer the reader to [7, Section 2.5]). Indeed, there exist a Hilbert space H, and a completely isometric homomorphism $\pi : A \to B(H)$ whose (unique) w^*-continuous extension $\tilde{\pi} : A^{**} \to B(H)$ is a completely isometric homomorphism. In this case, $A^{**} \cong \tilde{\pi}(A^{**}) = \overline{\pi(A)}^{w^*}$ [7, Corollary 2.5.6]. The *multiplier algebra* of A can be defined as $M(A) = \{\eta \in A^{**} : \eta A \subset A \text{ and } A\eta \subset A\}$ [7, Section 2.6].

A *right* (resp. *left*) *ideal* in an operator algebra A is a closed subspace J of A such that $Ja \subset J$ (resp. $aJ \subset J$), for all $a \in A$. The one-sided ideals are operator algebras as well. We use the term *ideal* for a two-sided ideal, that is, a closed subspace which is both a right and a left ideal. One-sided ideals in general operator algebras do not necessarily contain one-sided approximate identities. A subspace J of A is an *r-ideal* (resp. *ℓ-ideal*) if J is a closed right ideal (resp. left ideal) with a left (resp. right) contractive approximate identity.

Note that if J is a closed two-sided ideal of an operator algebra A, then the quotient algebra A/J is an operator algebra. That is, there exist a Hilbert space H and a completely isometric isomorphism $\pi : A/J \to B(H)$ [7, Proposition 2.3.4]. This fact can be easily seen as a corollary of the *BRS theorem* which is due to Blecher, Ruan and Sinclair [7, 2.3.2].

A projection in A will always mean an orthogonal projection. A projection is *$*$-minimal* if it dominates no nontrivial projection in A. We say that the projection p in A is *algebraically minimal* if $pAp = \mathbb{C}\,p$. Clearly, an algebraically minimal projection is $*$-minimal. In certain algebras the converse is true too, but this is not common.

A *hereditary subalgebra* (HSA) of an operator algebra A is an approximately unital subalgebra D of A such that $DAD \subset A$. These are also the subalgebras of the form $pA^{**}p \cap A$ for an open projection $p \in A^{**}$; we say that p is the *support projection* of D. Note that the cai of D converges weak* to p. The HSA D is one-dimensional if and only if its support projection $p \in A^{**}$ is algebraically minimal.

If S is a subset of a C^*-algebra B, then $C^*_B(S)$ denotes the C^*-subalgebra of B generated by S (that is, the smallest C^*-subalgebra of B containing S). A C^*-cover of an operator algebra A is a pair (B, j) consisting of a C^*-algebra B and a completely isometric homomorphism $j : A \to B$, such that $j(A)$ generates B as a C^*-algebra; $C^*_B(j(A)) = B$. If (B, j) and (B', j') are two C^*-covers of A, then we say that $(B, j) \leq (B', j')$ if and only if there is a $*$-homomorphism $\pi : B \to B'$ such that $\pi \circ j' = j$. Such a $*$-homomorphism π is automatically surjective and unique. If π is also one-to-one (and therefore a $*$-isomorphism), then we say that (B, j) is A-isomorphic to (B', j'). This is an equivalence relation and we let $\mathcal{C}(A)$ to be the set of equivalence classes of C^*-covers of A. Then, this set has a largest element which is called the *maximal C^*-algebra* (or *maximal C^*-cover*) of A, denoted as $\mathrm{C}^*_{\max}(A)$. By [7, Proposition 2.4.2], the maximal C^*-algebra exists and it has the following universal property: if $\pi : A \to D$ is any completely contractive homomorphism into a C^*-algebra D, then there exists a (necessarily unique) $*$-homomorphism $\tilde{\pi}$: $\mathrm{C}^*_{\max}(A) \to D$ such that $\tilde{\pi} \circ j = \pi$. The smallest element in the set $\mathcal{C}(A)$ is called the *C^*-envelope* of A and it is denoted as $C^*_e(A)$. The C^*-envelope of A is the quotient of any C^*-cover of A by a closed two-sided ideal. In particular, $C^*_e(A) \cong \mathrm{C}^*_{\max}(A)/I$ where I is an ideal in $\mathrm{C}^*_{\max}(A)$.

Chapter 3

Quotient Operator Algebras

The Correspondence Theorem and the First, Second, and Third Isomorphism Theorems are very useful tools in algebra. In this chapter, we present the operator algebra versions of these theorems. One of our goals is to understand the structure of quotient operator algebras and these theorems will be our tools while investigating the properties of quotient operator algebras. We will make frequent use of them while working with composition series of operator algebras in Chapter 5.

In this dissertation, the operator algebras are assumed to be approximately unital unless otherwise stated.

3.1 Isomorphism Theorems for Operator Algebras

An ideal of an operator algebra A is a closed two-sided ideal in A. We say that the ideal is approximately unital if it contains a contractive approximate identity (cai). It is a well-known fact that if J is an ideal of an operator algebra A, then the quotient algebra A/J is isometrically isomorphic to an operator algebra. An important result about the quotient operator algebras, known as the Factor Theorem, can be found in Chapter 2 of [7]. We include this result here for the reader's convenience.

Theorem 3.1.1 (Factor Theorem). *If $u : A \to B$ is a completely bounded homomorphism between operator algebras, and if J is an ideal in A contained in* $\mathrm{Ker}(u)$*, then the canonical map $\tilde{u} : A/J \to B$ is also completely bounded with $\|\tilde{u}\|_{cb} = \|u\|_{cb}$. If $J =$* $\mathrm{Ker}(u)$*, then u is a complete quotient map if and only if $\tilde{u}$ is a completely isometric isomorphism.*

We first present the Correspondence Theorem.

Theorem 3.1.2 (Correspondence Theorem). *Let A be a Banach algebra and J be an ideal in A. Every closed subalgebra of A/J is of the form I/J, where I is a closed subalgebra of A with $J \subset I \subset A$. Also, every ideal of A/J is of the form I/J, where I is an ideal of A with $J \subset I \subset A$.*

Proof. Let K be a closed subalgebra of A/J and let $I = \{a \in A : a + J \in K\}$. If $x \in J$, then $x + J = 0_{A/J} \in K$, and hence $x \in I$. That is, $J \subset I$. Clearly, I is a subalgebra of A. To see that I is closed, let (a_t) be a net in I that converges to

$a \in A$. Then,

$$\|(a_t + J) - (a + J)\|_{A/J} = \|(a_t - a) + J\|_{A/J} \leq \|a_t - a\|_A \to 0.$$

Hence, $a_t + J \to a + J$ and $a + J \in K$ since K is closed. Therefore, $a \in I$. Thus, I is a closed subalgebra of A. Since $K = I/J$ by definition, we have proved the first assertion.

Now, let K be a proper ideal of A/J and let $I = \{a \in A : a + J \in K\}$. We only need to prove that I is an ideal in A.

Let $x \in A$ and $a \in I$. Then $x + J \in A/J$ and $a + J \in A/J$. So, $(x + J)(a + J) = xa + J \in K$ since K is an ideal in A/J. That is, $xa \in I$. Similarly, $ax \in I$. By the first part of our proof, I is closed. Thus, I is a closed two-sided ideal in A. $\square$

The following result is well-known; we include it here for the sake of completeness. We omit the quite easy proof.

Theorem 3.1.3 (First Isomorphism Theorem). *Let $u : A \to B$ be a complete quotient map which is a homomorphism between operator algebras. Then,* $\mathrm{Ker}(u)$ *is an ideal in A and Im(u) is a closed subalgebra of B. Moreover, $A/\mathrm{Ker}(u) \cong$ Im(u) completely isometrically isomorphically. In particular, if u is surjective, then $A/\mathrm{Ker}(u) \cong B$ completely isometrically isomorphically. Conversely, every ideal of A is of the form* $\mathrm{Ker}(u)$ *for a complete quotient map $u : A \to B$, where A and B are operator algebras.*

When we need to work with the quotient of two quotient operator algebras, the Second Isomorphism Theorem gives us a simplified quotient.

Theorem 3.1.4 (Second Isomorphism Theorem)**.** *Let A be an approximately unital operator algebra, J be an ideal in A and I be an ideal in J. Then, $(A/I)/(J/I) \cong A/J$ completely isometrically isomorphically.*

Proof. Define a map $u : (A/I) \to (A/J)$ by $u(a + I) = a + J$. This is a well-defined map with $\mathrm{Ker}(u) = J/I$. Notice that u is completely bounded;

$$\|u\|_{cb} = \sup \left\{ \|[u(a_{i,j} + I)]\|_n : \|[a_{i,j} + I]\|_n \leq 1, n \in \mathbb{N} \right\} \leq 1.$$

Indeed, $\|u\| = \sup \{\|a + J\| : \|a + I\| \leq 1\}$. If $\|a + I\| \leq 1$, then $\|a + J\| \leq 1$ since $I \subset J$. Hence, $\|u\| \leq 1$. Similar argument works for each u_n; that is, u is a completely contractive map.

Let $x + J$ be in the open ball of A/J. Since $\|x + J\| < 1$, there exists $j \in J$ such that $\|x + j\| < 1$. Now, $x + j + I \in A/I$ and $\|x + j + I\| \leq \|x + j\| < 1$. That is, $x + j$ is in the open ball of A/I. Since $u(x + j + I) = x + j + J = x + J$, we conclude that u maps the open ball of A/I onto the open ball of A/J; u is a 1-quotient map. Similar argument shows that u is a complete quotient map.

Let $\tilde{u} : (A/I)/(J/I) \to A/J$ be the canonical map. By the Factor Theorem, $\tilde{u}$ is a completely isometric isomorphism. Hence, $(A/I)/(J/I) \cong A/J$ completely isometrically isomorphically. □

Now, we prove the Third Isomorphism Theorem for operator algebras.

Theorem 3.1.5 (Third Isomorphism Theorem)**.** *Let A be an approximately unital operator algebra and J and K be ideals in A, where J has a cai. Then, $J/(J \cap K) \cong$*

$(J+K)/K$ completely isometrically isomorphically. In particular, $(J+K)/K$ is closed.

Proof. Note that by [25, Proposition 2.4], $J+K$ is closed. Define a map $u : J/(J \cap K) \to (J+K)/K$ by $u(j + J \cap K) = j + K$. This is a well-defined map and u is one-to-one since $\mathrm{Ker}(u) = (0_{J/(J\cap K)})$. Moreover, u is onto since $x + K \in (J+K)/K$ implies that $x = j + k$, where $j \in J, k \in K$ and $x + K = j + K = u(j + J \cap K)$.

Since $\inf \{\|j+k\| : k \in K\} \le \inf \{\|j+k\| : k \in J \cap K\}$, u is a contraction. Let (e_t) be the cai for J and let $k \in K$. Then,

$$\|j+k\| \ge \|e_t j + e_t k\| \ge \|e_t j + J \cap K\| .$$

Hence, after taking the limit, we get $\|j + J \cap K\| \le \|j+k\|$. Now, since $\|j+K\| = \inf \{\|j+k\| : k \in K\}$, we conclude that $\|j + J \cap K\| \le \|j+K\|$. Hence, u is an isometry. Similarly, u is a complete isometry. Hence, $J/(J \cap K) \cong (J+K)/K$ completely isometrically isomorphically. $\square$

Finally, we want to give miscellaneous results about quotient operator algebras. We start with an algebraic result.

Proposition 3.1.6. *Let A be an operator algebra with a bai and I and J be ideals in A. If $A/I \cong A/J$ isomorphically as A-bimodules, then $I = J$.*

Proof. First assume that A is unital. Let $\phi : A/I \to A/J$ be the A-bimodule isomorphism. Then, $\phi(1+I) = x + J$, for some $x \in A$. This implies that $\phi(a+I) = ax + J$, for any $a \in A$. If $a \in I$, then $ax \in J$. Since ϕ is one-to-one, $ax - bx \in J$

implies $a - b \in I$. Now, since ϕ is surjective, $bx + J = 1 + J$, for some $b \in I$. That is, $bx - 1 \in J$. Hence, $x + J$ is left invertible in A/J. Similarly, it is right invertible. Hence,

$$a \in I \Leftrightarrow (a + J)(x + J) = 0 \Leftrightarrow a + J = 0 \Leftrightarrow a \in J.$$

Thus, $I = J$.

If A contains a bai, then its bidual A^{**} is unital. If $\phi : A/I \to A/J$ is an A-bimodule isomorphism, let $\tilde{\phi}$ be the unique weak*-continuous extension. By weak*-continuity of $\tilde{\pi}$, $\tilde{\pi}$ maps $(A/I)^{**} \cong A^{**}/I^{\perp\perp}$ into $(A/J)^{**} \cong A^{**}/J^{\perp\perp}$. Moreover, $\tilde{\pi}$ is an A^{**}-bimodule isomorphism by the separate weak*-continuity of the Arens product. Indeed, if $\eta = \lim_t (x_t) \in A^{**}$ and $\mu = \lim_s (y_s + I) \in (A/I)^{**}$, then $\tilde{\phi}(\eta\mu) = \tilde{\phi}(\lim_t (x_t) \lim_s (y_s + I)) = \lim_{t,s} (\phi(x_t y_s + I)) = \lim_{t,s} ((x_t)\phi(y_s + I)) = \eta \lim_s (\phi(y_s + I)) = \eta\tilde{\phi}(\mu)$. Hence, since $\tilde{\pi} : A^{**}/I^{\perp\perp} \to A^{**}/J^{\perp\perp}$ is an A^{**}-bimodule isomorphism and A^{**} is unital, by the first part of our proof, $I^{\perp\perp} = J^{\perp\perp}$. That is, $I = A \cap I^{\perp\perp} = A \cap J^{\perp\perp} = J$. $\square$

Remark 3.1.7. Note that the previous proposition is not true for general operator algebras; the existence of a bai is needed. For example, let $A = \overline{\text{span}}(x, y)$ where $xy = 0$. If $I = \overline{\text{span}}(x)$ and $J = \overline{\text{span}}(y)$, then $A/I \cong A/J$ as A-bimodules, but $I \neq J$.

Now, we present a result about the ∞-sum of quotient operator algebras. In the following result, $M \oplus N$ is the ∞-sum of M and N.

Proposition 3.1.8. *Let M and N be closed subalgebras of an operator algebra A. If*

J is an ideal in M and K is an ideal in N, then

$$(M \oplus N)/(J \oplus K) \cong (M/J) \oplus (N/K)$$

completely isometrically isomorphically.

Proof. Let $q_1 : M \to M/J$ and $q_2 : N \to N/K$ be the canonical maps. We know that q_1 and q_2 are complete quotient maps. Then, $q_1 \oplus q_2 : (M \oplus N) \to (M/J) \oplus (N/K)$ is a complete quotient map. To see this for $n = 1$, let $(m + J, n + K)$ be an element in the open ball of $(M/J) \oplus (N/K)$. Then, $\|m + J\| < 1$ and $\|n + K\| < 1$. There exist x and y in the open ball of M and N respectively, such that $q_1(x) = m + J$ and $q_2(y) = n + K$. Since $\|(x, y)\| = \sup\{\|x\|, \|y\|\} < 1$ and $(q_1 \oplus q_2)(x, y) = (m + J, n + K)$, $q_1 \oplus q_2$ maps the open ball of $M \oplus^\infty N$ onto the open ball of $(M/J) \oplus (N/K)$. Similar argument works for $n > 1$.

If $(q_1 \oplus q_2)(m + n) = 0$, then $q_1(m) = 0$ and $q_2(n) = 0$. So, $m \in J$ and $n \in K$, and hence $m + n \in J \oplus K$. Conversely, if $m + n \in J \oplus K$, then $m \in J$ and $n \in K$, so $(q_1 \oplus q_2)(m + n) = 0$. Hence, $\mathrm{Ker}(q_1 \oplus q_2) = J \oplus K$. Therefore, by the factor theorem, the corresponding map from $(M \oplus N)/(J \oplus K)$ to $(M/J) \oplus (N/K)$ is a complete isometry. $\square$

Remark 3.1.9. Recall that the maximal tensor product of two operator algebras A and B is an operator algebra. The maximal tensor product, $A \otimes_{\max} B$, is defined to be the completion of $A \otimes B$ in a new norm. If A and B are unital (resp. approximately unital), then $A \otimes_{\max} B$ is unital (resp. approximately unital). More details can be found in Chapter 6 of [7]. We want to note that if A is an approximately unital operator algebra and B is a C^*-algebra, J is an approximately unital ideal in A and K is an ideal in B, then by [6, Lemma 2.7], we have:

$$(A \otimes_{\max} B)/(J \otimes_{\max} B) \cong (A/J) \otimes_{\max} B,$$

and

$$(A \otimes_{\max} B)/(A \otimes_{\max} K) \cong A \otimes_{\max} (B/K),$$

completely isometrically.

3.2 Approximately Unital Ideals and Quotient Operator Algebras

When we study the quotient operator algebra A/J, we may get better results if J is an approximately unital ideal. Hence, in this section, we look at the approximately unital ideals in A and in A/J. We give the correspondence theorem for approximately unital ideals and present further results.

First, we want to point out an important fact about projections in the second dual of a quotient algebra. If J is an approximately unital ideal in A with support projection p, then $(A/J)^{**} \cong A^{**}(1-p)$. This gives a correspondence between the projections in A^{**} and $(A/J)^{**}$. In one direction, a projection q in A^{**} corresponds to the projection $q(1-p)$ in $(A/J)^{**}$. This map is surjective when considered between the projections in A^{**} and $(A/J)^{**}$. In the other direction, a projection q in $(A/J)^{**}$ is mapped to $q \in A^{**}$. This map is one-to-one when considered between the projections.

However, these two maps are not inverse functions. This fact is quite useful in proving results about approximately unital ideals.

We begin with the Correspondence Theorem for approximately unital ideals.

Theorem 3.2.1. *Let A be an approximately unital operator algebra, I be an approximately unital ideal in A and J be an approximately unital ideal in I. Then, I/J is an approximately unital ideal in A/J. Conversely, every approximately unital ideal of A/J is of the form I/J, where I is an approximately unital ideal in A with $J \subset I \subset A$.*

Proof. Clearly, I/J is an ideal in A/J. Since J is approximately unital, by [10, Proposition 3.1], I/J is approximately unital if and only if I is. Hence, I/J is an approximately unital ideal in A/J.

To prove the converse assertion, let K be an approximately unital ideal in A/J. Let $I = \{x \in A : x + J \in K\}$. Then, by definition, I is an ideal in A, $J \subset I$ and $K = I/J$. Again by [10, Proposition 3.1], I is approximately unital. Hence, I is an approximately unital ideal in A. □

Remark 3.2.2. Note that since [10, Proposition 3.1] is also valid for Arens regular Banach algebras, the previous result (and Corollary 3.2.4) can be stated for Arens regular Banach algebras.

We know that semiprimeness and semisimplicity descend to HSAs ([2] and [3]). We include this result for the sake of completeness; the proof belongs to D. P. Blecher.

Proposition 3.2.3. *If A is a semiprime (resp. semisimple) operator algebra, then every HSA in A is semiprime (resp. semisimple).*

Proof. Let D be a HSA in A. If A is semiprime and if J is an ideal in D with $J^2 = (0)$, then since D is approximately unital, we have $JAJ \subset JDADJ \subset JDJ \subset J^2 = (0)$. Thus, AJA is a nil ideal in A and hence it is zero. That is, $J \subset DJD \subset AJA = (0)$, and $J = (0)$.

Now, suppose that A is semisimple and let (f_t) be a cai for D. If $x \in J(D)$, then since $J(D)$ is a nondegenerate D-module, by Cohen's factorization, there exist $d \in D$ and $y \in J(D)$ with $x = dy$. Now, $yf_tad \in J(D)$ for all $a \in A$, since (f_t) is a cai for D. Since $J(D)$ is closed, we have $yad \in J(D)$. Thus, $r(yad) = r_A(dya) = r_A(xa)$ for all $a \in A$. Hence, $x \in J(A) = (0)$. That is, $J(D) = (0)$. $\square$

By using the correspondence theorem, we can show that semiprimeness descends to quotients by approximately unital ideals.

Corollary 3.2.4. *If A is a semiprime approximately unital operator algebra and J is an approximately unital ideal in A, then A/J is semiprime.*

Proof. Let K be an ideal in A/J such that $K^2 = (0)_{A/J}$. By Theorem 3.2.1, there exists an approximately unital ideal I in A such that $J \subset I \subset A$ and $K = I/J$. Since $K^2 = (I/J)(I/J) = I^2/J = (0)_{A/J}$, we conclude that $I^2 = J$. Hence, $I = J$. That is, $K = I/J = (0)_{A/J}$ and A/J is semiprime. $\square$

We will use the following lemma to prove some results about HSA's and approximately unital ideals. The fact that the intersection of a HSA and an approximately unital ideal contains a cai will be useful.

Lemma 3.2.5. *Let A be an approximately unital operator algebra, D be a HSA in A and J be an approximately unital ideal in A. Then, $D \cap J$ is a HSA in J.*

Proof. WLOG, assume that A is unital. Let $K = \overline{DA}$ be the associated r-ideal for D in A. Then, K and J are one-sided M-ideals. By [12, Proposition 5.30], $J \cap K$ is a one-sided M-ideal. So, $J \cap K$ is an r-ideal in A and hence in J. Let e and f be support projections of D and J; $D = eA^{**}e \cap A$ and $J = fA^{**} \cap A = A^{**}f \cap A$. Then, $K = eA^{**} \cap A$ and we claim that $J \cap K = efA^{**} \cap A$. Indeed, since e, f are open projections such that $ef = fe$, ef is an open projection (in a C^*-algebra containing A). If (e_t) and (f_r) are cais of K and J respectively, then by [8, Corollary 2.4], $e_s e_t \to e_s$ with t, for each s, and $f_p f_r \to f_p$ with r, for each p. Hence, after changing the indexing, $(e_t f_t)$ is a cai for $J \cap K$. Notice that $e_t f_t \to ef$ and hence $ef \in (J \cap K)^{\perp\perp}$. That is, ef is the support projection of $J \cap K$; $J \cap K = efA^{**} \cap A$.

Now, if $x \in D \cap J$, then $exe = x$ and $fx = xf = x$; that is, $efxef = efxf = efx = ex = x$ and $x \in efA^{**}ef \cap A$. If $x \in efA^{**}ef \cap A$, then $efxef = x$. Since $xf = x$, $x \in J$. Since $ef = fe$, $exe = x$ and $x \in D$ as well. That is, $x \in D \cap J$. Thus, $D \cap J = efA^{**}ef \cap A$ is a HSA in A with the support projection ef. □

We want to note that we will later prove that if D and J in the previous lemma have positive cais, then $D \cap J$ has a positive cai as well.

The following result is a correspondence theorem for HSAs.

Theorem 3.2.6. *Let A be an approximately unital operator algebra, D be a HSA* *in A and J be an approximately unital ideal in D. Then, D/J is a HSA in A/J. Conversely, every HSA of A/J is of the form D/J, where D is a HSA in A with $J \subset D \subset A$.*

Proof. Clearly, D/J is an inner ideal in A/J. By [10, Proposition 3.1], D/J is approximately unital since D is. Hence, D/J is a HSA in A/J.

To prove the converse assertion, let K be a HSA in A/J. Let $D = \{x \in A : x + J \in K\}$. Then, by definition, D is an inner ideal in A, $J \subset D$ and $K = D/J$. By [10, Proposition 3.1], D is approximately unital if and only if K is. Hence, D is a HSA in A. □

Corollary 3.2.7. *Let A be an operator algebra, J be an approximately unital ideal in A, and D be a HSA in A. If I is an approximately unital ideal in $D \cap J$, then $(D \cap J)/I$ is a HSA in J/I.*

Proof. By Lemma 3.2.5, $D \cap J$ is a HSA in J. Now, the result follows from Theorem 3.2.6. □

Proposition 3.2.8. *If A is an ideal (resp. a HSA) in its bidual and J is an approximately unital ideal in A, then A/J is an ideal (resp. a HSA) in its bidual.*

Proof. Assume that A is an ideal in A^{**}. Then, $A/J \subset (A/J)^{**} = A^{**}p^{\perp}$. Here, p is the support projection of J and an element $a + J \in A/J$ is identified with $\hat{a}p^{\perp} \in A^{**}p^{\perp}$. If $a+J \in A/J$ and $\eta p^{\perp} \in A^{**}p^{\perp}$, then $\hat{a}p^{\perp}\eta p^{\perp} = \hat{a}\eta p^{\perp} = \hat{b}p^{\perp}$ for some $b + J \in A/J$, since A is an ideal in A^{**} and since p is a central projection. That is, A/J is an ideal in $A^{**}p^{\perp}$.

If A is a HSA in A^{**}, and if $a+J, b+J \in A/J$, then for any $\eta p^{\perp} \in A^{**}p^{\perp}$, $\hat{a}p^{\perp}\eta p^{\perp}\hat{b}p^{\perp} = \hat{a}\eta\hat{b}p^{\perp} = \hat{c}p^{\perp}$ for some $c+J \in A/J$. That is, A/J is an inner ideal. We know that A/J is approximately unital (by [10, Proposition 3.1]); hence, A/J is a HSA in its bidual. $\square$

Finally, we want to point out a result from [9]. If A is an operator algebra with a cai and J is an approximately unital ideal in A, let $q_J : A \to A/J$ is the canonical quotient map. The open projections in $(A/J)^{**}$ are exactly the $q_J^{**}(p)$ for open projections p in A^{**}. Indeed, if p is an open projection in A^{**}, then it is the weak* limit of a net (x_t) in A with $x_t = px_tp$. Then, $q_J^{**}(p)$ is the weak* limit of the net $(x_t + J)$ in A/J; so, $q_J^{**}(p)$ is open in $(A/J)^{**}$. If p is an open projection in $(A/J)^{**}$, then it is the support projection of a HSA $D' = q_J^{-1}(D)$, where D is a HSA in A. Then, $q_J^{**}(p)$ is the support projection of D and hence is open in A^{**}.

We showed that the ideals and HSAs in A/J are exactly the images of the ideals and HSAs in A, under the quotient map. Moreover, by [9, Corollary 6.3], the r-ideals in A/J are the images of the r-ideals in A, under the quotient map q_J. An r-ideal (resp. HSA) in A/J of the form $x(A/J)$ (resp. $x(A/J)x$) for some $x \in A$ with $\|1-x\| \leq 1$, is the image of an r-ideal (resp. HSA) in A of the form yA (resp. yAy) for some $y \in A$ with $\|1-y\| \leq 1$.

3.3 Diagonal of a Quotient Operator Algebra

The diagonal of an operator algebra may give some useful information about the algebra. Since it is a C^*-algebra sitting inside the given operator algebra, it may be slightly easier to work with the diagonal to gather information about the operator algebra. In [3] for example, when the diagonal of an operator algebra is a dual C^*-algebra, many conclusions can be made about the operator algebra itself, with some additional conditions of course.

Some results in this section are from [2], which is a joint work with D. P. Blecher.

If A is an operator algebra, represented as a subalgebra of $B(H)$, then the diagonal of A is defined to be the C^*-algebra

$$\Delta(A) = \{a \in A : a^* \in A\}.$$

Note that a^* is the adjoint of the element a in $B(H)$; and this does not depend on a particular H. If A is w^*-closed, then $\Delta(A)$ is a W^*-algebra.

An important observation is that a contractive homomorphism from a C^*-algebra into an operator algebra actually maps into the diagonal of that operator algebra, and is a $*$-homomorphism [7, 2.1.2].

Proposition 3.3.1. *If A and B are operator algebras such that $A \cong B$ completely isometrically isomorphically, then $\Delta(A) \cong \Delta(B)$ completely isometrically isomorphically.*

Proof. If $u : A \to B$ is the complete isometry, then $u' = u|_{\Delta(A)} : \Delta(A) \to B$ maps onto $\Delta(B)$. That is, $u' : \Delta(A) \to \Delta(B)$ is a completely isometric isomorphism. $\square$

Proposition 3.3.2. *Let A, B be closed subalgebras of $B(H)$. Then, $\Delta(A \cap B) = \Delta(A) \cap \Delta(B)$.*

Proof. If $x \in \Delta(A) \cap \Delta(B)$, then $x \in A \cap B$ and $x^* \in A \cap B$, hence $x \in \Delta(A \cap B)$. Conversely, if $x \in \Delta(A \cap B)$, then $x^* \in A \cap B$ and $x, x^* \in A$ and $x, x^* \in B$. Hence, $x \in \Delta(A) \cap \Delta(B)$. $\square$

We want to understand the diagonal of a HSA or an ideal in A. One case of the following result was given in [3].

Proposition 3.3.3. *Let J be a closed subspace of A such that $JAJ \subset J$. Then, $\Delta(J) = \Delta(A) \cap J$.*

Proof. It is trivial that $\Delta(J)$ is a subalgebra of $\Delta(A) \cap J$. Conversely, if $JAJ \subset J$, then $(\Delta(A) \cap J)\Delta(A)(\Delta(A) \cap J) \subset \Delta(A) \cap J$. So, $\Delta(A) \cap J$ is a HSA in a C^*-algebra, and hence it is selfadjoint. Thus, $x \in \Delta(A) \cap J$ implies that $x^* \in \Delta(A) \cap J \subset J$. That is, $x \in \Delta(J)$. $\square$

It is important to point out that the diagonal of an ideal J in A is an ideal in $\Delta(A)$ if it is nonzero. Indeed, $\Delta(J) = \Delta(A) \cap J$ and $\Delta(J)\Delta(A) \subset \Delta(A) \cap (JA) \subset \Delta(A) \cap J = \Delta(J)$. Similarly, since J is a two-sided ideal, $\Delta(A)\Delta(J) \subset \Delta(J)$. We sometimes use this fact without any references.

We are interested in understanding the structure of the quotient operator algebras. Hence, we want to understand the structure of the diagonal of a quotient operator algebra.

Let A be an operator algebra and J be an ideal in A. Then, $\Delta(A)/\Delta(J)$ is a C^*-subalgebra of $\Delta(A/J)$.

Proposition 3.3.4. *Let A be an approximately unital operator algebra and J be an ideal in A. Then, $\Delta(A)/\Delta(J) \subset \Delta(A/J)$ completely isometrically.*

Proof. Let $u : A \to A/J$ be the canonical complete quotient map defined as $u(x) = x + J$. The restriction of u to $\Delta(A) \subset A$, u', is a complete contraction. By the observation mentioned earlier, since u' is a contractive homomorphism, it maps into $\Delta(A/J)$. Hence, we have a completely contractive map $u' = u|_{\Delta(A)} : \Delta(A) \to \Delta(A/J)$, where $\mathrm{Ker}(u') = \Delta(A) \cap J = \Delta(J)$. Since $\Delta(A)$ is a C^*-algebra, $\Delta(J)$ is an approximately unital ideal in $\Delta(A)$. Since approximately unital ideals in (unital) operator algebras are M-ideals, and hence are proximinal (see e.g. [7, Section 4.8] or [29]), $u'(\mathrm{Ball}(\Delta(A))) = \mathrm{Ball}(\Delta(A/J))$. Here, $\mathrm{Ball}(A)$ denotes the open ball of A. Similarly for all matrix levels. That is, u' is a complete quotient map. Hence, $\Delta(A)/\Delta(J) \subset \Delta(A/J)$ completely isometrically. $\square$

For any operator algebra A, the diagonal $\Delta(A)$ acts nondegenerately on A if and only if A has a positive cai, and if and only if $1_{\Delta(A)^{\perp\perp}} = 1_{A^{**}}$. The latter is equivalent to $1_{A^{**}} \in \Delta(A)^{\perp\perp}$. Hence, we may use the statements '$\Delta(A)$ acts nondegenerately on A' and 'A has a positive cai' interchangeably.

If $\Delta(A)$ acts nondegenerately on A, then this does not imply that for any ideal J of A, $\Delta(J)$ acts nondegenerately on J. To see this, take any approximately unital operator algebra J such that $\Delta(J)$ does not act nondegenerately on J. Then, J is an ideal in $A = M(J)$, and $\Delta(A)$ acts nondegenerately on A. However, if $\Delta(A)$ acts nondegenerately on A, then we can conclude that $\Delta(A/J)$ acts nondegenerately on A/J.

Proposition 3.3.5. *Let J be an ideal in an operator algebra A. If A has a positive cai, then A/J has a positive cai.*

Proof. Let (e_t) be a positive cai for A. Then, $q(e_t) \in \Delta(A/J)_+$, where $q : A \to A/J$ is the canonical quotient map. Indeed, by the fact we mentioned at the beginning of this section q maps into $\Delta(A/J)$ and since each e_t is positive, $q(e_t)$ is positive. Now, it is easy to see that $q(e_t)$ is a positive cai for A/J; $\|q(e_t)\| \leq \|e_t\| \leq 1$ and $q(e_t)(x + J) = q(e_t x) \to q(x) = x + J$, for each $x + J \in A/J$. □

We prove a lemma about the diagonal of an ideal that contains a positive cai.

Lemma 3.3.6. *Let A be an approximately unital operator algebra and J be an ideal in A that contains a positive cai. Then, $\Delta(A) + J$ is closed and $\Delta(J)^{\perp\perp} = \Delta(A)^{\perp\perp} \cap J^{\perp\perp}$.*

Proof. We know that $\Delta(J) = \Delta(A) \cap J$ is an ideal in $\Delta(A)$. The proof of [12, Lemma 5.2.9] shows that if $\Delta(A) + J$ is closed, or equivalently by appendix A.1.5 there, if $\Delta(A)^{\perp\perp} + J^{\perp\perp}$ is closed, then $(\Delta(A) \cap J)^{\perp\perp} = (\Delta(A)^{\perp} + J^{\perp})^{\perp} = \Delta(A)^{\perp\perp} \cap J^{\perp\perp}$. If J has a positive cai, then the proof of [25, Proposition 2.4] shows that $\Delta(A) + J$ is closed. Hence, $\Delta(J)^{\perp\perp} = \Delta(A)^{\perp\perp} \cap J^{\perp\perp}$. □

Under additional conditions, we can identify the second dual of the diagonal of a quotient algebra with the second dual of the quotient of the diagonals.

Proposition 3.3.7. *Let A be an approximately unital operator algebra such that $\Delta(A^{**}) = \Delta(A)^{**}$. If J is an ideal in A that contains a positive cai, then $\Delta(J^{\perp\perp}) = \Delta(J)^{\perp\perp}$ and $\Delta((A/J)^{**}) = (\Delta(A)/\Delta(J))^{**}$.*

Proof. By Lemma 3.3.6, $\Delta(J)^{\perp\perp} = \Delta(A)^{\perp\perp} \cap J^{\perp\perp}$. That is, $\Delta(J)^{\perp\perp} = \Delta(A^{**}) \cap J^{\perp\perp} = \Delta(J^{\perp\perp})$ since $J^{\perp\perp}$ is an ideal in A^{**}. Let $p \in A^{**}$ be the support projection of J. Then,

$$\Delta((A/J)^{**}) = \Delta(A^{**}(1-p)) = \Delta(A^{**})(1-p) = \Delta(A)^{**}(1-p) = (\Delta(A)/\Delta(J))^{**}.$$

□

We want to point out that if A is a HSA in its bidual and if it has positive cai, then the condition of the Proposition 3.3.7 is satisfied [2]; hence, the conclusion is valid for such operator algebras (for example, for σ-matricial algebras which will be defined in Chapter 4). In this case, we also get a corollary about the diagonal of A/J.

Corollary 3.3.8. *Let A be an operator algebra with a positive cai that is a HSA in its bidual. If J is an ideal in A that contains a positive cai, then $\Delta(A/J) = \Delta(A)/\Delta(J)$.*

Proof. Since A is a HSA in its bidual and it contains a positive cai, we have $\Delta(A^{**}) = \Delta(A)^{**}$ [2]. Also, A/J is a HSA in its bidual $(A/J)^{**}$ by Proposition 3.2.8 and it has a positive cai by Proposition 3.3.5. Hence, $\Delta((A/J)^{**}) = \Delta(A/J)^{**}$ [2]. Moreover, since A/J is a HSA in its bidual, it is nc-discrete [2] and hence, $\Delta(A/J)$ is an annihilator C^*-algebra. We know by Proposition 3.3.4 that $\Delta(A)/\Delta(J) \subset \Delta(A/J)$ completely isometrically; that is, $\Delta(A)/\Delta(J)$ is an annihilator C^*-algebra as well.

Using Proposition 3.3.7, we have $(\Delta(A)/\Delta(J))^{**} = \Delta((A/J)^{**}) = \Delta(A/J)^{**}$. Hence, $\Delta(A)/\Delta(J) = \Delta(A/J)$. $\square$

Now, we present a result about the intersection of a HSA and an approximately unital ideal. If the HSA and the ideal contain positive cais, then the intersection contains a positive cai.

Proposition 3.3.9. *Let A be an approximately unital operator algebra, D be a HSA in A and J be an approximately unital ideal in A. Then,*

$$(D \cap J)^{\perp\perp} = D^{\perp\perp} \cap J^{\perp\perp} \text{ and } \Delta(D \cap J)^{\perp\perp} = \Delta(D)^{\perp\perp} \cap \Delta(J)^{\perp\perp}.$$

If D and J have positive cais, then $D \cap J$ has a positive cai as well.

Proof. Let d and p be the support projections of D and J, respectively. Notice that $dp = pd = dpd \in D^{\perp\perp} \cap J^{\perp\perp}$. We know by Lemma 3.2.5 that $D \cap J$ is a HSA in J, and its support projection is $dpd \in J^{**}$. Hence, $D^{\perp\perp} \cap J^{\perp\perp} = dA^{**}d \cap pA^{**}p = dpdA^{**}dpd = dpA^{**}pd = dJ^{**}d = d(pJ^{**}p)d = dpdJ^{**}dpd = (D \cap J)^{\perp\perp}$.

It is easy to see that $\Delta(D \cap J) = \Delta(D) \cap \Delta(J)$. Hence, $\Delta(D \cap J)^{\perp\perp} = (\Delta(D) \cap \Delta(J))^{\perp\perp}$. Since $\Delta(D)$ is a HSA in $\Delta(A)$ and $\Delta(J)$ is an approximately unital ideal in $\Delta(A)$, by the first part of our proof, we can conclude that $(\Delta(D) \cap \Delta(J))^{\perp\perp} = \Delta(D)^{\perp\perp} \cap \Delta(J)^{\perp\perp}$. That is, $\Delta(D \cap J)^{\perp\perp} = \Delta(D)^{\perp\perp} \cap \Delta(J)^{\perp\perp}$.

Now, suppose that D and J have positive cais. We have;

$$1_{(D\cap J)^{\perp\perp}} = 1_{D^{\perp\perp}\cap J^{\perp\perp}} = 1_{D^{\perp\perp}} 1_{J^{\perp\perp}} \in \Delta(D)^{\perp\perp} \cap \Delta(J)^{\perp\perp} = \Delta(D \cap J)^{\perp\perp}.$$

That is, $1_{(D\cap J)^{\perp\perp}} \in \Delta(D \cap J)^{\perp\perp}$. Hence, $D \cap J$ has a positive cai. $\square$

3.4 Irreducible Representations of Operator Algebras

In this section, we want to work with irreducible representation of operator algebras and give some results about extensions and restrictions of irreducible representations. Similar to the C^*-algebra theory, irreducible representations of the operator algebra A restrict to irreducible representations of an approximately unital ideal J. Also, irreducible representations of the approximately unital ideal extend to irreducible representations of the operator algebra. We also give such a correspondence for irreducible representations of the quotient algebras (by approximately unital ideals).

A representation of an operator algebra A on a Hilbert space H is a completely contractive homomorphism $\pi : A \to B(H)$. We say that π is nondegenerate if $[\pi(A)H] = H$; which is equivalent to saying that H is a nondegenerate A-module. Also, this is equivalent to saying that $\pi(e_t) \to I_H$ strongly, if (e_t) is a cai for A ([7, add])

Definition 3.4.1. Let A be an approximately unital operator algebra and $\pi : A \to B(H)$ be a representation of A. A closed subspace K of H is said to be π-*reducing* if both K and $K^\perp$ are invariant under π. We say that π is *irreducible* if H does not contain any nontrivial π-reducing subspaces.

Note that K is π-reducing if and only if $\pi(a)$ commutes with $P_{K^\perp}$ for any $a \in A$ (see [7, p109]). Note also that every irreducible representation is nondegenerate.

We want to point out the important fact about the irreducible representations on A and on $C^*_{\max}(A)$. Every irreducible representation of A extends (uniquely) to an irreducible representation of $C^*_{\max}(A)$ [7, Proposition 2.4.2]. Moreover, every irreducible representation of $C^*_{\max}(A)$ restricts to an irreducible representation of A; if $P_{K^\perp}$ commutes with $T \in \pi(A)$, then it commutes with $T^* \in \pi(A)^*$.

We say that a vector $\eta \in H$ is cyclic if $[\pi(A)\eta] = H$. If $\pi : A \to B(H)$ is a nondegenerate representation and if every nonzero vector in H is cyclic, then π is irreducible. Indeed, if K is a π-reducing subspace of H and $0 \neq \eta \in K$, then $\pi(A)\eta \subset \pi(A)K \subset K$ and since K is closed, $H = [\pi(A)\eta] \subset K$. Hence, π does not have nontrivial reducing subspaces.

When an irreducible representation of a C^*-algebra is restricted to an ideal, this restriction is an irreducible representation of the ideal. Conversely, every irreducible representation of an ideal extends uniquely to an irreducible representation of the C^*-algebra [40, Theorem 5.5.1]. We prove that the same holds for irreducible representations of operator algebras and approximately unital ideals.

Proposition 3.4.2. *Let $\pi : A \to B(H)$ be an irreducible representation and J be an approximately unital ideal in A. The restriction of π to J is irreducible. Furthermore, every irreducible representation of J extends uniquely to an irreducible representation of A.*

Proof. Let $I = C^*_{\max}(J)$; I is an ideal in $C^*_{\max}(A)$. Since $\tilde{\pi}$, the extension of π to $C^*_{\max}(A)$, is irreducible, $\tilde{\pi}|_I$ is irreducible. Hence, the restriction of $\tilde{\pi}|_I$ to J, which is $\pi|_J$, is irreducible.

Moreover, if ρ is an irreducible representation of J, then it extends to an irreducible representation $\tilde{\rho}$ on I and that extends uniquely to an irreducible representation of $\mathrm{C}^*_{\max}(A)$. This last irreducible representation restricts to an irreducible representation of A which equals ρ when restricted to J. Hence, ρ is a restriction of an irreducible representation of A. $\square$

Now, we want to prove a result about the irreducible representations of quotient operator algebras.

Proposition 3.4.3. *Let $\pi : A \to B(H)$ be an irreducible representation and J be an approximately unital ideal contained in* $\mathrm{Ker}(\pi)$. *Then $\tilde{\pi} : A/J \to B(H)$ defined by $\tilde{\pi}(a + J) = \pi(a)$ for any $a \in A$, is an irreducible representation of A/J. Every irreducible representation of A/J arises in this way.*

Proof. Let $\tilde{\pi} : A/J \to B(H)$ be defined as $\tilde{\pi}(a + J) = \pi(a)$. Then, by the Factor Theorem, $\|\tilde{\pi}\|_{cb} = \|\pi\|_{cb}$. That is, $\tilde{\pi}$ is completely contractive. By definition of $\tilde{\pi}$, M is a $\tilde{\pi}$-reducing subspace of H if and only if it is π-reducing as well. Hence, $\tilde{\pi}$ is irreducible if and only if π is irreducible. $\square$

For a HSA D in A, we can not say that every irreducible representation of D extends to an irreducible representation of A. As a simple example, let

$$D = \left\{ \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix} : x \in \mathbb{C} \right\} \text{ and } A = M_2.$$

Then, $\pi : D \to \mathbb{C}$, where $\pi([a_{ij}]) = a_{11}$ is an irreducible representation but this does not extend to an irreducible representation from M_2 to $\mathbb{C}$; as such an extension would correspond to an ideal of M_2; however, M_2 has no nontrivial ideals.

We present a special case where $\mathrm{C}^*_{\max}(A)$ is simple. The representations are automatically completely isometric in this case.

Proposition 3.4.4. *Let A be an operator algebra such that $\mathrm{C}^*_{\max}(A)$ is simple. Then, any representation of A is completely isometric. Moreover, A does not have any nontrivial approximately unital ideals.*

Proof. If π is a completely contractive homomorphism on A, then π extends to $\mathrm{C}^*_{\max}(A)$ and the kernel of π is (0) since $\mathrm{C}^*_{\max}(A)$ is simple. That is, the extension of π is completely isometric. Hence, π is completely isometric. Now, let J be an approximately unital ideal in A and $q : A \to A/J$ be the canonical quotient map. By the first part of the proof, q is a complete isometry and hence $J = (0)$ or $J = A$. $\square$

Chapter 4

Matricial Algebras

The C^*-algebras of compact operators play an important role in operator theory and there are several characterizations of this class; some of these characterizations are given in Chapter 2. For example, these are the C^*-algebras that are ideals in their second duals. We want to define a new class of operator algebras that may play a similar role in the theory of non-selfadjoint operator algebras. Many results in this chapter are from [3], which is a joint work of the author with D. P. Blecher and S. Sharma; for those results, the proofs are not included.

4.1 Matricial Algebras

Note that, in this chapter, we only consider separable approximately unital operator algebras.

Definition 4.1.1. We say that an operator algebra A is *matricial* if it contains a full set of matrix units $\{T_{ij}\}$, whose span is dense in A. Hence, $T_{ij}T_{kl} = \delta_{jk}T_{il}$, where δ_{jk} is the Kronecker delta. Let $q_k = T_{kk}$. We say that a matricial operator algebra is 1-*matricial* if $\|q_k\| = 1$, for all k; that is, if and only if all q_k are orthogonal projections.

We will focus on 1-matricial operator algebras. We will consider two 1-matricial algebras as the same if there is a completely isometric isomorphism between them.

We are only interested in the separable algebras, and in this case we prefer using the following equivalent description of 1-matricial algebras. Consider a (finite or infinite) sequence $T_1, T_2, \ldots$ of invertible operators on a Hilbert space K, with $T_1 = I$. Set $H = \ell^2 \otimes^2 K = K^{(\infty)} = K \oplus^2 K \oplus^2 \ldots$ (in the finite sequence case, $H = K^{(n)}$). Define $T_{ij} = E_{ij} \otimes T_i^{-1}T_j \in B(H)$, for $i, j \in \mathbb{N}$. Let A be the closure of the span of the T_{ij}. Notice that $T_{ij}T_{kl} = \delta_{jk}T_{il}$, so that T_{ij} are matrix units for A. Let $q_k = T_{kk}$, then $\|q_k\| = 1$ for all k. Then, A is a 1-matricial algebra, and all separable or finite dimensional 1-matricial algebras arise in this way.

Definition 4.1.2. A σ-*matricial* algebra is a c_0-direct sum of 1-matricial algebras. Since we consider the separable case only, this sum will be a countable (or finite) direct sum.

We want to present some results about 1-matricial algebras. We start with a lemma that summarizes basic properties of 1-matricial algebras.

Lemma 4.1.3. *Any* 1*-matricial algebra* A *is approximately unital, topologically simple, hence semisimple and semiprime, and is a compact modular annihilator operator* *algebra. It is a HSA in its bidual, so has the unique Hahn-Banach extension property in [8, Theorem 2.10]. It also has dense socle, with the q_k algebraically minimal projections with $A = \oplus_k^c q_k A = \oplus_k^r A q_k$. The canonical representation of A on Aq_1 is faithful and irreducible, so that A is a primitive Banach algebra.*

Note that since a 1-matricial algebra A has the unique Hahn Banach extension property, the dual A^* has the Radon-Nikodým property. Moreover, A^{**} is a *rigid extension* of A (that is, there exists only one completely contractive map from A^{**} to itself extending the identity map on A).

Corollary 4.1.4. *A 1-matricial algebra A is a right (resp. left, two-sided) ideal in its bidual iff $q_1 A$ (resp. Aq_1, $q_1 A$ and Aq_1) is reflexive.*

An operator algebra A is *left* or *right essential* if the left or right multiplication by an element in A induces a bicontinuous injection of A in $B(A)$. Note that the existence of a bai implies that A is left and right essential. If $(q_k)_{k=1}^{\infty}$ are elements in a normed algebra A, then we say that $\sum_k q_k = 1$ *strictly* if $\sum_k q_k a = a$ and $\sum_k a q_k = a$ for all $a \in A$, with the convergence in the sense of nets (indexed by finite subsets of $\mathbb{N}$). The following gives a characterization of 1-matricial algebras in terms of the existence of a certain family of algebraically minimal idempotents or projections.

Theorem 4.1.5. *If A is a topologically simple left or right essential operator algebra with a sequence of nonzero algebraically minimal idempotents (q_k) with $q_j q_k = 0$ for $j \neq k$, and $\sum_k q_k = 1$ strictly, then A is similar to a 1-matricial algebra. If further the q_k are projections, then A is unitarily isomorphic to a 1-matricial algebra.*

For infinite dimensional 1-matricial algebras, being completely isometric to $\mathbb{K}(\ell^2)$ is equivalent to being topologically isomorphic to a C^*-algebra, or also equivalent to $\{\|T_k\| \left\|T_k^{-1}\right\|\}$ being bounded [3, Lemma 4.7].

An operator algebra is called a *subcompact* 1-*matricial algebra* if it is (completely isometrically isomorphic to) a 1-matricial algebra with the space K in the definition of a 1-matricial algebra (the alternative definition given after Definition 4.1.1) being finite dimensional. Subcompact 1-matricial algebras are subalgebras of $\mathbb{K}(\ell^2)$; and they are two-sided ideals in their biduals (which is not necessarily true for all 1-matricial algebras).

Lemma 4.1.6. *A* 1*-matricial algebra A is subcompact iff A is completely isometrically isomorphic to a subalgebra of $\mathbb{K}(\ell^2)$, and if and only if its C^*-envelope is an annihilator C^*-algebra. In this case, A is an ideal in its bidual, and q_kA (resp. Aq_k) is linearly completely isomorphic to a row (resp. column) Hilbert space. Here, q_k is as in Definition 4.1.1. Indeed, if a* 1*-matricial algebra A is bicontinuously (resp. isometrically) isomorphic to a subalgebra of $\mathbb{K}(\ell^2)$, then A is bicontinuously (resp. isometrically) isomorphic to a subcompact* 1*-matricial algebra.*

Now, we present some examples.

Example 4.1.7. Let $K = \ell_2^2$ and $T_k = \begin{bmatrix} k & 0 \\ 0 & 1/k \end{bmatrix}$. Notice that $T_{ij} = \begin{bmatrix} j/i & 0 \\ 0 & i/j \end{bmatrix}$

and $T_{ii} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. An element x in A looks like:

$$x = \begin{bmatrix} \alpha_{11}T_{11} & \alpha_{12}T_{12} & \alpha_{13}T_{13} & \dots \\ \alpha_{21}T_{21} & \alpha_{22}T_{22} & \alpha_{23}T_{23} & \dots \\ \alpha_{31}T_{31} & \alpha_{32}T_{32} & \alpha_{33}T_{23} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

That is,

$$x = \begin{bmatrix} \alpha_{11}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \alpha_{12}\begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} & \alpha_{13}\begin{bmatrix} 3 & 0 \\ 0 & \frac{1}{3} \end{bmatrix} & \dots \\ \alpha_{21}\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{bmatrix} & \alpha_{22}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \alpha_{23}\begin{bmatrix} \frac{3}{2} & 0 \\ 0 & \frac{2}{3} \end{bmatrix} & \dots \\ \alpha_{31}\begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 3 \end{bmatrix} & \alpha_{32}\begin{bmatrix} \frac{2}{3} & 0 \\ 0 & \frac{3}{2} \end{bmatrix} & \alpha_{33}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Hence,

$$x = \begin{bmatrix} \begin{bmatrix} \alpha_{11} & 0 \\ 0 & \alpha_{11} \end{bmatrix} & \begin{bmatrix} 2\alpha_{12} & 0 \\ 0 & \frac{1}{2}\alpha_{12} \end{bmatrix} & \begin{bmatrix} 3\alpha_{13} & 0 \\ 0 & \frac{1}{3}\alpha_{13} \end{bmatrix} & \dots \\ \begin{bmatrix} \frac{1}{2}\alpha_{21} & 0 \\ 0 & 2\alpha_{21} \end{bmatrix} & \begin{bmatrix} \alpha_{22} & 0 \\ 0 & \alpha_{22} \end{bmatrix} & \begin{bmatrix} \frac{3}{2}\alpha_{23} & 0 \\ 0 & \frac{2}{3}\alpha_{23} \end{bmatrix} & \dots \\ \begin{bmatrix} \frac{1}{3}\alpha_{31} & 0 \\ 0 & 3\alpha_{31} \end{bmatrix} & \begin{bmatrix} \frac{2}{3}\alpha_{32} & 0 \\ 0 & \frac{3}{2}\alpha_{32} \end{bmatrix} & \begin{bmatrix} \alpha_{33} & 0 \\ 0 & \alpha_{33} \end{bmatrix} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Since (α_{ij}) were arbitrary, we can change the coefficients in the matrix;

$$x = \begin{bmatrix} \begin{bmatrix} \alpha_{11} & 0 \\ 0 & \alpha_{11} \end{bmatrix} & \begin{bmatrix} \alpha_{12} & 0 \\ 0 & \frac{1}{4}\alpha_{12} \end{bmatrix} & \begin{bmatrix} \alpha_{13} & 0 \\ 0 & \frac{1}{9}\alpha_{13} \end{bmatrix} & \cdots \\ \begin{bmatrix} \alpha_{21} & 0 \\ 0 & 4\alpha_{21} \end{bmatrix} & \begin{bmatrix} \alpha_{22} & 0 \\ 0 & \alpha_{22} \end{bmatrix} & \begin{bmatrix} \alpha_{23} & 0 \\ 0 & \frac{4}{9}\alpha_{23} \end{bmatrix} & \cdots \\ \begin{bmatrix} \alpha_{31} & 0 \\ 0 & 9\alpha_{31} \end{bmatrix} & \begin{bmatrix} \alpha_{32} & 0 \\ 0 & \frac{9}{4}\alpha_{32} \end{bmatrix} & \begin{bmatrix} \alpha_{33} & 0 \\ 0 & \alpha_{33} \end{bmatrix} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

After canonical shuffling, we get:

$$x = \begin{bmatrix} \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \cdots \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \cdots \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} & \begin{bmatrix} 0 & \cdots \\ \vdots & \ddots \end{bmatrix} \\ \begin{bmatrix} 0 & \cdots \\ \vdots & \ddots \end{bmatrix} & \begin{bmatrix} \alpha_{11} & \frac{1}{4}\alpha_{12} & \frac{1}{9}\alpha_{13} & \cdots \\ 4\alpha_{21} & \alpha_{22} & \frac{4}{9}\alpha_{23} & \cdots \\ 9\alpha_{31} & \frac{9}{4}\alpha_{32} & \alpha_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \end{bmatrix}.$$

To have a better understanding of the elements of A, let $a = [\alpha_{ij}]$ and $x = [\alpha_{ij}T_{ij}] \in A$. Then, after canonical shuffling, the algebra A looks like:

$$A = \left\{ \begin{bmatrix} a & 0 \\ 0 & S^{-1}aS \end{bmatrix} : a, S^{-1}aS \in \mathbb{K}(\ell^2) \right\},$$

where $S = \operatorname{diag}\{1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16} \ldots\}$. Hence, A is a subcompact 1-matricial algebra. Moreover, A is an ideal in its bidual but it is not isomorphic to $\mathbb{K}(\ell^2)$ as Banach algebras by [3, Lemma 4.7]. Also, A is not an annihilator algebra by [41, Theorem 8.7.12], since $(q_1A)^*$ is not isomorphic to Aq_1 via the canonical pairing.

Example 4.1.8. Let $K = \ell^2$ and $T_k = E_{kk} + \frac{1}{k}I$. That is,

$$T_k = \operatorname{diag}\left\{\frac{1}{k}, \frac{1}{k}, \cdots \frac{1}{k}, \frac{k+1}{k}, \frac{1}{k}, \frac{1}{k}, \cdots\right\}.$$

Here, $q_1A \cong c_0$ by [3, Lemma 4.12] and q_1A is not reflexive. Note that T_k^{-1} has k in all diagonal entries but one; and that entry is $\frac{k}{k+1}$, which is positive and less than k. It follows that Aq_1 is a column Hilbert space; that is, Aq_1 is reflexive. By Corollary 4.1.4, A is a left ideal in its bidual, but it is not a right ideal in its bidual. This is interesting since any C^*-algebra which is a left ideal in its bidual is also a right ideal in its bidual. Moreover, this algebra is not an annihilator algebra by [41, Theorem 8.7.12], since $(q_1A)^*$ is not isomorphic to Aq_1. Also, by Lemma 4.1.6, A is not subcompact and A is not bicontinuously isomorphic to a subalgebra of $\mathbb{K}(\ell^2)$.

Example 4.1.9. Let A be the 1-matricial algebra generated by $T_k = I - \sum_{i=1}^{k}\left(1 - \frac{i}{k}\right)E_{ii}$. That is,

$$T_1 = \begin{bmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}, T_2 = \begin{bmatrix} 1/2 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

$$T_3 = \begin{bmatrix} 1/3 & 0 & 0 & \cdots \\ 0 & 2/3 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}, T_4 = \begin{bmatrix} 1/4 & 0 & 0 & \cdots \\ 0 & 2/4 & 0 & \cdots \\ 0 & 0 & 3/4 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}, etc.$$

Notice that $\|T_k\| = 1$ and $\left\|T_k^{-1}\right\| = k$; $\|T_k\| \left\|T_k^{-1}\right\| = k$. Since $\{\|T_k\| \left\|T_k^{-1}\right\|\}$ is not bounded, A is not subcompact by [3, Lemma 4.7].

Let $x = [\alpha_{ij} T_{ij}]$ be an element in A. For $k \leq \ell$, we have;

$$T_k^{-1} T_\ell = \sum_{i=1}^{k} (\frac{k}{\ell} - 1) E_{i,i} + \sum_{j=1}^{\ell-k} (\frac{k+j}{\ell} - 1) E_{k+j,k+j} + I.$$

In other words, for $k \leq \ell$, the $k\ell$-entry of the matrix $T_{k\ell}$ is a diagonal matrix where the first $\ell - 1$ entries in the diagonal are $\frac{k}{\ell}$ and the rest of the elements in the diagonal are 1.

That is,

$$x = \begin{bmatrix} \alpha_{11}T_{11} & \alpha_{12}T_{12} & \alpha_{13}T_{13} & \ldots \\ \alpha_{21}T_{21} & \alpha_{22}T_{22} & \alpha_{23}T_{23} & \ldots \\ \alpha_{31}T_{31} & \alpha_{32}T_{32} & \alpha_{33}T_{23} & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

where , for $k \leq \ell$, $T_{k\ell} = E_{k\ell} \otimes \text{diag}\{\overbrace{\frac{k}{\ell}, \frac{k}{\ell}, \cdots, \frac{k}{\ell}}^{\ell-1}, 1, 1, \cdots\}$.

After canonical shuffling, x can be viewed as a block diagonal matrix where the blocks $B_1, B_2, \cdots, B_k, \cdots$ are of the form;

$$B_1 = \begin{bmatrix} 1\alpha_{11} & \frac{1}{2}\alpha_{12} & \frac{1}{3}\alpha_{13} & \frac{1}{4}\alpha_{14} & \cdots \\ 2\alpha_{21} & 1\alpha_{22} & \frac{2}{3}\alpha_{23} & \frac{2}{4}\alpha_{24} & \cdots \\ 3\alpha_{31} & \frac{3}{2}\alpha_{32} & 1\alpha_{33} & \frac{3}{4}\alpha_{34} & \cdots \\ 4\alpha_{41} & \frac{4}{2}\alpha_{42} & \frac{4}{3}\alpha_{43} & 1\alpha_{44} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 1\alpha_{11} & 1\alpha_{12} & \frac{2}{3}\alpha_{13} & \frac{2}{4}\alpha_{14} & \cdots \\ 1\alpha_{21} & 1\alpha_{22} & \frac{2}{3}\alpha_{23} & \frac{2}{4}\alpha_{24} & \cdots \\ \frac{3}{2}\alpha_{31} & \frac{3}{2}\alpha_{32} & 1\alpha_{33} & \frac{3}{4}\alpha_{34} & \cdots \\ \frac{4}{2}\alpha_{41} & \frac{4}{2}\alpha_{42} & \frac{4}{3}\alpha_{43} & 1\alpha_{44} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

$$B_3 = \begin{bmatrix} 1\alpha_{11} & 1\alpha_{12} & 1\alpha_{13} & \frac{3}{4}\alpha_{14} & \cdots \\ 1\alpha_{21} & 1\alpha_{22} & 1\alpha_{23} & \frac{3}{4}\alpha_{24} & \cdots \\ 1\alpha_{31} & 1\alpha_{32} & 1\alpha_{33} & \frac{3}{4}\alpha_{34} & \cdots \\ \frac{4}{3}\alpha_{41} & \frac{4}{3}\alpha_{42} & \frac{4}{3}\alpha_{43} & 1\alpha_{44} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, etc.$$

We can write each block as $B_k = S_k^{-1} a S_k$, where $S_k = \text{diag}\{\overbrace{1, 1, \ldots 1}^{k}, \frac{k}{k+1}, \frac{k}{k+2}, \ldots\}$, and $a = [\alpha_{ij}] \in \mathbb{K}(\ell^2)$. That is,

$$A = \left\{\text{diag}\left\{S_1 a S_1^{-1}, S_2 a S_2^{-1}, \cdots, S_k a S_k^{-1}, \cdots\right\} : a \in \mathbb{K}(\ell^2)\right\}.$$

Following the same arguments in [49, Example 4.2.11], we conclude that $C^*(A) = c_o^1 \otimes_{\min} \mathbb{K}(\ell^2)$. Thus, the C^*-envelope of A is not an annihilator C^*-algebra.

For an operator algebra A, let f be the join of all the algebraically minimal projections in A. Then, h-soc(A) is the HSA with support projection f; that is, h-soc$(A) = fA^{**}f \cap A$ (more details can be found in [3, Section 4]).

Theorem 4.1.10. *Let A be a semiprime operator algebra. Then h-soc(A) is a σ-matricial algebra.*

Hence, any semiprime operator algebra contains a canonical σ-matricial algebra; namely, h-soc(A).

Remark 4.1.11. Let D be a HSA in an operator algebra A. If e is an idempotent in D, then e is algebraically minimal in D if and only if e is algebraically minimal in A. Indeed, if e is algebraically minimal in D, then for $a \in A$, $eae = e(eae)e = \lambda e$ for some $\lambda \in \mathbb{C}$. That is, e is algebraically minimal in A as well.

Remark 4.1.12. We believe that if D is a HSA in A, then h-soc$(D) = h$-soc$(A) \cap D$. Also, if J is an ideal in A then h-soc$(D \cap J) = D \cap h$-soc(J). We hope to present the details of these soon.

Note that if A is a σ-matricial algebra, then the r-ideals, ℓ-ideals, and HSAs of A are of a very nice form; $J = \oplus^c_{k\in E} f_k A$ or $J = \oplus^r_{k\in E} A f_k$ [3, Proposition 4.25]. Moreover, every HSA in a 1-matricial algebra (resp. in a σ-matricial algebra) is a 1-matricial algebra (resp. σ-matricial algebra) [3, Corollary 4.26]. Furthermore, the quotients of σ-matricial algebras by approximately unital ideals are σ-matricial.

Proposition 4.1.13. *Let A be a σ-matricial algebra and J be an approximately unital ideal in A. Then, A/J is a σ-matricial algebra.*

Proof. Notice that the only approximately unital closed two-sided ideals of A are blocks consisting of 1-matricial algebras. If A is a σ-matricial algebra such that $A = \oplus^0 A_k$, where A_k are 1-matricial algebras, let J be an ideal in A. Set $J_k = J \cap A_k$; for each k, $J_k A \subset J \cap A_k = J_k$ and J_k is an ideal in A_k. We claim that $J = \oplus^0_k J_k$. Indeed, if $x \in J \subset \oplus^0_k A_k$, then $x = (x_k)$ where $x_k \in J \cap A_k$ and hence $x = (x_k) \in \oplus^0_k J_k$. If $(x_k) \in \oplus^0_k J_k$, then for each k, $x_k \in J \cap A_k$. So, $x = (x_k) \in J$. Since A_k is simple by

Lemma 4.1.3, for each k, we have $J_k = (0)$ or $J_k = A_k$. If $J \neq (0)$, then there exists k such that $J_k = A_k$. So, J consists of blocks where each block is (0) or is A_k. If we look at the quotient algebra $B = A/J$, then B consists of blocks where each block is (0) (if $J_k = A_k$) or is A_k (if $J_k = (0)$). Hence, A/J consists of blocks of 1-matricial algebras and it is a σ-matricial algebra. $\square$

We know that the center of M_n is trivial. The same is true for 1-matricial algebras.

Proposition 4.1.14. *Let A be a* 1*-matricial algebra. The center of A is trivial.*

Proof. Let x be a nonzero element in $Z(A)$, the center of A, and let $\{q_k\}$ be the family of mutually orthogonal algebraically minimal projections as in the definition of matricial algebras. Since $q_i x q_j = x q_i q_j = 0$ for $i \neq j$, all the off-diagonal entries of x are zero. Moreover, since each q_k is algebraically minimal, $q_k x q_k = \lambda_k q_k$ gives the diagonal entries. Since A is not unital, x is not a multiple of the identity. Take the matrix $T \in A$ with $T = q_1 + T_{12}$ (T_{12} is as in the definition of matricial algebras). Then, $xT \in A$ is a matrix where the 1-2 entry is $\lambda_1 T_{12}$ and $Tx \in A$ is a matrix where the 1-2 entry is 0. That is, $\lambda_1 = 0$. Similarly, we can show that each $\lambda_k = 0$. Hence, $x = 0$ and $Z(A)$ is trivial. $\square$

We say that an operator algebra is *nc-discrete* if it satisfies the equivalent conditions in the following result.

Proposition 4.1.15. *For an approximately unital operator algebra A, the following are equivalent.*

(i) Every open projection e in A^{**} is also closed (in the sense that $1 - e$ is open).

(ii) The open projections in A^{**} are exactly the projections in $M(A)$.

(iii) Every r-ideal (resp. ℓ-ideal) J of A is of the form eA (resp. Ae) for a projection $e \in M(A)$.

(iv) The left (resp. right) annihilator of every nontrivial r-ideal (ℓ-ideal) of A is a nontrivial ℓ-ideal (resp. r-ideal).

(v) Every HSA of A is of the form eAe for a projection $e \in M(A)$.

If any of these hold, then $\Delta(A)$ is an annihilator C^*-algebra.

A nice property of nc-discrete operator algebras is that they are somewhat similar to Kaplansky's dual algebras; the 'left (resp.) right annihilator' operation is a lattice anti-isomorphism between the lattices of one-sided M-ideals of A [3, Corollary 2.11].

An approximately unital operator algebra A which is an ℓ-ideal in its bidual is nc-discrete. Moreover, every projection in the second dual of such an operator algebra is open and is in $M(A)$ [3, Proposition 2.12].

We say that an operator algebra is Δ-*dual* if $\Delta(A)$ is an annihilator C^*-algebra and $\Delta(A)$ acts nondegenerately on A.

In regards to the relation between being Δ-dual and nc-discrete, we want to mention that being Δ-dual is far from implying nc-discrete. For example, the disk algebra $A(\mathbf{D})$ is Δ-dual but not nc-dicrete. However, under some conditions, being nc-discrete implies being Δ-dual.

Corollary 4.1.16. *Let A be an approximately unital operator algebra which is nc-discrete. The following are equivalent.*

(i) A is Δ-dual.

(ii) $\Delta(A)$ acts nondegenerately on A.

(iii) Every nonzero projection in $M(A)$ dominates a nonzero positive element in A.

(iv) If p is a nonzero projection in $M(A)$, then there exists a with pap selfadjoint.

(v) $1_{A^{**}} \in \Delta(A)^{\perp\perp}$.

It is important to note that σ-matricial algebras are Δ-dual [3, Proposition 4.14]. That is, σ-matricial algebras are nc-discrete and Δ-dual; however, Example 4.1.20 shows that the converse is not true.

We now consider a class of algebras which are a commutative variant of matricial operator algebras, and are ideals in their biduals.

Proposition 4.1.17. *Let A be a commutative operator algebra with no nonzero annihilators in A, and possessing a sequence of nonzero algebraically minimal idempotents (q_k) with $q_j q_k = 0$, for $j \neq k$, and $\overline{\sum_k A q_k} = A$. Then, A is a semisimple annihilator algebra with dense socle, and A is an ideal in its bidual. If further the q_k are projections (resp. $\sum_k q_k = 1$ strictly), and if A is left essential, then $A \cong c_0$ isometrically (resp. $A \cong c_0$ isomorphically).*

In case of uniform algebras (which are commutative), we prove that being nc-discrete implies being isomorphic to c_0.

Proposition 4.1.18. *Let A be an approximately unital function algebra. If A is nc-discrete, then A is isomorphic to $c_0(I)$, for some set I.*

Proof. We will prove the unital case; the approximately unital case is in [2]. Suppose that A is a unital function algebra. If A is nc-discrete, then $\Delta(A)$ is an annihilator C^*-algebra. Hence, there exist minimal idempotents (q_k) in $\Delta(A)$ such that $\overline{\sum_k q_k \Delta(A)} = \Delta(A)$. Since $\Delta(A)$ acts nondegenerately on A, we get $\overline{\sum_k q_k A} = A$.

We will prove that each q_j is algebraically minimal in A. Notice that Aq_j is a unital function algebra with the unit q_j. If J is an r-ideal in Aq_j with support projection $p \neq 0$, then J is an r-ideal in A and $p \in M(A) = A$. Since $J = Jq_j$, we have $p \in Aq_j$ and consequently $p \leq q_j$. But since $p \in \Delta(A)$, $p \leq q_j$ implies that $p = q_j$. That is, $J = Aq_j$. Hence, Aq_j is a unital function algebra that does not have any nontrivial r-ideals. Note that every nontrivial uniform algebra contains proper closed ideals with cai (for example, the ideals associated with Choquet boundary points). Therefore, Aq_j is one dimensional; $Aq_j = \mathbb{C}\, q_j$. We conclude that every q_j is algebraically minimal in A.

Now, $A = \overline{\sum \mathbb{C}\, q_k} = \Delta(A)$. Hence, A is an annihilator C^*-algebra. Since A is commutative as well, A is isomorphic to $c_0(I)$, for some set I. □

Remark 4.1.19. If A is a function algebra which is nc-discrete, then $\Delta(A^{**}) = \Delta(A)^{**}$. Hence, every projection in A^{**} is in A and every projection is both open and closed. Every projection is both open and closed in all unital finite dimensional algebras and also in all 1-matricial algebras. However, this property alone does not characterize the 1-matricial algebras.

We present an example of a commutative operator algebra which is Δ-dual and nc-discrete, but not σ-matricial.

Example 4.1.20. Let $e_k = [1:1] \otimes [-k : k+1] = \begin{bmatrix} -k & k+1 \\ -k & k+1 \end{bmatrix}$ and $f_k = [(k+1)/k : 1] \otimes [k : -k] = \begin{bmatrix} k+1 & -(k+1) \\ k & -k \end{bmatrix}$, which are idempotents in M_2. Consider the idempotents

$$q_{2k} = 0 \oplus \ldots \oplus 0 \oplus e_k \oplus 0 \oplus \ldots$$

and

$$q_{2k+1} = 0 \oplus \ldots \oplus 0 \oplus f_k \oplus 0 \oplus \ldots,$$

inside $B = M_2 \oplus^\infty M_2 \oplus^\infty \ldots$.

Let A be the closure of the span of these idempotents (q_k). Then A has a cai and it may be viewed as a subspace of $\mathbb{K}(\ell^2)$.

Notice that A is a commutative algebra since the idempotents q_k commute; $q_j q_k = q_k q_j = 0$, for each $j \neq k$. By Proposition 4.1.17, A is an ideal in its bidual. Hence, by [3, Proposition 2.12], A is nc-discrete. That is, $\Delta(A)$ is an annihilator C^*-algebra. Since $\Delta(A)$ acts nondegenerately on A , A is also Δ-dual. However, A is not isomorphic to a σ-matricial algebra. Notice that the algebraically minimal idempotents in A are not uniformly bounded, which is the case in c_0. Hence, A can't be isomorphic to c_0. Here, we have $\overline{\sum_k q_k A} = \overline{\sum_k A q_k} = A$. However, the idempotents do not satisfy the condition $\sum_k q_k = 1$ strictly, which characterizes algebras that are isomorphic or similar to a σ-matricial algebra by Theorem 4.1.5.

This example illustrates that a semisimple approximately unital algebra which is dual in the sense of Kaplansky [33] need not be isomorphic to a σ-matricial algebra.

To see that A is dual in the sense of Kaplansky, we will look at the closed ideals of A since A is commutative. Let J be a closed ideal in A. We will prove that $R(L(J)) = J$. The inclusion $J \subset R(L(J))$ is easy since $x \in J$ and $b \in L(J)$ implies that $bx = 0$, hence $L(J)x = 0$. To prove the other inclusion by contradiction, assume that there is $q_k \in R(L(J))$ such that $q_k \notin J$. That is, $L(J)q_k = 0$, but $q_k \notin J$. The latter implies that $q_k J = 0$, which in turn implies that $q_k \in L(J)$. This is a contradiction to $L(J)q_k = 0$ since $L(J)$ is itself a two sided ideal in A. This proves that $\{k : q_k \in J\} = \{k : q_k \in R(L(J))\}$. Remember that A is defined to be the closure of the span of the (q_k). So every closed ideal is the closure of the span of the q_k that are contained in the ideal. That is, $J = \overline{\text{span}\{q_k : q_k \in J\}}$. Hence, for every closed ideal J in A, $R(L(J)) = J$. That is, A is a dual algebra. Therefore, we can conclude that being semisimple and dual (in the sense of Kaplansky) does not imply being σ-matricial or being isomorphic to a σ-matricial algebra.

Another point about this example is that it illustrates the necessity of the minimal projection condition in [3, Theorem 4.19]. The socle of A is dense, but A is not isomorphic to a σ-matricial algebra. To see that the minimal projection condition is not satisfied, look at $q_k A q_k = A q_k$ which contains a nontrivial idempotent.

Finally, we want to make an attempt to generalize the notion of having real rank zero to operator algebras. Since having real rank zero is defined in terms of the selfadjoint elements (which are already in $\Delta(A)$), we want to work with the case that $\Delta(A)$ is large enough. Recall from Chapter 2 that a C^*-algebra has real rank zero if and only if it has property (HP) (that is, every HSA in A has a cai of projections), if and only if it has property (FS) (that is, the elements with finite spectrum are dense in A_{sa}).

Definition 4.1.21. Let A be an operator algebra with a positive cai; or equivalently, assume that $\Delta(A)$ acts nondegenerately on A. We say that A *has real rank zero* if the invertible selfadjoint elements are dense in A^1_{sa}.

This definition is equivalent to $\Delta(A)$ being a C^*-algebra with real rank zero since A has positive cai and $A_{sa} = \Delta(A)_{sa}$.

We say that an operator algebra has property (HP') if every HSA in A that contains a positive cai contains a cai consisting of projections.

Proposition 4.1.22. *Let A be an operator algebra with a positive cai. The following are equivalent.*

(i) A has real rank zero.

(ii) $\Delta(A)$ has real rank zero.

(iii) A has property (FS).

(iv) A has property (HP').

Proof. Clearly $(i) \Leftrightarrow (i) \Leftrightarrow (iii)$. If A has real rank zero and D is a HSA in A with positive cai, then $\Delta(D)$ is a nonzero HSA in $\Delta(A)$ by Proposition 3.3.3, and hence contains a cai of projections. This cai of projections is also a cai for D since $\Delta(D)$ acts nondegenerately on D. Conversely, assume that A has property (HP'). If D is a HSA in $\Delta(A)$, then it has a positive cai. Let $p \in \Delta(A)^{**}$ be the support projection of D; $D = p\Delta(A)^{**}p \cap \Delta(A)$. Let $D' = pA^{**}p \cap A$. Then, D' is a HSA in A with support projection $p \in A^{**}$ and it contains a cai of projections by assumption. This cai of projections is contained in D and it is a cai for D as well. Hence, $\Delta(A)$ has real rank zero. $\square$

Example 4.1.23. If A is a σ-matricial algebra, then it has real rank zero. Clearly, every σ-matricial algebra contains a cai of projections, and every HSA in A (being a σ-matricial algebra itself) contains a cai of projections.

We say that the operator algebra A has property (IP) if for any mutually orthogonal projections p, q in A^{**}, where p is compact and q is closed, there exists a projection $r \in A$ such that $p \leq r \leq 1 - q$. A C^*-algebra has real rank zero if and only if it has property (IP) [15, Theorem 1].

Proposition 4.1.24. *Let A be an algebra with positive cai such that $\Delta(A)^{**} = \Delta(A^{**})$. Then, A has real rank zero if and only if A has property (IP).*

Proof. Since $\Delta(A)^{**} = \Delta(A^{**})$, $p, q \in A^{**}$ if and only if $p, q \in \Delta(A^{**}) = \Delta(A)^{**}$. Since $\Delta(A)$ is a C^*-algebra, it has real rank zero if and only if it has (IP) by [15, Theorem1]. $\square$

Notice that if A is σ-matricial, then $\Delta(A)^{**} = \Delta(A^{**})$ (for example by Lemma 4.3.2 that will be presented later in this chapter). That is, σ-matricial algebras have this interpolation property (IP).

Remark 4.1.25. In our attempt of generalizing the notion of having real rank zero, the notion becomes equivalent to $\Delta(A)$ having real rank zero and we may not say much about the operator algebra structure of A. If A is an operator algebra with a cai, let

$\mathcal{F}_A = \{a \in A : \|1 - a\| \leq 1\}$. In [9], Blecher and Read investigated the properties of $\mathcal{F}_A$ and showed that $\mathbb{R}^+ \mathcal{F}_A$ is an analogue of the positive cone in a C^*-algebra. That is, $\mathbb{R}\,\mathcal{F}_A$ may be a good candidate to generalize A_{sa}; the selfadjoint elements in a C^*-algebra. We conjecture that, in order to say more about the operator algebra structure, the following may be a better definition: 'We say that A has real rank zero if the invertible elements in $\mathbb{R}\,\mathcal{F}_A$ are dense in $\mathbb{R}\,\mathcal{F}_A$'. Of course, to study the consequences of this definition, one needs to study the properties of $\mathbb{R}\,\mathcal{F}_A$ first.

4.2 A Wedderburn Type Structure Theorem

In this section, we give a 'Wedderburn type' structure theorem for σ-matricial algebras.

As mentioned in the previous section, σ-matricial algebras are nc-dicrete and Δ-dual, but these two terms alone do not characterize σ-matricial algebras. Also, every projection in a σ-matricial algebra is both open and closed; again, this alone does not characterize the σ-matricial algebras. The following theorem lists the characterizations of σ-matricial algebras in terms of the existence of algebraically minimal projections, existence of minimal right ideals, being Δ-dual and being nc-discrete. This theorem is basically [3, Theorem 4.18], we add some new equivalent conditions. An operator algebra is said to have *Property (M)* if every nonzero projection in A dominates a nonzero algebraically minimal projection in A. We will study this property in more details in the next section.

Theorem 4.2.1. *Let A be an approximately unital semiprime operator algebra. The* *following are equivalent.*

(i) A is completely isometrically isomorphic to a σ-matricial algebra.

(ii) A is the closure of $\sum_k q_k A$ for mutually orthogonal algebraically minimal projections $q_k \in A$.

(iii) A is the closure of the joint span of the minimal right ideals which are also r-ideals (these are the qA, for algebraically minimal projections $q \in A$).

(iv) A is Δ-dual, and every $*$-minimal projection in A is algebraically minimal.

(v) A is Δ-dual, and A has Property (M).

(vi) A is nc-discrete, and every nonzero projection in $M(A)$ dominates a nonzero algebraically minimal projection in A.

(vii) A is nc-discrete, and every nonzero HSA D in A containing no nonzero projections of A except possibly an identity for D is one-dimensional.

(viii) A is a HSA in its bidual and every HSA D in A with dim$(D) > 1$ contains a nonzero projection which is not an identity for D.

(ix) Every projection in A^{**} is both open and closed, A has a positive cai and has Property (M).

(x) A is a HSA in its bidual and A^{**} has Property (M).

Proof. The equivalences of $(i)-(vii)$ are given in [3, Theorem 4.18]. The equivalence of those with $(viii)$ is given in [2].If (i) holds, then A every projection in A^{**} is both open and closed by Lemma 4.3.2. By the equivalence of (i) and (v), we know that A has a positive cai and it has Property (M). That is, (i) implies (ix). If every projection in A^{**} is open and closed, then A is nc-discrete and $\Delta(A)$ is an annihilator C^*-algebra by Proposition 4.1.15. That is, (ix) implies (v). The equivalence of (i) and (x) follows from Proposition 4.3.15 and Theorem 4.3.17 (which will be presented in the next section) and the fact that (i) implies that A is a HSA in its bidual. $\square$

One may ask whether we need A to be semiprime in the hypothesis. In fact, we need this condition. For example, if A is the algebra of upper triangular 2×2 matrices, then it is not semiprime. Notice that A is nc-discrete and satisfies (vi) in Theorem 4.2.1. It is also Δ-dual and satisfies some other conditions listed in Theorem 4.2.1. However, A is not a σ-matricial algebra since any finite dimensional σ-matricial algebra is of the form M_n (or c_0-sum of some copies of M_n).

We want to mention that in [3, Theorem 4.19], we presented another 'Wedderburn type' structure theorem. That theorem shows that, under certain conditions, being σ-matricial is equivalent to being compact or being a modular annihilator algebra.

4.3 The Second Duals of Matricial Algebras

We want to understand the second duals of matricial algebras. We start with second duals of 1-matricial algebras and we will present some facts about the second duals of σ-matricial algebras as well.

In the case of 1-matricial algebras, the second duals have a quite simple form.

Lemma 4.3.1. *If A is a 1-matricial algebra defined by a system of matrix units $\{T_{ij}\}$ in $B(K^{(\infty)})$ (as in Definition 4.1.1), then*

$$A^{**} = \{T \in B(K^{\infty}) : q_i T q_j \in \mathbb{C}\, T_{ij}, \forall i, j\}.$$

Thus, A^{**} is the collection of infinite matrices $[\beta_{ij} T_i^{-1} T_j]$, for scalars β_{ij}, which are bounded operators on $K^{(\infty)}$.

Every projection in the second dual of a σ-matricial algebra is open and closed.

Lemma 4.3.2. *Let A be a σ-matricial algebra. If p is a projection in A^{**}, then $p \in M(A)$ and $p \in M(\Delta(A))$, and thus is open. Hence, A is nc-discrete. Also,*

$$\Delta(A^{**}) = \Delta(A)^{**} = M(\Delta(A)) = \Delta(M(A)).$$

This result states that if A is σ-matricial, then $\Delta(A^{**}) = \Delta(A)^{**}$. Note that this also follows from the fact that σ-matricial algebras have positive cai and are HSAs in their biduals [2].

It is well known that $\mathbb{K}(H)$ is an ideal in its bidual $B(H)$. In fact, it is the unique closed two-sided ideal of $B(H)$ if H is separable, and it is contained in every closed two-sided ideal of $B(H)$ if H is not separable. We have a similar result for 1-matricial algebras; any closed two-sided ideal of the bidual contains the 1-matricial algebra.

Proposition 4.3.3. *Let A be a 1-matricial algebra and J be a nonzero closed ideal of A^{**}. Then, $A \subset J$.*

Proof. Let J be a nonzero closed ideal in A^{**}. If $0 \neq T \in J$, then $q_j T q_k = \beta_{jk} T_{jk} \neq 0$ for some j, k, since T is not zero. Hence, $\beta_{jk} T_{jk} \in J$ since J is an ideal. Thus, $T_{pq} \in J$ for all p, q, since these are matrix units. That is, $A \subset J$. □

Remark 4.3.4. Following the previous result, we want to remark on the relation between $\Delta(A)$ and $\Delta(J)$. Since A is 1-matricial, $\Delta(A)$ is an annihilator C^*-algebra. If $\Delta(J) \neq (0)$, then since J is a closed ideal in A^{**}, by Proposition 3.3.3, $\Delta(J)$ is a nonzero closed ideal in $\Delta(A^{**}) = \Delta(A)^{**}$. If M is an ideal in $\Delta(J)$, then M is an ideal in $\Delta(A)^{**}$. Since $\Delta(A)$ is a c_0-sum of elementary C^*-algebras, M consists of blocks where each block is either a block of $\Delta(A)$ or it is a bidual of a block of $\Delta(A)$.

We want to prove that the second dual of a σ-matricial algebra is semisimple and DMA. First, we present a lemma that proves that the ∞-sum of semisimple Banach algebras is semisimple.

Lemma 4.3.5. *If $\{A_k\}$ is a family of semisimple Banach algebras, then $\oplus_k^\infty A_k$ is semisimple.*

Proof. Let $B = \oplus_k^\infty A_k$ and $x = (x_k) \in \mathrm{Rad}(B)$. First assume that B is unital; as a consequence, each A_k is unital. By [41, Theorem 4.3.6], a characterization of the Jacobson radical states that $\mathrm{Rad}(B) = \{a \in B : Ba \subset qi(B)\}$. That is, we know that $yx \in qi(B)$, for each $y \in B$. Let $y_k \in A_k$; and set $y = (0, 0, \cdots, y_k, 0, \cdots) \in B$. Since $yx \in qi(B)$, there exists a $z \in B$ such that $yx + z = yxz$. That is, $y_k x_k + z_k = y_k x_k z_k$. Hence, $y_k x_k$ is quasi-invertible in A_k. Thus, for each k, $x_k \in \mathrm{Rad}(A_k) = (0)$. That is, $x = 0$.

If B is not unital, then let B^1 be the unitization of B and let $x = (x_k) \in \mathrm{Rad}(B) = \{a \in B : B^1 a \subset qi(B)\}$. Let $y_k + \mu_k e_k \in A_k^1$ and set $y = (0, 0, \cdots, y_k, 0, \cdots) \in B$. Since $(y + \mu_k e_k)x \in qi(B)$, we have $(y_k + \mu_k e_k)x_k \in qi(A_k)$ and hence $A_k^1 x_k \subset qi(A_k)$. That is, for each k, $x_k \in \mathrm{Rad}(A_k) = (0)$. Hence, $x = 0$ and $\mathrm{Rad}(B) = (0)$. $\square$

Note that this lemma gives us an immediate corollary about the c_0-sum of a family of Banach algebras.

Corollary 4.3.6. *If $\{A_k\}$ is a family of semisimple Banach algebras, then $\oplus_k^0 A_k$ is semisimple.*

Proof. Since $\oplus_k^0 A_k$ is an ideal in $\oplus_k^\infty A_k$ and since semisimplicity is a property that descends to ideals, the result follows from Lemma 4.3.5. $\square$

Proposition 4.3.7. *Let A be a σ-matricial algebra. Then, A^{**} is semisimple.*

Proof. First, suppose that A is a 1-matricial algebra. We know that $\mathrm{Rad}(A^{**})$ is a closed ideal in A^{**}. Assume that $\mathrm{Rad}(A^{**}) \neq (0)$. Then, by Proposition 4.3.3, $A \subset \mathrm{Rad}(A^{**})$. We know by [22, Proposition 2.6.25] that $\mathrm{Rad}(A^{**}) \cap A \subset \mathrm{Rad}(A)$. Note that A is semisimple; that is, $\mathrm{Rad}(A) = (0)$ and as a consequence, $\mathrm{Rad}(A^{**}) \cap A = (0)$. This contradicts to the fact that $A \subset \mathrm{Rad}(A^{**})$. Hence, $\mathrm{Rad}(A^{**}) = (0)$ and A^{**} is semisimple.

If A is σ-matricial, then it is the c_0-sum of 1-matricial algebras and hence A^{**} is the ∞-sum of the second duals of those 1-matricial algebras. By the first part of our proof, each of those second duals is semisimple. Now, by Lemma 4.3.5, A^{**} is semisimple. $\square$

Lemma 4.3.8. *Let $\{A_k\}$ be a family of semisimple DMA algebras. Then, $\oplus_k^\infty A_k$ is DMA.*

Proof. Let $B = \oplus_k^\infty A_k$ and $x \in LA(B_F)$. Let e_k be an algebraically minimal idempotent in A_k. Then, $(0, 0, \ldots, e_k, 0, \ldots)$ is an algebraically minimal idempotent in B

and hence $x(0, 0, \ldots, e_k, 0, \ldots) = (0)$. That is, $x_k e_k = 0$. Since this is true for any algebraically minimal idempotent in A_k, $x_k \in LA((A_k)_F) = (0)$. That is, $x_k = 0$ for each k and hence $x = (0)$. Similarly, $RA(A_F) = (0)$. Since B is semisimple by Lemma 4.3.5, we conclude that B is DMA. $\square$

Proposition 4.3.9. *Let A be a σ-matricial algebra. Then, A^{**} is DMA.*

Proof. First, assume that A is 1-matricial. Let $B = A^{**} = \{T \in B(K^\infty) : q_i T q_j \in \mathbb{C}\, T_{ij}, \forall i, j\}$ and B_F be the socle of B. We know by Proposition 4.3.7 that B is semisimple. If $T \in LA(B_F)$, then for every minimal projection p in B, $Tp = 0$. In particular, since q_i is a minimal projection in B, $Tq_i = 0$, for all i. Hence, $q_i T q_j = 0$, for all i, j. Hence, $T = 0$. That is, $LA(B_F) = (0)$. Similarly, $RA(B_F) = (0)$. Hence, $B = A^{**}$ is DMA.

If A is σ-matricial, then A is a c_0-sum of 1-matricial algebras A_k say. Then, A^{**} is the infinity sum of the second duals of these 1-matricial algebras; $A^{**} = \oplus_k^\infty A_k^{**}$. By the first part of our proof, each A_k^{**} is DMA. Moreover, each A_k^{**} is semisimple by Lemma 4.3.7. Hence, A^{**} is DMA by Lemma 4.3.8. $\square$

Definition 4.3.10. We say that an operator algebra A has *Property (M)* if every nonzero projection in A dominates a nonzero algebraically minimal projection in A.

Remark 4.3.11. If A has Property (M), then every $*$-minimal projection in A is algebraically minimal. Indeed, if p is $*$-minimal, then it majorizes an algebraically minimal projection $q \in A$ and since p is $*$-minimal, $p = q$.

We know that σ-matricial algebras have Property (M). Also, they have the property that for every nonzero projection $p \in A$, there exists an algebraically minimal projection $q \in A$ such that $pq \neq 0$. We show that these two conditions are equivalent.

Proposition 4.3.12. *Let A be an approximately unital algebra. Then, A has Property (M) if and only if for every nonzero projection $p \in A$, there exists an algebraically minimal projection $q \in A$ such that $pq \neq 0$.*

Proof. One direction is clear. For the other implication, let p be a nonzero projection in A. Then, by our assumption, there exists an an algebraically minimal projection q such that $pq \neq 0$. This implies that for all $x \in A$, $(pqp)x(pqp) = pq(pxp)qp = p(\lambda q)p = \lambda pqp$ for some $\lambda \in \mathbb{C}$. Hence, $\frac{1}{t}pqp$ is an algebraically minimal projection for some $t > 0$. Since $\frac{1}{t}pqp \leq p$, the proof is complete. $\square$

It is important to note that Property (M) descends to HSAs.

Proposition 4.3.13. *Let A be an operator algebra that has Property (M). Then any HSA of A has Property (M).*

Proof. Let D be a HSA in A. If p is a nonzero projection in D, then it is a projection in A, and dominates a nonzero algebraically minimal projection q in A. Then, $p \geq q$ implies that $pqp = qp = q \in D$ since D is a HSA. Also, q is algebraically minimal in D by Remark 4.1.11; so, p majorizes an algebraically minimal projection in D. Hence, D has Property (M). $\square$

Remark 4.3.14. We want to point out that Property (M) does not descend to the quotients A/J. To see this, let $A = C_0((0,3)) \oplus \mathbb{C}$ and let $J = \{f \in A : f|_{[1,2]} = 0\}$. Then A has Property (M) since the only projection in A, $(0,1)$, is algebraically minimal. However, $A/J \cong C([1,2]) \oplus \mathbb{C}$ does not have this property; $\chi_{[1,2]}$ is a projection that does not dominate any algebraically minimal projections.

Now, we give some results about σ-matricial algebras and Property (M).

Proposition 4.3.15. *Let A be a σ-matricial algebra. Then, A^{**} has Property (M).*

Proof. Let p be a projection in A^{**}. By Theorem 4.2.1, p is open and closed and also $p \in M(A)$ since A is nc-discrete. Hence, by Theorem 4.2.1, there exists an algebraically minimal projection q in A such that $q \leq p$. Since A is a HSA in A^{**}, q is algebraically minimal in A^{**} as well by Remark 4.1.11. □

Remark 4.3.16. If A is an approximately unital operator algebra, then the algebraically minimal projections in $M(A)$ are exactly the algebraically minimal projections in A. Indeed, if p is algebraically minimal in A, then it is algebraically minimal in $M(A)$ by Remark 4.1.11. If p is a nonzero algebraically minimal projection in $M(A)$, then $pAp \neq (0)$ by weak*-density. So, $pAp = \mathbb{C}\,p \subset A$ since A is an ideal in $M(A)$. Hence, $p \in A$. If further A is an ideal in its bidual, then we can replace $M(A)$ above by A^{**}.

Theorem 4.3.17. *Let A be a semiprime approximately unital operator algebra that is a HSA in its bidual. If A^{**} (or $M(A)$) has Property (M), then A is σ-matricial.*

Proof. If A is a HSA in its bidual, then it is nc-discrete [2]. If $M(A)$ has Property (M) and if p is a nonzero projection in $M(A)$, then it dominates a nonzero algebraically minimal projection $q \in M(A)$. By Remark 4.3.16, q is an algebraically minimal projection in A. Hence, by Theorem 4.2.1 (vi), A is σ-matricial.

If A^{**} has Property (M) and if p is a nonzero projection in $M(A)$, then it dominates a nonzero algebraically minimal projection $q \in A^{**}$. Since A is nc-discrete and q is open, $q \in M(A)$. Hence, $M(A)$ has Property (M) and the proof follows from the first part. $\square$

Hence, we have the another characterization of σ-matricial algebras, which was given as the last item in Theorem 4.2.1; A is a HSA in its bidual and A^{**} has Property (M).

4.4 Characterizations of the C^*-algebras of Compact Operators

We gave several characterizations of the C^*-algebras of compact operators in Chapter 2. Now, we want to give a characterization in terms of all closed left ideals being A-complemented.

An interesting question is whether every approximately unital operator algebra with the property that all closed right ideals have a left cai (similarly for left ideals), is a C^*-algebra. The following result is [3, Theorem 5.1] and it gives a partial answer to this question.

Theorem 4.4.1. *Let A be a semisimple approximately unital operator algebra. The following are equivalent:*

(i) Every minimal right ideal of A has a left cai (or equivalently equals pA for a

projection $p \in A$).

(ii) *Every algebraically minimal idempotent in A has range projection in A.*

If any of these hold, and if A has dense socle, then A is completely isometrically isomorphic to an annihilator C^*-algebra.

Definition 4.4.2. We say that a left ideal in A is *A-complemented* if it is the range of a bounded idempotent left A-module map. We say that the left ideals in A are uniformly A-complemented if there exists $K > 0$ such that $\|p\| \leq K$ for every such idempotent p.

The following result is from [3].

Corollary 4.4.3. *Let A be a semisimple approximately unital operator algebra such that every closed left ideal in A is contractively A-complemented; or equivalently, equals $J = Ap$ for a projection $p \in M(A)$. Then, A is completely isometrically isomorphic to an annihilator C^*-algebra.*

In [51] and [52], Tomiuk studied the structure of complemented Banach algebras, which are defined below.

Definition 4.4.4. Let A be a Banach algebra and M_ℓ be the set of all closed left ideals of A. We say that A is *left complemented* if there is a map (called a *complementor*) $p : J \to J^p$ of M_ℓ into itself such that:

(i) $J \cap J^p = (0)$;

(ii) $J + J^p = A$, for all $J \in M_\ell$;

(iii) $(J^p)^p = J$, for all $J \in M_\ell$;

(iv) $J_1 \subset J_2$ implies $J_2^p \subset J_1^p$, for all $J_1, J_2 \in M_\ell$.

Of course, the term *right complemented* is defined symmetrically, in terms of right ideals. A semisimple left (or right) complemented Banach algebra has dense socle [51, Lemma 5] and is an annihilator algebra if it contains a bai [52, Corollary 4.3]. Following Tomiuk's results about complemented Banach algebras, we get the following corollary.

Corollary 4.4.5. *Let A be a semisimple operator algebra with a bai. If A is left complemented, then A is topologically isomorphic to a sum of $\mathbb{K}(H_i)$, for Hilbert spaces H_i.*

Proof. Since A is semisimple and left complemented, A has dense socle and A is the direct sum of its minimal closed ideals each of which is simple [51, Theorem 4]; $A = \oplus_i A_i$. Since A is approximately unital, $L_A = A$ and by [52, Corollary 4.2] each minimal ideal A_i in $A = L_A$ is isomorphic to $\mathbb{K}(H_i)$ for some Hilbert space H_i. Hence, A is the direct sum of $\mathbb{K}(H_i)$. $\square$

Now, we prove that some left complemented operator algebras are annihilator C^*-algebras.

Theorem 4.4.6. *Let A be a semisimple operator algebra with a bai. If A is left complemented and if the closed left ideals of A are uniformly A-complemented, then A is completely isomorphic to an annihilator C^*-algebra.*

Proof. By the last corollary, and the fact that A has a bai, so that $A = L_A$, we have that A is the closure of $\sum_i S_i$, where $S_i \cong K(H_i)$ for some Hilbert space H_i. This isomorphism is bicontinuous, and in fact completely bicontinuous, since any operator algebra which is bicontinuously isomorphic to $\mathbb{K}(H)$ is completely isomorphic to it. Also, $S_i S_j = S_i \cap S_j = (0)$. Since $K(H_i)$ has a bai, so does S_i. If (r_t) is a bai for S_i with weak* limit point $r_i \in A^{**}$, then r_i is an identity for $S_i^{\perp\perp}$. Note that $r_i r_j = 0$ if $i \neq j$. And $S_i^{\perp\perp} = A^{**} r_i = r_i A^{**}$, and r_i is in the center of A^{**}. By [3, Lemma 5.2(2)], we also have $S_i = Ae$ for an idempotent $e \in A^{**}$ and $S_i^{\perp\perp} = A^{**}e$, so $e = er_i = r_i e = r_i$. Thus, $r_i \in M(A)$, and $S_i = Ar_i$.

Since the ideals are uniformly complemented, we know that $\|r_i\| \leq K$ for some $K > 0$. Then, for any finite $J \subset I$, we have $A(\sum_{i \in J} r_i)$ is complemented too, so that $\left\|\sum_{i \in J} r_i\right\| \leq K$. That is, all finite partial sums of $\sum_i r_i$ are uniformly bounded. We can use the similarity trick from [3, Section 4] to get projections. There exists an invertible operator S such that $S^{-1} r_i S = p_i$ is an orthogonal projection on H for all i. And $B = S^{-1} A S$ is the closure of the sum of the $B_i = S^{-1} S_i S$. Moreover, the canonical map $\oplus_i^f B_i \to B$ is an isometric homomorphism with respect to the ∞-norm on the direct sum, since if $b_i \in B_i$ for i in a finite set J, then

$$\| \sum_{i \in J} b_i \| = \| \sum_{i \in J} p_i b_i p_i \| = \max_{i \in J} \|b_i\|.$$

So, $B \cong \oplus_i^0 B_i$ completely isometrically isomorphically, hence $A \cong B \cong \oplus_i^0 B_i \cong \oplus_i^0 K(H_i)$ completely bicontinuously. Hence, A is completely isomorphic to an annihilator C^*-algebra. $\square$

Chapter 5

Scattered Operator Algebras

In this chapter, we define scattered operator algebras and study their structure. Jensen [30] and Lazar [37] showed that scattered C^*-algebras are built up by using a composition series where each consecutive quotient is isometric to $\mathbb{K}(H)$ for some Hilbert space H. In the light of this result, we want to define scattered operator algebras using a composition series where each consecutive quotient is completely isometrically isomorphic to a 1-matricial algebra. We investigate the structure of such operator algebras.

We start this chapter by proving some results we haven't seen in the literature about scattered C^*-algebras. In the second section, we define composition series for operator algebras. In section 3, we introduce scattered operator algebras. In section 4, we study operator algebras with a scattered maximal C^*-cover.

5.1 Scattered C^*-algebras

Recall that a C^*-algebra A is *scattered* if every positive functional on A is a sum of a sequence of pure functionals on A [30]. Some characterizations of scattered C^*-algebras are given in Chapter 2; several others can be found in [30], [37], [35] or [36].

Remark 5.1.1. Every countable compact Hausdorff space is scattered [39, Corollary 1.5.9]. It is well-known that for a compact Hausdorff space K, K is scattered if and only if $C(K)$ is scattered; or, if and only if $C(K)^*$ is separable. Let K be a compact metric space which is scattered. Assume that K is uncountable. By functional analysis, if a compact space K is uncountable then $C(K)^*$ is nonseparable. But since K is scattered, $C(K)^*$ separable; we have a contradiction. Hence, if K is compact scattered, then it is countable. Hence, for a compact metric space K, $C(K)$ is scattered if and only if K is countable.

Remark 5.1.2. Let K be a locally compact Hausdorff space. The algebraically minimal projections in $C_0(K)$ correspond to isolated points in K. Notice that $\chi_{\{x\}}$ is continuous (that is, $\chi_{\{x\}} \in C_0(K)$) if and only if $x \in K$ is an isolated point. If $\chi_E \in C_0(K)$ is a minimal projection, and if $y \neq z \in E$, then by Urysohn's lemma for locally compact Hausdorff spaces, there exists $f \in C_0(K)$ such that $f(y) = 1$ and $f(z) = 0$. Since χ_E is algebraically minimal, $\chi_E f \chi_E = \lambda \chi_E$ for some $\lambda \in \mathbb{C}$. Then, $\chi_E(y)f(y)\chi_E(y) = 1 = 1\chi_E(y)$ and $\lambda = 1$. But, $\chi_E(z)f(z)\chi_E(z) = 0$ gives us a contradiction. Hence, E is a singleton. That is, any algebraically minimal projection in $C_0(K)$ is of the form $\chi_{\{x\}}$ for an isolated point $x \in K$. Thus, commutative scattered C^*-algebras contain algebraically minimal projections.

Recall that an algebra is a Duncan modular annihilator algebra (DMA) if it is semiprime and $LA(A_F) = RA(A_F) = A_J$ [41]. If the operator algebra is semisimple, then A is DMA if and only if $LA(A_F) = RA(A_F) = (0)$. We prove that scattered C^*-algebras are DMA.

Theorem 5.1.3. *If A is a scattered C^*-algebra, then A is DMA.*

Proof. Let A be a scattered C^* algebra. Let $I = LA(A_F)$; notice that this is a closed ideal in A. Suppose that $I \neq (0)$. If no element of I has spectrum different from (0), then by [41, Theorem 4.3.6(b)], $I \subset A_J = (0)$.

Suppose that there is a selfadjoint element $b \in I$ such that $Sp(b) \neq (0)$. Since $Sp(b)$ is countable and not equal to (0), it contains a nonzero isolated point. The C^*-algebra generated by the selfadjoint element b is isomorphic to $C_0(Sp(b) \setminus \{0\})$. Since $Sp(b)$ contains a nonzero isolated point w, we get a nonzero projection in $C^*(b)$ corresponding to $w \in Sp(b)$. Notice that $C^*(b) \subset I$. Hence, there exists a nonzero projection in I. Let E be a maximal family of commuting projections in I and let C be the closed $*$-algebra generated by E. Since E only contains commuting projections, C is commutative. Since I is selfadjoint, C is a $*$-subalgebra of I. We know that the ideals and C^*-subalgebras of scattered C^*-algebras are scattered [30]. Hence, C is a commutative scattered C^*-algebra. Since commutative scattered C^*-algebras contain minimal projections (by Remark 5.1.2), there exists a nonzero algebraically minimal projection $e \in C$. By following the argument of [41, 8.6.2], we get that the unital C^*-algebra eIe contains no nonzero projections except its identity e. Hence, the spectrum of each selfadjoint element in eIe is connected. Since eIe is a spectral subalgebra of A, each selfadjoint element in eIe has countable spectrum. That is, if $a \in eIe$ is selfadjoint, then $Sp(a)$ is a singleton. Hence, if $a \in eIe$ is a nonzero selfadjoint element, then $Sp(a) = \{\lambda\}$, where $\lambda \neq 0$. This means that every nonzero selfadjoint element a in eIe satisfies $a = \lambda e$ for $\lambda \in Sp(a)$. Since eIe is a C^*-algebra, any element $b \in eIe$ can be written as $b = x_1 + ix_2$ where x_j are selfadjoint elements in eIe. This implies that $b = \lambda_1 e + i\lambda_2 e = \mu e \in \mathbb{C}\, e$. That is, $eIe = \mathbb{C}\, e$. Moreover, for any $a \in A$, $eae = e(eae)e \in eIe$, since I is an ideal in A. That is, $eAe \subset eIe = \mathbb{C}\, e$. Hence, e is algebraically minimal in A.

Thus, e is a minimal projection which is contained in I. This is a contradiction since e also belongs to A_F. Hence, for every selfadjoint element $b \in I$, $Sp(b) = (0)$.

Notice that I is a two sided ideal in A; $b \in I$ implies that $bb^* \in I$. So, $b \in I$ implies that $Sp(bb^*) = (0)$ and hence $bb^* = 0$ by the spectral radius formula. By the C^*-identity, we get that $b = 0$. That is, $I = (0)$.

Hence, $LA(A_F) = (0)$. Similarly, $RA(A_F) = (0)$. We conclude that A is DMA.

□

For commutative C^*-algebras, we want to include the following result about the C^*-algebra being DMA.

Proposition 5.1.4. *Let K be a locally compact and Hausdorff space. Then, $C_0(K)$ is DMA if and only if the isolated points in K are dense in K.*

Proof. Let $A = C_0(K)$. Assume that the isolated points in K are dense in K and $I = L(A_F) \neq (0)$. Notice that I is an ideal in A and I does not contain any nonzero minimal projections. Moreover, $I = \{f \in C_0(K) : f|_{E^c} = 0\}$ where E is a nonempty open set in K. If w is an isolated point in E, then $\chi_{\{w\}} \in C_0(K)$ is 0 on E^c. So $\chi_{\{w\}}$ is a nonzero minimal projection contained in I. This is not possible. Hence, E is an open subset of K which does not contain any isolated points. So, if x is an element in E, then x has an open neighborhood $E \subset K$ where E does not contain any isolated points. This is a contradiction to the fact that the isolated points in K are dense in K. Thus, $I = (0)$.

Now, suppose that $A = C_0(K)$ is DMA. Then, $L(A_F) = (0)$. Let $J \subset K$ be the set of all isolated points in K. If $\overline{J} \neq K$, then let $x \in K \setminus \overline{J}$. Since $\{x\}$ and $\overline{J}$ are disjoint closed subsets of K and $\{x\}$ is compact, by Urysohn's lemma for locally compact Hausdorff spaces, we get a continuous function $f \in A$ such that $f(x) = 1$ and $f|_{\overline{J}} = 0$. That is, $f(x) = 1$ and $f(w) = 0$ for all isolated points $w \in K$. By Remark 5.1.2, every minimal projection in $C_0(K)$ corresponds to an isolated point $w \in K$. Hence, $fe = 0$ for every minimal projection in A. That is, $f \in L(A_F) = (0)$. This is a contradiction. Hence, $\overline{J} = K$. That is, the isolated points in K are dense in K. □

Corollary 5.1.5. *A commutative scattered C^*-algebra has Property (M).*

Proof. Let $A = C_0(K)$ be a commutative scattered C^*-algebra where K is a locally compact Hausdorff space. If $p \in A$ is a nonzero projection, then it corresponds to χ_E for a set $E \subset K$. Since K is scattered, E contains an isolated point. Indeed, if E is closed, then the isolated points in E are dense in E. If it is not closed, let $x \in E$; since K is regular, there exists a neighborhood V of x such that $\overline{V} \subset E$. Since $\overline{V}$

is closed, the isolated points in $\overline{V}$ are dense in it. Hence, there exists an isolated point $w \in E$. Then, $q = \chi_{\{w\}} \in A$ is a minimal projection and $q \leq p$. Hence, A has Property (M). $\square$

The existence and abundance of projections in a C^*-algebra is important. Scattered C^*-algebras have real rank zero; every HSA in a scattered C^*-algebra contains a cai of projections, and hence they are very rich in terms of projections. Moreover, whether the projections in the quotient algebra lift or not is also a problem that has been studied deeply (see for example [20], [14], [42], and [54]). That is, if J is an ideal in A and $p+J$ is a projection in A/J, we want to know if $p+J$ lifts to a projection in A. There are several C^*-algebras with affirmative answers; for example, projections in the Calkin algebra $B(H)/\mathbb{K}(H)$ lift to projections in $B(H)$. However, there are also many C^*-algebras that do not have this lifting property. If we take $A = C([0,3])$ and $J = \{f \in A : f|_{[0,1]\cup[2,3]} = 0\}$, then $\chi_{[0,1]}$ is a projection in $A/J \cong C([0,1]\cup[2,3])$ that does not lift to a projection in A; A does not contain any nontrivial projections. We note that scattered C^*-algebras have the projection lifting property.

Corollary 5.1.6. *Let A be a scattered C^*-algebra and J be an ideal in A. Then, the projections in A/J lift to projections in A.*

Proof. If J and A/J have real rank zero, then projections in A/J lift if and only if A has real rank zero [16, Theorem 3.14]. If A is scattered; then J and A/J are scattered by [30, Proposition 2.4]. That is, A, J and A/J have real rank zero (by [36, Theorem 2.3] for example). Hence, the projections in A/J lift to projections in A. $\square$

5.2 Composition Series of an Operator Algebra

Out of several equivalent definitions of scattered C^*-algebras, we want to focus on the definition that uses a composition series where each quotient is an elementary C^*-algebra. For this reason, we define the composition series of operator algebras and present some facts that will be used in the coming section.

Blackadar defines the composition series of a C^*-algebra [5, Section IV.1]; analogously, we define the composition series of an operator algebra.

Definition 5.2.1. Let A be an approximately unital operator algebra. We say that A has a *subcomposition series* if there exists a family of closed two sided approximately unital ideals $\{I_\alpha\}$, indexed by ordinals α such that $I_0 = (0)$, $I_\alpha \subset I_\beta$ if $\alpha < \beta$, and $I_\gamma = \overline{\cup_{\alpha<\gamma} I_\alpha}$ if γ is a limit ordinal. If there is a γ such that $I_\alpha = I_\gamma$ for all $\alpha > \gamma$, then I_γ is the limit of the series. If the limit of the series is A, then the series is called a *composition series for* A.

Following Pedersen [43, Section 6.2], we can define the composition series in a slightly different way.

Definition 5.2.2. Let A be an approximately unital operator algebra. We say that A has a *composition series* if there exists a strictly increasing family of closed two sided approximately unital ideals $\{I_\alpha\}$, indexed by a segment $\{0 \leq \alpha \leq \beta\}$ of the ordinals such that $I_0 = (0)$ and $I_\beta = A$, and for each limit ordinal γ, we have $I_\gamma = \overline{\cup_{\alpha<\gamma} I_\alpha}$.

Note that these two definitions are equivalent. If $\{I_\alpha\}$ is a composition series as in

Definition 5.2.2, then for each ordinal $\alpha > \beta$, let $I_\alpha = A$, and then it is a composition series in the sense of Definition 5.2.1. Conversely, if $\{I_\alpha\}$ is a composition series as in Definition 5.2.1, then let β be the smallest ordinal such that $I_\alpha = I_\beta$ for each $\alpha > \beta$. Consider the series to be indexed by the segment $\{0 \leq \alpha \leq \beta\}$. If $I_{\alpha_0} = I_{\alpha_0+1}$ for an ordinal $\alpha_0 < \beta$, then there exists an ordinal γ such that I_{α_0} is properly contained in I_γ. Indeed, if there does not exist such an ordinal, then $I_{\alpha_0} = I_\beta = A$ and hence $\alpha_0 = \beta$. Now, relabel the indexing so that $I_{\alpha_0+1} = I_\gamma$. After the relabeling, the series $\{I_\alpha\}_{0\leq\alpha\leq\beta}$ is strictly increasing and is a composition series in the sense of Definition 5.2.2. Hence, we can use either definition for the composition series of an operator algebra.

We remark that if A is a separable operator algebra, then the composition series must be countable [4, Section 1.5]. Indeed, if A is separable with a strictly increasing composition series $\{I_\alpha\}_{0\leq\alpha\leq\beta}$, then for each α, we can choose an element $x_\alpha \in I_{\alpha+1}$ such that $\|x_\alpha - z\| \geq 1$ for all $z \in I_\alpha$. This can be done since the series is strictly increasing. Hence, if $\alpha \neq \gamma$, then $\|x_\alpha - x_\gamma\| \geq 1$ since $\{I_\alpha\}$ is a well-ordered set of ideals in A. Since A is separable, $\{x_\alpha : 0 \leq \alpha < \beta\}$ must be a countable set. Hence, β is a countable ordinal.

Note that if A has a composition series $\{I_\alpha\}$, then the second dual of A can be expressed in terms of the second duals of the quotients of the consecutive ideals. This fact will be useful in the next section. First we present a lemma.

Lemma 5.2.3. *If M is a dual operator algebra and q_k are mutually orthogonal central projections in M such that $\sum_k q_k = 1_M$, then $M \cong \oplus_k^\infty q_k M$.*

Proof. Let N be the enveloping von Neumann Algebra for M. Assume that $1_N = 1_M \in M$. By von Neumann algebra theory, $N \cong \oplus_k^\infty q_k N$. Let $\theta : M \to \oplus_k^\infty q_k M$ be the restriction of this map. Here, $\theta(x) = (q_k x)_k \in \oplus_k^\infty q_k M$. Since $\sum_k q_k = 1_M$, the map θ is onto. Hence, it is a complete isometry and $M \cong \oplus_k^\infty q_k M$. ☐

Now, we prove that if A has a composition series, then A^{**} is the ∞-sum of the second duals of the consecutive quotients.

Theorem 5.2.4. *Let A be an approximately unital operator algebra with composition series $\{I_\alpha\}_{\{0\leq\alpha\leq\beta\}}$. Then, $A^{**} = \oplus_{\alpha<\beta}^\infty (I_{\alpha+1}/I_\alpha)^{**}$.*

Proof. First assume that the series is countable with $\beta = \omega$. Let p_k be the support projection of I_k and let $q_k = p_{k+1} - p_k$. Note that since the series is increasing, $p_k \leq p_{k+1}$ for each k. Then, for each k,

$$(I_{k+1}/I_k)^{**} \cong I_{k+1}^{**}/I_k^{**} \cong (p_{k+1}A^{**})/(p_k A^{**}) \cong (p_{k+1} - p_k)A^{**} = q_k A^{**}.$$

Since $A = \overline{\cup_{k=0}^\infty I_k}$, we know that $\sup_k p_k = 1_{A^{**}}$. Moreover, $\sum_{k=0}^n q_k = \sum_{k=0}^n (p_{k+1} - p_k) = p_n$ which converges weak* to $\sup_k p_k = 1_{A^{**}}$. That is, $\sum_{k=0}^\infty q_k = 1_{A^{**}}$. Now, by Lemma 5.2.3, $A^{**} = \oplus^\infty q_k A^{**} = \oplus^\infty (I_{k+1}/I_k)^{**}$.

Now, for the general case, assume that the series is indexed by ordinals $\{0 \leq \alpha \leq \beta\}$. For each ordinal α, let p_α be the support projection of I_α in A^{**} and let $q_\alpha = p_{\alpha+1} - p_\alpha$. Then, for each α,

$$(I_{\alpha+1}/I_\alpha)^{**} \cong (I_{\alpha+1})^{**}/(I_\alpha)^{**} \cong (p_{\alpha+1}A^{**})/(p_\alpha A^{**}) \cong (p_{\alpha+1} - p_\alpha)A^{**} = q_\alpha A^{**}.$$

Since the series is infinite, WLOG we can assume that β is a limit ordinal; for if not, then we can choose the smallest limit ordinal $\beta_0 > \beta$ and let $I_\alpha = I_\beta$ for each $\alpha > \beta$. Since $I_\beta = A$, we know that $p_\beta = 1_{A^{**}}$.

Note that ordinals are well-ordered sets, and hence are directed sets; so the projections above are increasing net of projections, and by the theory of the correspondence between open projections and ideals, the support projection of $I_\beta = \overline{\cup_{\alpha<\beta} I_\alpha}$ must be the sup of the support projections of I_α, for $\alpha < \beta$. That is, $p_\beta = \sup_{\alpha<\beta} p_\alpha$.

If r is a projection such that $r \geq q_\alpha$ for all $\alpha < \beta$, then we want to show that $r \geq p_\beta$; that is, $r \geq p_\alpha$ for all $\alpha < \beta$. We will prove this by transfinite induction. If $\gamma + 1$ is a successor ordinal, then $p_{\gamma+1} = p_\gamma + q_{\gamma+1} \leq r$. If γ is a limit ordinal and if $r \geq p_\alpha$ for all $\alpha < \gamma$, then $p_\gamma = \sup_{\alpha<\gamma} p_\alpha$ and $p_\gamma \leq r$ by the inductive hypothesis. Hence, by transfinite induction, we conclude that if $q_\alpha \leq r$ for all $\alpha < \beta$, then $p_\beta \leq r$.

Now, we claim that $\oplus^\infty_{\alpha<\beta}(q_\alpha A^{**}) = p_\beta A^{**}$. Indeed, $\oplus^\infty_{\alpha<\beta}(q_\alpha A^{**}) = eA^{**}$ where $e = \sup_{\alpha<\gamma} q_\alpha$. So, $e \leq p_\beta$, but by the previous paragraph, we see that $e \geq p_\beta$; that is, $e = p_\beta$. Thus, we have:

$$\oplus^\infty_{\alpha<\beta}(I_{\alpha+1}/I_\alpha)^{**} \cong \oplus_{\alpha<\beta}(q_\alpha A^{**}) = p_\beta A^{**} = A^{**}.$$

$\square$

If A is an operator algebra that has a composition series, we list some methods to construct composition series for certain related operator algebras.

Remark 5.2.5. Let $\{I_\alpha\}$ and $\{J_\alpha\}$ be countable composition series for approximately unital operator algebras A and B, respectively. Since A is the limit of the series $\{I_\alpha\}$, there exists an ordinal β such that $I_\beta = A$ and $I_\alpha = I_\beta$ for all $\alpha > \beta$. Similarly, there exists μ such that $J_\mu = B$ and $J_\alpha = J_\mu$ for all $\alpha > \mu$. WLOG, we can assume that $\beta = \mu$.

(i) If D is a HSA in A, then $D_\alpha = D \cap I_\alpha$ is a composition series for D. Since I_α is an ideal in A, each $D \cap I_\alpha$ is an ideal in D and is approximately unital by Lemma 3.2.5. If γ is a limit ordinal, then $D_\gamma = D \cap I_\gamma = D \cap \overline{\cup_{\alpha<\gamma} I_\alpha} = \overline{\cup_{\alpha<\gamma} D \cap I_\alpha} = \overline{\cup_{\alpha<\gamma} D_\alpha}$. Indeed, if $x \in D \cap \overline{\cup_{\alpha<\gamma} I_\alpha}$, then $x = \lim_t x_t$ where $x_t \in I_{\alpha_t}$. If (e_s) is a cai for D, then $e_s x_t e_s \in D \cap I_{\alpha_t}$ and $x = \lim_{s,t} e_s x_t e_s \in \overline{\cup_{\alpha<\gamma} D \cap I_\alpha}$.

(ii) If $\{M_\alpha\}_{0\le\alpha\le\beta_0}$ is a composition series for I_1, then

$$\{(0) = M_0, M_1, M_2, \cdots, M_{\beta_0} = I_1, I_2, I_3 \cdots, I_\beta = A\}$$

is a composition series for A. Notice that the length of the new series is different.

(iii) For each ordinal α, let $M_\alpha = I_\alpha \oplus^\infty J_\alpha$. Then, each M_α is an ideal in $A \oplus^\infty B$ by Proposition 3.1.8 and is approximately unital since (e_t, f_t) is a cai for M_α if (e_t) and (f_t) are cais for I_α and J_α, respectively. If γ is a limit ordinal, then $M_\gamma = \overline{\cup_{\alpha<\gamma} M_\alpha}$ since $\overline{\cup_{\alpha<\gamma} I_\alpha} \oplus^\infty \overline{\cup_{\alpha<\gamma} J_\alpha} = \overline{\cup_{\alpha<\gamma} I_\alpha \oplus^\infty J_\alpha}$.

(iv) If $C = \mathrm{C}^*_{\max}(A)$, then for each ordinal α, let $M_\alpha = \mathrm{C}^*_{\max}(I_\alpha)$. Notice that each M_α is an approximately unital ideal in $\mathrm{C}^*_{\max}(A)$. Since $I_\alpha \subset I_{\alpha+1}$ implies that $\mathrm{C}^*_{\max}(I_\alpha) \subset \mathrm{C}^*_{\max}(I_{\alpha+1})$, the series is increasing. If γ is a limit ordinal, then $M_\gamma = \overline{\cup_{\alpha<\gamma} M_\alpha}$ since $\overline{\cup_{\alpha<\gamma} \mathrm{C}^*_{\max}(I_\alpha)} = \mathrm{C}^*_{\max}(\overline{\cup_{\alpha<\gamma} I_\alpha})$. To see that $\overline{\cup_{\alpha<\gamma} \mathrm{C}^*_{\max}(I_\alpha)}$ is a maximal C^*-cover, let $\pi : \overline{\cup_{\alpha<\gamma} I_\alpha} \to B(H)$ be a completely contractive homomorphism. Then, for each α, the restriction π_α of π to I_α is a completely contractive homomorphism and hence, by the universal property of $\mathrm{C}^*_{\max}(I_\alpha)$, there exists a unique $*$-homomorphism $\tilde{\pi}_\alpha : \mathrm{C}^*_{\max}(I_\alpha) \to B(H)$. Define a map $\rho : \cup_{\alpha<\gamma}\mathrm{C}^*_{\max}(I_\alpha) \to B(H)$ by $\rho(x) = \tilde{\pi}_\alpha(x)$ where α is the smallest ordinal such that $x \in \mathrm{C}^*_{\max}(I_\alpha)$. Notice that for each $\mu > \alpha$, $\mathrm{C}^*_{\max}(I_\alpha) \subset \mathrm{C}^*_{\max}(I_\mu)$ and hence we have $\tilde{\pi}_\mu(x) = \tilde{\pi}_\alpha(x)$, by the uniqueness of the extensions. To see that this is a homomorphism, note that if $x \in \mathrm{C}^*_{\max}(I_\alpha)$ and $y \in \mathrm{C}^*_{\max}(I_\mu)$, for $\alpha < \mu$ say, then $xy \in \mathrm{C}^*_{\max}(I_\alpha)$ and $\rho(xy) = \tilde{\pi}_\alpha(xy)$, since $\alpha < \mu$. By the fact mentioned above, we have $\tilde{\pi}_\alpha(xy) = \tilde{\pi}_\mu(xy)$, so that $\rho(xy) = \tilde{\pi}_\mu(xy)$. Since $\tilde{\pi}_\mu$ is a homomorphism, $\tilde{\pi}_\mu(xy) = \tilde{\pi}_\mu(x)\tilde{\pi}_\mu(y) = \tilde{\pi}_\alpha(x)\tilde{\pi}_\mu(y) = \rho(x)\rho(y)$. That is, $\rho(xy) = \rho(x)\rho(y)$. It is easy to see that ρ is a *-homomorphism. Then, the extension $\tilde{\rho} : \overline{\cup_{\alpha<\gamma}\mathrm{C}^*_{\max}(I_\alpha)} \to B(H)$ is a *-homomorphism as well. Since $\tilde{\rho}$ extends π, $\overline{\cup_{\alpha<\gamma}\mathrm{C}^*_{\max}(I_\alpha)}$ has the universal property of the maximal C^*-cover.

5.3 Scattered Operator Algebras

In Chapter 4, we introduced a new class of operator algebras; namely, 1-matricial operator algebras, to generalize the elementary C^*-algebras to a non-selfadjoint setting. We want to define scattered operator algebras using a composition series where the building blocks are 1-matricial algebras.

Definition 5.3.1. Let A be an approximately unital operator algebra with a composition series $\{I_\alpha\}$. We say that A is a *scattered operator algebra* if for each α, $I_{\alpha+1}/I_\alpha$ is completely isometrically isomorphic to a 1-matricial algebra. If A is a scattered operator algebra with such a composition series $\{I_\alpha\}$, we say that $\{I_\alpha\}$ is a *scattered composition series* for A.

Since we are interested in the separable operator algebras, we often assume that the series is countable, but the general case is almost identical.

Before we investigate the properties of such operator algebras, we want to present some examples.

Example 5.3.2. The elementary C^*-algebra $\mathbb{K}(H)$ is a scattered operator algebra; for the composition series, take $I_0 = (0)$ and $I_1 = \mathbb{K}(H)$. Hence, any annihilator C^*-algebra is scattered since annihilator C^*-algebras are c_0-sums of elementary C^*-algebras.

Example 5.3.3. Every scattered C^*-algebra has a composition series of ideals such that each consecutive quotient is an elementary C^*-algebra [37]. Hence, every scattered C^*-algebra, considered as an operator algebra, is scattered.

Example 5.3.4. If H is an infinite dimensional Hilbert space, then $B(H)$ is not scattered. Indeed, $B(H)$ is not a type I C^*-algebra if H is infinite dimensional [5, IV.1.1.5], and any scattered C^*-algebra is a Type I C^*-algebra by [30, Theorem 2.3]. Moreover, the Calkin algebra $B(H)/\mathbb{K}(H)$ is not scattered. One way to see this is to observe that the Calkin algebra is antiliminal and hence not of Type I [5, IV.1.1.6]. Or, assume that the Calkin algebra is scattered, then $B(H)$ is scattered by [30, Proposition 2.4], which is a contradiction.

Example 5.3.5. Let A be a 1-matricial algebra. Since A is topologically simple, it does not contain nontrivial closed two-sided ideals. Let $I_0 = (0)$ and $I_1 = A$; we get a composition series for A. That is, A is a scattered operator algebra. Moreover, every σ-matricial algebra is scattered as a σ-matricial algebra is the c_0-sum of 1-matricial algebras. If $A = \oplus_k^0 A_k$, then we can get a countable composition series

$B_n = \oplus^0_{0 \leq k \leq n} A_k$, where each quotient $B_{n+1}/B_n \cong A_{n+1}$ is 1-matricial.

Example 5.3.6. Let A be a 1-matricial algebra. Any extension of A in the sense of [10] by another 1-matricial algebra C is scattered. Such extensions are given by the Busby invariant; any completely contractive homomorphism $\tau : C \to M(A)/A = A^{**}/A$.

Now, we want to present some results about the structure of scattered operator algebras.

We proved in Chapter 3 that semiprimeness and semisimplicity descend to HSAs. Now, we prove that these properties pass to direct limits.

Proposition 5.3.7. *Let A be a Banach algebra with no nonzero left annihilators and $\{I_\alpha\}$ be an increasing family of approximately unital ideals in A such that $A = \overline{\cup I_\alpha}$. If each I_α is semiprime (resp. semisimple), then A is semiprime (resp. semisimple).*

Proof. For each ordinal α, assume that I_α is semiprime. Let J be a nil ideal in A. Then, for each α, $J \cap I_\alpha$ is a nil ideal in I_α. Since each I_α is semiprime, $J \cap I_\alpha = (0)$ for each α. Notice that this implies that $J = (0)$. Indeed, if $x \in J$ and $y \in I_\alpha$, then $xy \in J \cap I_\alpha = (0)$. That is, $xy = 0$ for each $y \in I_\alpha$ and hence for each $y \in A$. Thus, $x = 0$ and $J = (0)$.

Now, suppose that each I_α is semisimple. Notice that $J = \text{Rad}(A)$ is an ideal in A and for each α, $J \cap I_\alpha$ is an ideal in I_α. Then, for each α, $J \cap I_\alpha = \text{Rad}(I_\alpha)$ by [41, Theorem 4.3.2] and thus $J \cap I_\alpha = (0)$. Hence, as in the first paragraph, $J = (0)$ and A is semisimple. $\square$

Using this lemma, we prove that scattered operator algebras are semisimple and semiprime.

Proposition 5.3.8. *A scattered operator algebra is semisimple and semiprime.*

Proof. Let A be an operator algebra with scattered composition series $\{I_\alpha\}_{0\leq\alpha\leq\beta}$. Let $\mathrm{Rad}(A)$ be the Jacobson radical of A. We will use transfinite induction to prove that A is semisimple. For $\alpha = 0$, $I_0 = (0)$ is semisimple. Let $\alpha + 1$ be a successor ordinal and assume that I_α is semisimple. Notice that $I_{\alpha+1}/I_\alpha$ is semisimple since it is 1-matricial. Then, by [41, Theorem 4.3.2], $\mathrm{Rad}(I_{\alpha+1}) = \mathrm{Rad}(I_\alpha) = (0)$, and hence $I_{\alpha+1}$ is semisimple. Let γ be a limit ordinal and assume that I_α is semisimple for each $\alpha < \gamma$. Then, $I_\gamma = \overline{\cup_{\alpha<\gamma} I_\alpha}$ is semisimple by Proposition 5.3.7. Hence, by transfinite induction, we conclude that I_α is semisimple for any ordinal α. Since $A = I_\beta$, A is semisimple, and hence semiprime. $\square$

As one would expect, HSAs and approximately unital ideals in a scattered operator algebra are scattered.

Proposition 5.3.9. *Let D be a HSA of a scattered operator algebra A. Then, D is a scattered operator algebra.*

Proof. Let $\{I_\alpha\}$ be a scattered composition series for A. For each α, let $D_\alpha = D \cap I_\alpha$. Then, by Remark 5.2.5, $\{D_\alpha\}$ is a composition series for D. Notice that since $D_{\alpha+1} \cap I_\alpha = D_\alpha$, we have

$$D_{\alpha+1}/D_\alpha = D_{\alpha+1}/(D_{\alpha+1} \cap I_\alpha) \cong (D_{\alpha+1} + I_\alpha)/I_\alpha,$$

completely isometrically isomorphically. In the last displayed equation, the congruence follows from Theorem 3.1.5. Let $M_{\alpha+1} = D_{\alpha+1} + I_\alpha$. Then, since $D_{\alpha+1}$ is a HSA in $I_{\alpha+1}$ by Lemma 3.2.5 and since I_α is an approximately unital ideal in $I_{\alpha+1}$, $M_{\alpha+1}$ is a HSA in $I_{\alpha+1}$. Notice that I_α is an approximately unital ideal in $M_{\alpha+1}$. Hence, $M_{\alpha+1}/I_\alpha$ is a HSA in $I_{\alpha+1}/I_\alpha$ by Theorem 3.2.6. Now, by [3, Corollary 4.26], $M_{\alpha+1}/I_\alpha$ is 1-matricial. We conclude that $D_{\alpha+1}/D_\alpha$ is 1-matricial for each α. Hence, D is scattered. □

Remark 5.3.10. We want to note that any C^*-subalgebra of a scattered C^*-algebra is scattered [36]. However, we can not say that any subalgebra of a scattered operator algebra is scattered. For example, $A = \mathcal{T}_2$, the upper triangular 2×2 matrices, is a subalgebra of M_2 which is not scattered. To see that A is not scattered, note that A is not semiprime.

Proposition 5.3.11. *Let A be an approximately unital operator algebra. Then, A is scattered if and only if A^1 is scattered.*

Proof. Since A is an ideal in A^1, if A^1 is scattered, then A is scattered by Proposition 5.3.9. Assume that A is scattered with composition series $\{I_\alpha\}$ where $I_\beta = A$. Each I_α can be considered as an approximately ideal in A^1. Extend this composition series by defining $I_{\beta+1} = A^1$. Then, $\{I_\alpha\}_{0\leq\alpha\leq\beta+1}$ is a composition series for A^1 and $I_{\beta+1}/I_\beta \cong \mathbb{C}$ is 1-matricial. Note that we already know that $I_{\alpha+1}/I_\alpha$ is 1-matricial for all $\alpha < \beta$. Hence, $\{I_\alpha\}_{0\leq\alpha\leq\beta+1}$ is a scattered composition series for A^1. □

Example 5.3.12. The previous proposition gives us an interesting example of scattered operator algebras. If A is a σ-matricial algebra, then its unitization A^1 is scattered.

In the non-selfadjoint case, the simplest scattered operator algebra is a 1-matricial algebra. A general scattered operator algebra is built up by using a composition series of 1-matricial operator algebras. Moreover, each operator algebra which is built up using composition series of scattered operator algebras is again scattered.

Theorem 5.3.13. *Let $\{I_\alpha\}_{0\leq\alpha\leq\beta}$ be a strictly increasing family of approximately unital ideals in A such that $A = I_\beta = \overline{\cup_{\alpha<\beta} I_\alpha}$. If I_α is scattered for each $\alpha < \beta$, then A is scattered.*

Proof. For each $\alpha < \beta$, let $\{J_\rho^{(\alpha)}\}$ be a scattered decomposition series for I_α. For any successor ordinal $\alpha+1$, consider the composition series $\{I_\alpha + J_\rho^{(\alpha+1)}\}_\rho$, which is a composition series running from I_α to $I_{\alpha+1}$. When we extend these series, we get a new composition series for I_β:

$$
\begin{aligned}
(0) = I_0 \ &\subset \ J_1^{(1)} \subset J_2^{(1)} \subset \cdots \subset I_1 = I_1 + J_0^{(2)} \\
&\subset \ I_1 + J_1^{(2)} \subset I_1 + J_2^{(2)} \subset \cdots \subset I_2 = I_2 + J_0^{(3)} \\
&\subset \ I_2 + J_1^{(3)} \subset I_2 + J_2^{(3)} \subset \cdots \subset I_3 = I_3 + J_0^{(4)} \\
&\subset \ \cdots \\
&\ \vdots \\
&\subset \ I_\beta .
\end{aligned}
$$

Now, to see that this composition series is scattered, for notational purposes, consider I_1 and I_2 and let $K_\rho = J_\rho^{(2)}$ be the composition series for I_2. Then, $\{I_1 + K_\rho\}_\rho$

is a composition series running from I_1 to I_2. We claim that for each ordinal ρ, $(I_1 + K_{\rho+1})/(I_1 + K_\rho)$ is 1-matricial. Notice that

$$(I_1 + K_{\rho+1})/(I_1 + K_\rho) = (I_1 + K_\rho + K_{\rho+1})/(I_1 + K_\rho) \cong K_{\rho+1}/((I_1 + K_\rho) \cap K_{\rho+1}),$$

completely isometrically isomorphically, by Theorem 3.1.5. Since $(I_1 + K_\rho) \cap K_{\rho+1} = (I_1 \cap K_{\rho+1}) + K_\rho$, we conclude that $(I_1 + K_{\rho+1})/(I_1 + K_\rho) \cong K_{\rho+1}/((I_1 \cap K_{\rho+1}) + K_\rho)$.

Let $q : K_{\rho+1} \to K_{\rho+1}/K_\rho$ be the canonical quotient map. Since $I_1 \cap K_{\rho+1}$ is an ideal in $K_{\rho+1}$, $\overline{q(I_1 \cap K_{\rho+1})}$ is an ideal in $K_{\rho+1}/K_\rho$, which is 1-matricial and hence simple. That is, $\overline{q(I_1 \cap K_{\rho+1})} = (0)_{K_{\rho+1}/K_\rho}$ or $\overline{q(I_1 \cap K_{\rho+1})} = K_{\rho+1}/K_\rho$. If the former case is true, then $I_1 \cap K_{\rho+1} \subset K_\rho$. This implies that $(I_1 + K_{\rho+1})/(I_1 + K_\rho) \cong K_{\rho+1}/((I_1 \cap K_{\rho+1}) + K_\rho) = K_{\rho+1}/K_\rho$ is 1-matricial. In the latter case, $(I_1 \cap K_{\rho+1}) + K_\rho = K_{\rho+1}$, and this implies that $(I_1 + K_{\rho+1})/(I_1 + K_\rho) \cong K_{\rho+1}/((I_1 \cap K_{\rho+1}) + K_\rho) = K_{\rho+1}/K_{\rho+1} \cong (0)$. That is $\{I_1 + K_\rho\}_\rho$ is a composition series running from I_1 to I_2 where each consecutive quotient is 1-matricial.

Similarly, we can show that each consecutive quotient in $\{I_\alpha + J_\rho^{(\alpha+1)}\}_\rho$ is 1-matricial. Hence, the composition series we obtained for I_β is scattered. □

We know that the quotients of scattered C^* algebras are scattered [30]. We obtain a similar result for quotients of scattered operator algebras.

Theorem 5.3.14. *If A is a scattered operator algebra, then each quotient of A by an approximately unital ideal is scattered.*

Proof. Let $\{I_\alpha\}_{\alpha \leq \beta}$ be a scattered composition series for A. For $\alpha = 0$, every quotient of $I_0 = (0)$ is scattered. For a successor ordinal $\alpha + 1$, assume that every quotient of

I_α is scattered and let J be an approximately unital ideal in $I_{\alpha+1}$. Since $(I_\alpha+J)/J \cong I_\alpha/(I_\alpha \cap J)$ by Theorem 3.1.5, $(I_\alpha+J)/J$ is a quotient of I_α and hence it is scattered by the inductive hypothesis. Let $\{K_\rho\}_{\rho\leq\alpha}$ be a scattered composition series running from (0) to $K_\alpha = (I_\alpha + J)/J$. Extend this composition series by defining $K_{\alpha+1} = I_{\alpha+1}/J$. Notice that

$$K_{\alpha+1}/K_\alpha \cong I_{\alpha+1}/(I_\alpha + J) \cong (I_{\alpha+1}/I_\alpha)/((I_\alpha + J)/I_\alpha).$$

Since $I_{\alpha+1}/I_\alpha$ is 1-matricial and hence simple, $(I_\alpha + J)/I_\alpha = (0)$ or $(I_\alpha + J)/I_\alpha = I_{\alpha+1}/I_\alpha$. In the former case, $K_{\alpha+1}/K_\alpha \cong I_{\alpha+1}/I_\alpha$ and it is 1-matricial and we get a scattered composition series for $I_{\alpha+1}/J$. In the latter case, $I_\alpha + J = I_{\alpha+1}$ and $K_{\alpha+1}/K_\alpha \cong (0)$ and again we get a scattered composition series for $I_{\alpha+1}/J$. Hence, $I_{\alpha+1}/J$ is scattered.

Let γ be a limit ordinal and assume that for each $\alpha < \gamma$, every quotient of I_α is scattered. Let J be an approximately unital ideal in I_γ. Then, $\{J + I_\alpha\}_{\alpha\leq\gamma}$ is a composition series running from J to I_γ. Hence, $\{(J+I_\alpha)/J\}$ is a composition series running from $(0)_{I_\gamma/J}$ to I_γ/J. To see that I_γ/J is scattered, let $K_\alpha = (J + I_\alpha)/J$. Notice that $K_\alpha \cong I_\alpha/(J \cap I_\alpha)$ by Theorem 3.1.4, and K_α is a quotient of I_α and hence scattered by the inductive hypothesis. So, $\{K_\alpha\}$ is a composition series for I_γ/J where each K_α is scattered. Hence, by Theorem 5.3.13, I_γ is scattered.

Hence, by transfinite induction, we conclude that for each ordinal α, any quotient of I_α by an approximately unital ideal is scattered. Thus, any quotient of A by an approximately unital ideal is scattered. $\square$

This proposition gives us a corollary about the homomorphic images of scattered algebras.

Corollary 5.3.15. *Let $u : A \to B$ be a complete quotient map which is a homomorphism between operator algebras. Suppose that $Ker(u)$ contains a cai. If A is scattered, then $Im(u)$ is scattered. In particular, if u is surjective, then B is scattered.*

Proof. The assertions follow from the Theorems 3.1.3 and 5.3.14. □

If A is a C^*-algebra and J is any closed two-sided ideal in A, then we know that A is a scattered C^*-algebra if and only if J and A/J are both scattered [30, Proposition 2.4]. We prove a similar result for scattered operator algebras and approximately unital ideals.

Theorem 5.3.16. *Let A be an operator algebra and J be an approximately unital ideal in A. Then, A is scattered if and only if J and A/J are scattered.*

Proof. Suppose that A is scattered. Then, J is scattered by Proposition 5.3.9 since J is a HSA in A. Also, A/J is scattered by Theorem 5.3.14.

Conversely, assume that J and A/J are scattered and let $\{M_\alpha\}_{\alpha\leq\beta}$ be a scattered composition series for A/J. By Theorem 3.2.1, for each α, $M_\alpha = N_\alpha/J$, where N_α is an approximately unital ideal in A such that $J \subset N_\alpha \subset A$. Notice that by Theorem 3.1.4,

$$M_{\alpha+1}/M_\alpha = (N_{\alpha+1}/J)/(N_\alpha/J) \cong N_{\alpha+1}/N_\alpha,$$

completely isometrically isomorphically. Hence, each $N_{\alpha+1}/N_\alpha$ is 1-matricial.

Since $M_0 = (0)_{A/J} = J$, we conclude that $N_0 = J$. Since $M_\alpha \subset M_{\alpha+1}$, we have $N_\alpha \subset N_{\alpha+1}$. That is, we have an increasing series of approximately unital ideals $J = N_0 \subset N_1 \subset N_2 \subset \cdots \subset A$. If γ is a limit ordinal, then $M_\gamma = \overline{\cup_{\alpha<\gamma} M_\alpha} = \overline{\cup_{\alpha<\gamma} N_\alpha / J} = \overline{\cup_{\alpha<\gamma} N_\alpha}/J$. Since $M_\gamma = N_\gamma/J$, we have $\overline{\cup_{\alpha<\gamma} N_\alpha}/J = N_\gamma/J$ and $N_\gamma/\overline{\cup_{\alpha<\gamma} N_\alpha} \cong (0)$ by Theorem 3.1.4. Hence, $N_\gamma = \overline{\cup_{\alpha<\gamma} N_\alpha}$. Since $M_\beta = A/J$, we have $N_\beta = A$. Hence, $\{(0), J = N_0, N_1, N_2, \cdots, N_\beta\}$ is a composition series for A. Now, let $\{J_\alpha\}$ be a scattered composition series for the operator algebra J. By Remark 5.2.5, we get a new composition series of approximately unital ideals;

$$\{(0), J_1, J_2, \cdots, J = N_0, N_1, N_2, \cdots, A\}.$$

Since $N_1/J = N_1/N_0$, $J_{\alpha+1}/J_\alpha$, $N_{\alpha+1}/N_\alpha$ are 1-matricial algebras for each α, we conclude that A is scattered. □

Corollary 5.3.17. *Let $\{I_\alpha\}$ be a composition series for an approximately unital operator algebra A. If each $I_{\alpha+1}/I_\alpha$ is scattered, then A is scattered.*

Proof. For $\alpha = 0$, $I_1/I_0 \cong I_1$ is scattered. Let $\alpha+1$ be a successor ordinal and assume that I_α is scattered. Since $I_{\alpha+1}/I_\alpha$ is scattered by hypothesis, $I_{\alpha+1}$ is scattered by Theorem 5.3.16. Let γ be a limit ordinal and assume that I_α is scattered for each $\alpha < \gamma$. By Theorem 5.3.13, $I_\gamma = \overline{\cup_{\alpha<\gamma} I_\alpha}$ is scattered. Hence, by transfinite induction, we can conclude that for each ordinal α, I_α is scattered. Thus, $I_\beta = A$ is scattered. □

For an operator algebra A, let f be the join of all the algebraically minimal projections in A. Then, h-soc(A) is the HSA with support projection f; that is,

h-soc$(A) = fA^{**}f \cap A$ (more details can be found in [3, Section 4]). Observe that for any semiprime operator algebra A, h-soc(A) is a scattered HSA in A. Also, if q is an algebraically minimal projection in the center of A, then $J = Aq = qA = qAq = \mathbb{C}\, q$ is a scattered ideal contained in A. Hence, the existence of algebraically minimal projections in an operator algebra implies the existence of scattered ideals or scattered HSAs.

We want to note that any operator algebra contains a unique maximal scattered ideal.

Theorem 5.3.18. *Let A be an approximately unital operator algebra. There exists a unique maximal scattered ideal K such that A/K has no nontrivial scattered ideals.*

Proof. If A has no nontrivial scattered ideals, then the result follows with $K = (0)$. Note that the sum of any two scattered ideals is closed (since they are approximately unital) and scattered. Indeed, if I and J are scattered ideals in A, then $I/(I \cap J) \cong (I + J)/J$ by Theorem 3.1.5. By Lemma 3.2.5, $I \cap J$ is approximately unital and hence $I/(I \cap J)$ is scattered by Theorem 5.3.14. Hence, $(I + J)/J$ is scattered by Corollary 5.3.15. Now, since J and $(I + J)/J$ are scattered, $I + J$ is scattered by Theorem 5.3.16.

Moreover, if $\{I_\alpha\}$ is a totally ordered family of scattered ideals , then $\overline{\sum_\alpha I_\alpha} = \overline{\cup_\alpha I_\alpha}$ is scattered by Theorem 5.3.13. Hence, every chain of scattered ideals has an upper bound.

Let $\{I_\alpha\}$ be the family of all scattered ideals in A. By Zorn's lemma, the family has a maximal element K. Then, K is a maximal scattered ideal in A and A/K does not contain any nontrivial scattered ideals. Indeed, if M is a nontrivial scattered ideal in A/K, we know by Theorem 3.2.1 that $M = J/K$ where J is an approximately unital ideal in A such that $K \subset J \subset A$. Since M and K are scattered, J is scattered by Theorem 5.3.16. By maximality of K, this implies that $J = K$. □

If A is a 1-matricial operator algebra, then its second dual has a nice form (see Chapter 4 for details). We want to present some results about the bidual of a scattered operator algebra. Note that if A has a scattered composition series $\{I_\alpha\}$, then $A^{**} = \oplus_\alpha^\infty (I_{\alpha+1}/I_\alpha)^{**}$ by Theorem 5.2.4. That is, the second dual of A is the ∞-sum of the second duals of 1-matricial algebras. First, we prove that the second dual of a scattered operator algebra is semisimple and semiprime.

Proposition 5.3.19. *Let A be a scattered operator algebra. Then, A^{**} is semisimple.*

Proof. We know by Theorem 5.2.4 that $A^{**} = \oplus_\alpha^\infty (I_{\alpha+1}/I_\alpha)^{**}$. By Proposition 4.3.7, each $(I_{\alpha+1}/I_\alpha)^{**}$ is semisimple. Hence, by Lemma 4.3.5, A^{**} is semisimple. □

Proposition 5.3.20. *Let A be a scattered operator algebra. Then, A^{**} is DMA.*

Proof. By Theorem 5.2.4, $A^{**} = \oplus_\alpha^\infty (I_{\alpha+1}/I_\alpha)^{**}$. Since the second duals of 1-matricial algebras are DMA by Proposition 4.3.9 and semisimple by Proposition 4.3.7, we conclude that A^{**} is DMA by Lemma 4.3.8. □

Recall that we say that an operator algebra A has *Property (M)* if every nonzero projection in A dominates a nonzero algebraically minimal projection in A.

Notice that if A is an elementary C^*-algebra then A^{**} has Property (M). In fact, by [30, Theorem 2.2], A is a scattered C^*-algebra if and only if A^{**} has Property (M). This property appears often in our 'Wedderburn type' structure theorem (Theorem 4.2.1), which lists some characterizations of σ-matricial algebras. In Chapter 4, we showed that the second duals of 1-matricial algebras have Property (M). The second dual of a scattered operator algebra has Property (M).

Proposition 5.3.21. *Let A be a scattered operator algebra. Then, A^{**} has Property (M).*

Proof. Let $A^{**} = \oplus_{\alpha}^{\infty} q_\alpha A^{**}$ as in the proof of Theorem 5.2.4. If $p = (p_\alpha)$ is a nonzero projection in A^{**}, then there exists an ordinal α which is not a limit ordinal such that $p_\alpha \neq 0$. Then, p_α is a nonzero projection in $q_\alpha A^{**}$. Notice that $q_\alpha A^{**} \cong (I_{\alpha+1}/I_\alpha)^{**}$ is the second dual of a 1-matricial algebra and hence it has Property (M) by Proposition 4.3.15. There exists an algebraically minimal projection $r_\alpha \in q_\alpha A^{**}$ such that $r_\alpha \leq p_\alpha$. Let $r = (0, 0, \cdots, r_\alpha, 0, \cdots) \in A^{**}$. Then r is a projection and $rxr = (0, 0, \cdots, r_\alpha x_\alpha r_\alpha, 0, \cdots) = (0, 0, \cdots, \lambda_\alpha r_\alpha, 0, \cdots) \in \mathbb{C}\, r$, for $x \in A^{**}$. That is, r is an algebraically minimal projection in A^{**}. Since $pr = (0, 0, \cdots, p_\alpha r_\alpha, 0, \cdots) = (0, 0, \cdots, r_\alpha, 0, \cdots)$, r is dominated by p. Hence, A^{**} has Property (M). $\square$

We know that any σ-matricial algebra is a HSA in its bidual. A natural question to ask is whether scattered algebras are HSAs in their biduals. The following result shows that if a scattered algebra is a HSA in its bidual, then it is a σ-matricial algebra.

Theorem 5.3.22. *Let A be a scattered operator algebra. If A is a HSA in its bidual,* *then A is σ-matricial.*

Proof. Since A is scattered, A is semisimple by Proposition 5.3.8 and A^{**} has Property (M) by Proposition 5.3.21. If A is an HSA in its bidual, then by Theorem 4.3.17, A is σ-matricial. □

Recall the definition of Radon-Nikodým Property (RNP) from Chapter 2. By [21, Theorem 3], a C^*-algebra A is scattered if and only if the dual A^* has RNP. We show that the duals of scattered operator algebras have this property.

Theorem 5.3.23. *Let A be a scattered operator algebra. Then the dual A^* has RNP and A is an Asplund space.*

Proof. Let $\{I_\alpha\}$ be the scattered composition series for A. For each α, the dual of $I_{\alpha+1}/I_\alpha$ has RNP by [3, Lemma 4.4]. Note that A^* is completely isometric to the ℓ_1-direct sum $\sum_k \oplus (I_{\alpha+1}/I_\alpha)^*$. Hence, A^* has RNP since an ℓ_1-direct sum of Banach spaces with RNP also has this property [21, p. 534]. By [50, Theorem 1], a Banach space is an Asplund space if and only if its dual has RNP. Hence, A is an Asplund space. □

As mentioned earlier, $\Delta(A)$ is a C^*-algebra sitting inside A and we are interested in understanding the diagonal with the motivation that it can give us further information about A. We show that the diagonal of a scattered operator algebra is a scattered C^*-algebra.

Theorem 5.3.24. *Let A be a scattered operator algebra. Then, $\Delta(A)$ is a scattered C^*-algebra.*

Proof. Note that being an Asplund space is hereditary. To see this, recall that a Banach space X is an Asplund space if and only if every separable subspace of X has a separable dual [50]. Since $\Delta(A)$ is a subalgebra of A and A is an Asplund space by Theorem 5.3.23, the dual of $\Delta(A)$ has RNP. By [21, Theorem 3], the dual of a C^*-algebra has RNP if and only if it is a scattered C^*-algebra. Hence, $\Delta(A)$ is a scattered C^*-algebra. □

We say that an operator algebra A is Δ-*scattered* if A has a positive cai and $\Delta(A)$ is a scattered C^*-algebra. By Theorem 5.3.24, if A is scattered with a positive cai, then A is Δ-scattered.

Notice that (by [37, Theorem 2.2]) every selfadjoint element in A has countable spectrum if and only if $\Delta(A)$ is a scattered C^*-algebra. Hence, if every selfadjoint element in A has countable spectrum and A has a positive cai, then A is Δ-scattered.

Remark 5.3.25. In Chapter 4, we defined the notion of having real rank zero for operator algebras containing a positive cai. If A is Δ-scattered, then A has real rank zero and contains a cai of projections. If A is scattered with positive cai, then again, A contains a cai consisting of projections. In [17], a C^*-algebra is said to have *generalized real rank zero* if it has a composition series where each consecutive quotient has real rank zero. If we generalize this to operator algebras, then a scattered algebra has generalized real rank zero since each consecutive quotient in its composition series is 1-matricial and hence has real rank zero.

We want to present a result about scattered function algebras.

Proposition 5.3.26. *Let A be an approximately unital function algebra. If A is* *scattered, then it is a C^*-algebra.*

Proof. We only prove the unital case. Let $\{I_\alpha\}$ be a scattered composition series for a unital function algebra A. We know by Theorem 5.2.4 that $A^{**} = \oplus^\infty (I_{\alpha+1}/I_\alpha)^{**}$. Notice that each $I_{\alpha+1}/I_\alpha$ is a function algebra. Indeed, $I_{\alpha+1}/I_\alpha$ is a semisimple Q-algebra and hence is isomorphic to $\{f|_E : f \in I_{\alpha+1}\}$ where E is a closed subset of the maximal ideal space of $I_{\alpha+1}$ [23]. Thus, for each α, $I_{\alpha+1}/I_\alpha$ is an nc-discrete function algebra. Hence, by Proposition 4.1.18, for each α, $I_{\alpha+1}/I_\alpha \cong c_0(J_\alpha)$ for some set J_α. That is, $A^{**} = \oplus^\infty c_0(J_\alpha)^{**} = \oplus^\infty \ell^\infty(J_\alpha)$ and A is a C^*-algebra. □

In the following result, $A \oplus B$ is the ∞-direct sum of the operator algebras A and B.

Proposition 5.3.27. *Let A and B be scattered operator algebras. Then, $A \oplus B$ is scattered.*

Proof. Let $\{I_\alpha\}$ and $\{J_\alpha\}$ be the scattered composition series for A and B, respectively. Let $M_\alpha = I_\alpha \oplus J_\alpha$. Then, by Remark 5.2.5, $\{M_\alpha\}$ is a composition series for $A \oplus B$. By Proposition 3.1.8, we know that

$$M_{\alpha+1}/M_\alpha = (I_{\alpha+1} \oplus J_{\alpha+1})/(I_\alpha \oplus J_\alpha) \cong I_{\alpha+1}/I_\alpha \oplus J_{\alpha+1}/J_\alpha,$$

completely isometrically isomorphically. Since the ∞-direct sum of two 1-matricial algebras is 1-matricial, we conclude that $A \oplus B$ is scattered. □

We now prove that the maximal tensor product of a scattered operator algebra with a scattered C^*-algebra is scattered.

Lemma 5.3.28. *Let A be a scattered operator algebra and $B = \mathbb{K}(H)$ for some Hilbert space H. Then, $A \otimes_{\max} B$ is scattered.*

Proof. Let $\{I_\alpha\}$ be a scattered composition series for A. Let $M_\alpha = I_\alpha \otimes_{\max} B$. Then, $\{M_\alpha\}$ is a composition series for $A \otimes_{\max} B$. By [6, Lemma 2.7], we have;

$$M_{\alpha+1}/M_\alpha = (I_{\alpha+1} \otimes_{\max} B)/(I_\alpha \otimes_{\max} B) \cong (I_{\alpha+1}/I_\alpha) \otimes_{\max} B,$$

completely isometrically. Here $I_{\alpha+1}/I_\alpha$ is a 1-matricial algebra and $B = \mathbb{K}(H)$ is 1-matricial as well.

If M and N are two 1-matricial algebras with matrix units $\{E_{ij}\}$ and $\{F_{kl}\}$, respectively, then $\{E_{ij} \otimes F_{kl}\}$ are matrix units for $M \otimes_{\max} N$. In other words, the maximal tensor product of two 1-matricial algebras is 1-matricial. Hence, for each α, $M_{\alpha+1}/M_\alpha$ is a 1-matricial algebra. That is, $A \otimes_{\max} B$ is scattered. □

Proposition 5.3.29. *Let A be a scattered operator algebra and B be a scattered C^*-algebra. Then, $A \otimes_{\max} B$ is scattered.*

Proof. Let $\{J_\alpha\}$ be a scattered composition series for B and for each ordinal α, let $M_\alpha = A \otimes_{\max} J_\alpha$. Then, $\{M_\alpha\}$ is a composition series for $A \otimes_{\max} B$. Also,

$$M_{\alpha+1}/M_\alpha = (A \otimes_{\max} J_{\alpha+1})/(A \otimes_{\max} J_\alpha) \cong A \otimes_{\max} (J_{\alpha+1}/J_\alpha),$$

by [6, Lemma 2.7]. Since A is scattered and $J_{\alpha+1}/J_\alpha$ is isometric to $\mathbb{K}(H_\alpha)$ for some Hilbert space H_α, by Lemma 5.3.28, $M_{\alpha+1}/M_\alpha$ is scattered. Hence, $A \otimes_{\max} B$ has a composition series where each consecutive quotient is scattered. By Corollary 5.3.17, $A \otimes_{\max} B$ is scattered. □

Finally, we want to comment on the relation between A being scattered and A having a C^*-cover which is scattered. Unfortunately, if A is scattered, we can't say that $\mathrm{C}^*_{\max}(A)$ is scattered since we do not know much about the maximal C^*-cover of a 1-matricial algebra.

If A is an operator algebra which is a subalgebra of a scattered C^*-algebra B, then $\Delta(A)$ is a C^*-subalgebra of B and hence $\Delta(A)$ is scattered. In particular, if we know that $C^*(A)$, $C^*_{\max}(A)$ or $C^*_e(A)$ is scattered, then we can conclude that $\Delta(A)$ is scattered.

If A has a scattered C^*-cover, then the C^*-envelope is a scattered C^*-algebra. Indeed, the C^*-envelope is a quotient of any C^*-cover B. Since every quotient of a scattered C^*-algebra is scattered, $C^*_e(A)$ is scattered. Hence, if we know that $\mathrm{C}^*_{\max}(A)$ is scattered, then $C^*_e(A)$ is scattered as well.

Remark 5.3.30. Recall that by [3, Lemma 4.8], a 1-matricial algebra is subcompact if and only if its C^*-envelope is an annihilator C^*-algebra. We believe that if A is scattered and $C^*_e(A)$ is an annihilator C^*-algebra, then each consecutive quotient in the scattered composition series is a subcompact 1-matricial algebra. We hope to present the details soon.

5.4 Maximally Scattered Operator Algebras

In non-selfadjoint operator algebra theory, one often studies the maximal C^*-cover of A, $\mathrm{C}^*_{\max}(A)$, to understand some facts about A as $\mathrm{C}^*_{\max}(A)$ is a C^*-algebra and one has a lot of tools to unveil the properties of $\mathrm{C}^*_{\max}(A)$. Consequently, some of those properties will give one information about A. With this motivation, we want to study the case that the maximal C^*-cover of an operator algebra is scattered. This could be considered as another approach to generalizing the notion of being scattered to the non-selfadjoint setting.

Definition 5.4.1. We say that an approximately unital operator algebra A is *maximally scattered* (or *max-scattered*) if $C^*_{\max}(A)$ is a scattered C^*-algebra.

We conjecture that a maximally scattered operator algebra is a scattered C^*-algebra. That is, if $\mathrm{C}^*_{\max}(A)$ is scattered, our conjecture is that this forces A to be selfadjoint.

The first implication we get from the definition is that if A is max-scattered, then $\Delta(A)$ is scattered. Hence, if A has a positive cai, A is max-scattered implies that A is Δ-scattered. Moreover, if A is max-scattered, then the C^*-envelope of A is a scattered C^*-algebra as well since it is a quotient of $\mathrm{C}^*_{\max}(A)$.

It is easy to see that the approximately unital ideals and quotients of max-scattered operator algebras are max-scattered.

Proposition 5.4.2. *Let A be max-scattered operator algebra and J be an approximately unital ideal in A. Then, J is max-scattered and A/J is max-scattered.*

Proof. Since $C^*_{\max}(J)$ is an ideal in $C^*_{\max}(A)$, $C^*_{\max}(J)$ is scattered. Moreover, by [10, Lemma 2.7], $C^*_{\max}(A/J) = C^*_{\max}(A)/C^*_{\max}(J)$. Since every quotient of a scattered C^*-algebra is scattered, $C^*_{\max}(A/J)$ is scattered. Hence, A/J is scattered. $\square$

We know that HSAs of scattered operator algebras are scattered Proposition 5.3.9. However, this is not necessarily true for HSAs of max-scattered operator algebras.

Remark 5.4.3. Let A be an approximately unital operator algebra which is contained in a C^*-algebra B. Let D be a HSA in A. Then, $\mathrm{C}^*_{\max}(D)$ (or $C^*_e(D)$) is not necessarily a HSA in B.

To see this, let

$$A = \left\{ \begin{bmatrix} \lambda I_2 & x \\ 0 & \lambda I_2 \end{bmatrix} \in M_4 \right\} \subset M_4,$$

and let $e = E_{11} \otimes I_2 = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix} \in A$. Notice that $eAe = \mathbb{C}\, e$ is a HSA in A. However, $C^*_e(eAe) = \mathbb{C}\, e$ is not a HSA in $C^*_e(A) = C^*_{M_4}(A) = M_4$. Moreover, we claim that $\mathrm{C}^*_{\max}(eAe)$ is not a HSA in $\mathrm{C}^*_{\max}(A)$. Indeed, if $\mathbb{C}\, e$ is a HSA in $\mathrm{C}^*_{\max}(A)$, and if $q : \mathrm{C}^*_{\max}(A) \to C^*_e(A)$ is the canonical quotient map, then $q(\mathbb{C}\, e\, \mathrm{C}^*_{\max}(A)\, \mathbb{C}\, e) \subset \mathbb{C}\, e$. However, $q(\mathbb{C}\, e\, \mathrm{C}^*_{\max}(A)\, \mathbb{C}\, e) = \mathbb{C}\, e C^*_e(A)\, \mathbb{C}\, e$ which is not contained in $\mathbb{C}\, e$. This example shows that $\mathrm{C}^*_{\max}(D)$ (resp. $C^*_e(D)$) is not a HSA in $\mathrm{C}^*_{\max}(A)$ (resp. $C^*_e(A)$). We conclude that a HSA in a max-scattered operator algebra is not necessarily max-scattered.

We now prove that if $\mathrm{C}^*_{\max}(A)$ is an annihilator C^*-algebra, then this forces A to be self adjoint.

Theorem 5.4.4. *Let A be an approximately unital operator algebra. Then, $\mathrm{C}^*_{\max}(A)$ is an annihilator C^*-algebra if and only if A is an annihilator C^*-algebra.*

Proof. Annihilator C^*-algebras are c_0-sums of elementary C^*-algebras. First we show that if $\mathrm{C}^*_{\max}(A)$ is elementary, then A is a C^*-algebra. In general, if $\mathrm{C}^*_{\max}(A) = \oplus^0_k B_k$ where B_k are elementary C^*-algebras, then $A = \oplus^0_k A_k$ where for each k, $\mathrm{C}^*_{\max}(A_k) = B_k$. To see this, set $A_k = A \cap B_k \subset B_k$. If $x \in A \subset B$, then $x = (x_k)$ where $x_k \in B_k$ and $\|x_k\| \to 0$. That is, $x_k \in A \cap B_k$ and hence $x \in \oplus^0_k A_k$. If $x \in \oplus^0_k A_k$, then $x = (x_k)$ where for each k, $x_k \in A \cap B_k$. That is, $x \in A$. To see that $\mathrm{C}^*_{\max}(A_k) = B_k$, let $\pi_k : A_k \to B(H)$ be a completely contractive homomorphism. Then, the extension $\pi : A \to B(H)$ where $\pi|_{A_k} = \pi_k$ and $\pi_{A_j} = 0$ for any $j \neq k$, is a completely contractive homomorphism. Hence, it extends uniquely to a $*$-homomorphism $\tilde{\pi} : B \to B(H)$. The restriction of $\tilde{\pi}$ to B_k is a $*$-homomorphism which extends π_k. That is, B_k has the universal property and $\mathrm{C}^*_{\max}(A_k) = B_k$.

It follows that each A_k is selfadjoint, and this implies that $A = \oplus^0_k A_k$ is a C^*-algebra. So, we assume that $C^*_{\max}(A)$ is an elementary C^*-algebra. For simplicity, assume $C^*_{\max}(A) = \mathbb{K}(\ell^2)$, the general case is almost identical. Write $\sigma : A \to B(\ell^2)$ for the canonical inclusion, or for its canonical extension to $C^*_{\max}(A)$. This is an irreducible representation, since if a projection commutes with A, then it commutes with $C^*_{\max}(A) = \mathbb{K}(\ell^2)$, and hence is trivial. By Arveson's boundary theorem, σ is a boundary representation of A.

Let $\theta : A \to B(H)$ be a nondegenerate representation of A, and let K be a $\theta(A)$-invariant subspace of H. Let ρ be the compression of θ to K. Thus, we can write

$$\theta(a) = \begin{bmatrix} \rho(a) & V(a) \\ 0 & \pi(a) \end{bmatrix},$$

where V and π are some maps on A. Now, K is a $C^*_{\max}(A)$-module and $C^*_{\max}(A) =$

$\mathbb{K}(\ell^2)$. Hence, there exists a unitary $U : K \to \ell^2$ such that $\rho(a) = U^*(\sigma(a) \otimes I_\infty)U$. Thus,

$$\theta(a) = \begin{bmatrix} U & 0 \\ 0 & I \end{bmatrix}^* \begin{bmatrix} \sigma(a) \otimes I_\infty & UV(a) \\ 0 & \pi(a) \end{bmatrix} \begin{bmatrix} U & 0 \\ 0 & I \end{bmatrix}.$$

Since σ is a boundary representation, we can conclude that $UV(\cdot) = (0)$. That is, $V = 0$. Hence, K reduces θ. By [7, Theorem 7.2.5(1)], A is a C^*-algebra; that is, $A = C^*_{\max}(A) = \mathbb{K}(\ell^2)$. □

If $\mathrm{C}^*_{\max}(A)$ is scattered with a composition series $\{J_\alpha\}$, we can't conclude that A has a composition series $\{I_\alpha\}$ where $\mathrm{C}^*_{\max}(I_\alpha) = J_\alpha$. On the other hand, if A has a composition series $\{I_\alpha\}$, then $\{J_\alpha\}$, where $J_\alpha = \mathrm{C}^*_{\max}(I_\alpha)$ for each α, is a composition series for $\mathrm{C}^*_{\max}(A)$ by Remark 5.2.5. However, if A is scattered (that is, if $I_{\alpha+1}/I_\alpha$ is 1-matricial), we can't conclude that $\mathrm{C}^*_{\max}(A)$ is a scattered (that is, $J_{\alpha+1}/J_\alpha$ is an annihilator C^*-algebra). We can give the following result if such a nice relation exists.

Proposition 5.4.5. *If A has a composition series $\{I_\alpha\}$ such that $\{\mathrm{C}^*_{\max}(I_\alpha)\}$ is a composition series for $\mathrm{C}^*_{\max}(A)$ where each consecutive quotient is an annihilator C^*-algebra, then A is a scattered C^*-algebra.*

Proof. We know by [10, Lemma 2.7] that $\mathrm{C}^*_{\max}(I_{\alpha+1}/I_\alpha) \cong \mathrm{C}^*_{\max}(I_{\alpha+1})/\mathrm{C}^*_{\max}(I_\alpha)$ and by our hypothesis $\mathrm{C}^*_{\max}(I_{\alpha+1})/\mathrm{C}^*_{\max}(I_\alpha)$ is an annihilator C^*-algebra. So, by Theorem 5.4.4, $I_{\alpha+1}/I_\alpha$ is an annihilator C^*-algebra. Hence, A has a composition series where for each α, $I_{\alpha+1}/I_\alpha$ is an annihilator C^*-algebra. This forces A to be a C^*-algebra. Indeed, notice that $A^{**} = \oplus_\alpha^\infty (I_{\alpha+1}/I_\alpha)^{**}$. Since each $I_{\alpha+1}/I_\alpha$ is an annihilator C^*-algebra, we know that $(I_{\alpha+1}/I_\alpha)^{**} = \oplus_i^\infty B(H_i)$ for some Hilbert spaces H_i and hence A^{**} is selfadjoint. That is, by [7, Theorem 7.2.5(1)] and [7, 7.2.4], A is selfadjoint. Hence, A is a C^*-algebra with a composition series where each quotient is an annihilator C^*-algebra; that is, by [30, Proposition 2.6], A is a scattered C^*-algebra. □

We know that a C^*-algebra is scattered if and only if every nondegenerate representation on the algebra is unitarily equivalent to a subrepresentation of a sum of irreducible representations [30]. We prove that the same relation holds for maximally scattered operator algebras.

Proposition 5.4.6. *Let A be an approximately unital operator algebra. Then, A is max-scattered if and only if every nondegenerate representation on A is unitarily equivalent to a subrepresentation of a sum of irreducible representations.*

Proof. Suppose that every nondegenerate representation on A is unitarily equivalent to a subrepresentation of a sum of irreducible representations. Let π be a nondegenerate representation on $C^*_{\max}(A)$. Then, it restricts to a nondegenerate representation π' on A. Hence, π' is equivalent to a subrepresentation of a sum of irreducible representations on A. Since each nondegenerate representation (resp. irreducible representation) on A extends to a nondegenerate representation (resp. irreducible representation) on $C^*_{\max}(A)$, we conclude that π is equivalent to a subrepresentation of a sum of irreducible representations on $C^*_{\max}(A)$. Hence, $C^*_{\max}(A)$ is scattered. The proof of the converse statement is almost identical. □

Note that, by the previous result, if A is max-scattered, then the universal representation of A is equivalent to a sum of irreducible representations on A.

The following result shows that being max-scattered passes to the maximal tensor product with a scattered C^*-algebra.

Proposition 5.4.7. *Let B be a scattered C^*-algebra and A be a max-scattered operator algebra. Then, $B \otimes_{\max} A$ is max-scattered.*

Proof. By [10, Lemma 2.8], $\mathrm{C}^*_{\max}(B \otimes_{\max} A) = B \otimes_{\max} \mathrm{C}^*_{\max}(A)$. Note that, by Proposition 5.3.29, $B \otimes_{\max} \mathrm{C}^*_{\max}(A)$ is scattered. Hence, $B \otimes_{\max} A$ is max-scattered. $\square$

There is an important class of C^*-algebras called the Type I C^*-algebras. This class has a tractable representation theory and therefore at times this class has been regarded as the class of reasonable C^*-algebras (by Glimm's results [28]). Type I C^*-algebras are built up by using elementary C^*-algebras and the structure of Type I C^*-algebras is clearly understood. Note that there are several characterizations of Type I C^*-algebras. For example, a C^*-algebra is Type I if it has a composition series $\{I_\alpha\}$ such that $I_{\alpha+1}/I_\alpha$ is liminal for each α. For more details, we refer the reader to [28] or [5, Section IV.1].

We define Type I operator algebras using the maximal C^*-cover.

Definition 5.4.8. An approximately unital operator algebra A is said to be of Type I if $\mathrm{C}^*_{\max}(A)$ is a Type I C^*-algebra.

The class of Type I operator algebras includes the class of max-scattered operator algebras, as one would expect. Indeed, any scattered C^*-algebra is of Type I by [30, Theorem 2.3]. Hence, if A is max-scattered, then $\mathrm{C}^*_{\max}(A)$ is scattered and of Type I.

Any ideal and any quotient of a Type I C^*-algebra is again Type I. We have a similar result for Type I operator algebras. However, note that, by Remark 5.4.3, a HSA in a Type I operator algebra is not necessarily of Type I.

Proposition 5.4.9. *Let A be a Type I operator algebra and J be an approximately unital ideal in A. Then, J is of Type I and A/J is of Type I.*

Proof. Since $C^*_{\max}(J)$ is an ideal in $C^*_{\max}(A)$, $C^*_{\max}(J)$ is of Type I. Moreover, by [10, Lemma 2.7], $C^*_{\max}(A/J) = C^*_{\max}(A)/C^*_{\max}(J)$. Since every quotient of a Type I C^*-algebra is Type I, $C^*_{\max}(A/J)$ is Type I. Hence, A/J is Type I. □

An operator algebra which is built up using a composition series where each quotient is of Type I is again Type I.

Proposition 5.4.10. *Let A be an approximately unital operator algebra with a composition series $\{I_\alpha\}$ such that each $I_{\alpha+1}/I_\alpha$ is of Type I. Then, A is of Type I.*

Proof. Let $B = C^*_{\max}(A)$. Then, by Remark 5.2.5, $\{\mathrm{C}^*_{\max}(I_\alpha)\}$ is a composition series for B. Moreover, for each ordinal α, $C^*_{\max}(I_{\alpha+1})/C^*_{\max}(I_\alpha) = C^*_{\max}(I_{\alpha+1}/I_\alpha)$, by [10, Lemma 2.7]. Since $I_{\alpha+1}/I_\alpha$ is of Type I, $C^*_{\max}(I_{\alpha+1})/C^*_{\max}(I_\alpha)$ is of Type I. Hence, by [5, Proposition IV.1.1.11], B is of Type I. That is, A is of Type I. □

Chapter 6

Further Directions

In this dissertation we tried to generalize the notion of scattered C^*-algebras to a non-selfadjoint setting. We defined scattered operator algebras using a composition series where each consecutive quotient is completely isometrically isomorphic to a 1-matricial algebra. We presented many results about their structure. Scattered C^*-algebras have many nice characterizations. In later studies, we hope to find some characterizations of scattered operator algebras.

By some recent work by M. Kusuda, we see that scattered C^*-algebras are connected to the notion of having real rank zero [36]. This is expected as the notion of having real rank zero is a noncommutative analogue of the topological notion of being totally disconnected. Indeed, for a locally compact Hausdorff space K, $C_0(K)$ has real rank zero if and only if K is totally disconnected. A C^*-algebra A is scattered if and only if every C^*-subalgebra of A has real rank zero. This gives a lot of information on the abundance of the projections in A, and also on the projection lifting property. In a future work, we plan to study the non-selfadjoint analogue of the notion of having real rank zero. We believe that the study of operator algebras with real rank zero will be somewhat related to our matricial algebras and hence to scattered operator algebras. We made a tiny attempt on this subject in this dissertation; however, our conjecture is that there is a better approach.

Approximation plays a fundamental role in mathematics, especially while working in infinite dimensions. Without the approximation theory, one can say very little about C^*-algebras (which are infinite dimensional in most cases). Approximation theory has been used for many of the fundamental results such as Voiculescu's free entropy theory, Elliott's classification program, Choi, Effros and Kirchberg's work on nuclear and exact C^*-algebras, and so on. Approximately finite-dimensional (AF) C^*-algebras have been studied deeply (see for example [18], [19], [26], [27] or [14]). Scattered C^*-algebras can be characterized as the C^*-algebras for which every C^*-subalgebra is AF. Another project that we consider is studying approximation theory for non-selfadjoint operator algebras which may be connected to the work of S. Power and others on this subject (e.g. in [45]).

Note that we defined matricial algebras using a full set of matrix units. In future, we hope to study operator algebras generated by a set of matrix units that is not full. We hope to connect this to the work of S. Power and others (e.g. D. Larson and K. Davidson).

Bibliography

[1] F. E. Alexander, *On complemented and annihilator algebras,* Glasgow Math. J. **10** (1969), 38–45.

[2] M. Almus and D. P. Blecher, in preparation.

[3] M. Almus, D. P. Blecher, and S. Sharma, *Ideals and structure of operator algebras*, to appear Journal of Operator Theory, (ArXiv: math.OA/0907.3103v5).

[4] W. Arveson, *An invitation to C^*-algebra,* Graduate Texts in Mathematics, No. 39. Springer-Verlag, New York-Heidelberg, 1976.

[5] B. Blackadar, *Theory of C^*-algebras and von Neumann Algebras,* Encyclopaedia of Mathematical Sciences, 122. Operator Algebras and Non-commutative Geometry, III. Springer-Verlag, Berlin, 2006.

[6] D. P. Blecher and B. L. Duncan, *Nuclearity-related properties for nonselfadjoint algebras,* J. Operator Theory **65** (2011), 47–70.

[7] D. P. Blecher and C. Le Merdy, *Operator algebras and their modules- an operator space approach,* London Mathematical Society Monographs. New Series, 30. Oxford Science Publications. The Clarendon Press, Oxford University Press, Oxford, 2004.

[8] D. P. Blecher, D. Hay, and M. Neal, *Hereditary subalgebras of operator algebras,* J. Operator Theory **59** (2008), no. 2, 333–357.

[9] D. P. Blecher and C. J. Read, *Operator algebras with contractive approximate identities,* Journal of Functional Analysis **261** (2011), 188–217.

[10] D. P. Blecher and M. K. Royce, *Extensions of operator algebras I,* J. Math. Anal. Appl. **339** (2008), no. 2, 1451–1467.

[11] D. P. Blecher, Z-J. Ruan, and A. M. Sinclair, *A characterization of operator algebras,* J. Funct. Anal. **89** (1990), 188–201.

[12] D. P. Blecher and V. Zarikian, *The calculus of one-sided M-ideals and multipliers in operator spaces,* Memoirs Amer. Math. Soc. **179** (2006), no. 842.

[13] F. F. Bonsall and A. W. Goldie, *Annihilator algebras,* Proc. London Math. Soc. (3) **4** (1954), 154–167.

[14] L. G. Brown, *Extensions of AF algebras: the projection lifting problem,* Amer. Math. Soc., Proc. Symp. Pure Math., Vol. 38 of Operator algebras and applications, Part II, (1982), 175 - 176.

[15] L. G. Brown, *Interpolation by projections in C^*-algebras with real rank zero,* J. Operator Theory **26** (1991), no. 2, 383–387.

[16] L. G. Brown and G. K. Pedersen, *C^*-algebras of real rank zero,* J. Funct. Anal. **99** (1991), no. 1, 131–149.

[17] L. G. Brown and G. K. Pedersen, *Limits and C^*-algebras of low rank and dimension,* J. Operator Theory **61** (2009), no. 2, 381–417.

[18] N. P. Brown and N. Ozawa, C^*-algebras and finite-dimensional approximations, Graduate Studies in Mathematics, **88**, AMS, 2008.

[19] O. Bratteli, *Inductive limits of finite dimensional C^*-algebras,* Trans. Amer. Math. Soc. **171** (1972), 195–234.

[20] M. Choi, *Lifting projections from quotient C^*-algebras,* J. Operator Theory **10** (1983), no. 1, 21–30.

[21] C. H. Chu, *A note on scattered C^*-algebras and Radon-Nikodým Property,* J. London Math. Soc. (2) **24** (1981), no. 3, 533–536.

[22] H. G. Dales, *Banach algebras and automatic continuity,* London Mathematical Society Monographs. New Series, **24**. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 2000.

[23] A. M. Davie, *Quotient algebras of uniform algebras,* J. London Math. Soc. **7** (1973), 31–40.

[24] J. Dixmier, *C^*-algebras,* North-Holland Mathematical Library, Vol. 15. North-Holland Publishing Co., Amsterdam-New York-Oxford, 1977.

[25] P. G. Dixon, *Non-closed sums of closed ideals in Banach algebras,* Proc. Amer. Math. Soc. **128** (2000), no. 12, 3647–3654.

[26] G. A. Elliott, *On the classification of inductive limits of sequences of semisimple finite-dimensional algebras,* J. Algebra **38** (1976), 29–44.

[27] G. A. Elliott, *The Mackey Borel Structure on the Spectrum of an Approximately Finite-Dimensional Separable C^*-Algebra,* Transactions of the American Mathematical Society **233** (1977), 59–68.

[28] J. Glimm, *Type I C^*-algebras,* Ann. of Math. (2) **73** (1961), 572–612.

[29] P. Harmand, D. Werner, and W. Werner, *M-ideals in Banach spaces and Banach algebras,* Lecture Notes in Math., 1547, Springer-Verlag, Berlin–New York, 1993.

[30] H. E. Jensen, *Scattered C^*-algebras,* Math. Scand. **41** (1977), no. 2, 308–314.

[31] H. E. Jensen, *Scattered C^*-algebras. II,* Math. Scand. **43** (1978), no. 2, 308–310.

[32] I. Kaplansky, *Dual rings,* Ann. of Math. (2) **49** (1948), 689–701.

[33] I. Kaplansky, *The structure of certain operator algebras,* Trans. Amer. Math. Soc. **70** (1951), 219–255.

[34] E. Kirchberg and M. Rordam, *Non-simple purely infinite C^*-algebras,* Amer. J. Math. **122** (2000), no. 3, 637–666.

[35] M. Kusuda, *A characterization of scattered C^*-algebras and its application to C^*-crossed products,* J. Operator Theory **63** (2010), no. 2, 417–424

[36] M. Kusuda, *C^*-algebras whose every C^*-subalgebra is AF,* submitted (ArXiv: math.OA/1008.3748v1).

[37] A. J. Lazar, *On scattered C^*-algebras,* preprint, University of Montana, 1980.

[38] H. Lin, *The structure of quasi-multipliers of C^*-algebras,* Transactions of the American Mathematical Society Vol. **315**, No. 1 (1989), 147–172.

[39] P. Lin, *Köthe-Bochner function spaces,* Birkháuser Boston, Inc., Boston, MA, 2004.

[40] G. J. Murphy, *C^*-algebras and operator theory,* Academic Press, Inc., Boston, MA, 1990.

[41] T. W. Palmer, *Banach algebras and the general theory of $*$-algebras, Vol. I. Algebras and Banach algebras,* Encyclopedia of Math. and its Appl., 49, Cambridge University Press, Cambridge, 1994.

[42] C. Pasnicu and M. Rordam, *Purely infinite C^*-algebras of real rank zero,* J. Reine Angew. Math. **613** (2007), 51–73.

[43] G. K. Pedersen, *C^*-algebras and their automorphism groups,* London Mathematical Society Monographs, **14**. Academic Press, London-New York, 1979.

[44] A. Pelczynski and Z. Semadeni, *Spaces of continuous functions. III. Spaces $C(\Omega)$ for Ω without perfect subsets,* Studia Math. **18** (1959), 211–222.

[45] S. C. Power, *Approximately finitely acting operator algebras,* J. Funct. Anal. **189** (2002), no. 2, 409–468.

[46] M. L. Rothwell, *Scattered C^*-algebras,* preprint, University of Pennsylvania, 1980.

[47] W. Rudin, *Continuous functions on Banach spaces without perfect subsets,* Proc. Amer. Math. Soc. **8** (1957), 39–42.

[48] Z. Semadeni, *Banach spaces of continuous functions,* Vol. I. Monografie Matematyczne, Tom 55. PWN–Polish Scientific Publishers, Warsaw, 1971.

[49] S. Sharma, *One-sided M-structure of Operator Spaces,* Ph. D. dissertation, University of Houston, 2009.

[50] C. Stegall, *The duality between Asplund spaces and spaces with the Radon-Nikodým property,* Israel J. Math. **29** (1978), no.4, 408–412.

[51] B. J. Tomiuk, *Structure theory of complemented Banach algebras,* Canad. J. Math. **14**(1962), 651–659.

[52] B. J. Tomiuk, *Multipliers on complemented Banach algebras,* Proc. Amer. Math. Soc. **115** (1992), no. 2, 397–404.

[53] B. J. Tomiuk, *Isomorphisms of multiplier algebras,* Glasgow Math. J. **28** (1986), no. 1, 73–77.

[54] S. Zhang, *On the structure of multiplier algebras,* Ph. D. dissertation, Purdue University, 1988.