CONSTRUCTING COMPLEX EQUIANGULAR PARSEVAL FRAMES

A Dissertation

Presented to

the Faculty of the Department of Mathematics

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In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

By

Helen J. Elwood

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ABSTRACT

A frame is a family of vectors which provides stable expansions for vectors in Hilbert spaces. Equiangular Parseval frames are a special class of frames possessing optimality properties for many applications. Here we explore two different approaches to constructing equiangular Parseval frames. One is combinatorial; the inner products between any two frame vectors is assumed to be a multiple of the pth roots of unity. This setting allows us to derive necessary conditions for the existence of certain complex equiangular Parseval frames, and frames of size p^2 are confirmed.

The second construction method involves an optimization problem on the matrix manifold of Parseval frames. An energy function is defined for which equiangular Parseval frames are the minimizers, if they exist. We compute the gradient in the manifold setting using equivalences between one-parameter subgroups and tangent vectors. Finally, we establish that all accumulation points of the gradient descent are fixed points.

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Chapter 1

Introduction

1.1 Background

Orthonormal bases are commonly used for representing vectors or operators on Hilbert spaces. Bases, however, are at times restrictive allowing for no linear dependence and requiring orthogonality. Therefore, it is preferable to use an overcomplete, non-orthogonal family of vectors instead of an orthonormal basis, incorporating redundancy in the representation. Frames are such families which provide stable embeddings of Hilbert spaces. They were introduced in 1952 by Duffin and Schaeffer [22] in order to generalize Fourier expansions. Daubechies, Grossman, and Meyer [20] initiated the use of frames in signal processing. More recently, frames have become popular as the flexibility in their design [38, 39] has yielded applications in coding theory [16, 29, 42] loss-insensitive data transmissions [28, 16, 34, 11, 36], engineering [48, 47], quantum communication [2, 43, 54], telecommunications [53], sigma-delta quantization [4, 5, 6, 7], and sparse reconstructions [21, 41, 52].

A family of vectors $\{f_j\}_{j\in J}$ is a frame for a Hilbert space H if the map from each vector in H to the sequence of its inner products with the frame vectors has a bounded inverse on its range. If the map is an isometry, then we speak of a Parseval frame. Parseval frames satisfy the reconstruction formula [19]

$$x = \sum_{j=1} \langle x, f_j \rangle f_j$$

for all $x \in H$, a valuable property for signal processing applications. Furthermore, Parseval frames are projections of orthonormal bases from a larger space [31] and are therefore closest in idea to an orthonormal basis. As the many applications of frame theory are ultimately performed in real-world circumstances, corresponding to a finite dimensional vector space, we are mainly concerned with finite frames.

Another structural condition of interest in applications is equiangularity. It has been shown that the special class of equiangular Parseval frames has certain optimality properties for this purpose [16, 34, 11]. These frames were initially studied by Strohmer and Heath, and Holmes and Paulsen. Holmes and Paulsen demonstrated that equiangular Parseval frames provide optimal error correction for two erasures [34]. Sustik, Tropp, Dhillon, and Heath derived necessary conditions for their existence [49]. However, the construction of these frames can be challenging.

1.2 Summary

We will consider two constructions for equiangular Parseval frames; a combinatorial approach and a gradient descent. In Chapter 2 we take the combinatorial view. In the real case, the work of Seidel and his collaborators [27, 40, 45] remains the standard source of constructions. Meanwhile the few known examples in the complex case [32, 44, 30, 54] leave fundamental, unanswered questions such as whether maximal families of complex equiangular tight frames exist in any dimension and whether they can always be generated with a group action [55]. The existence of equiangular Parseval frames is known to be equivalent to the existence of a Seidel matrix with two eigenvalues [40] which we call a signature matrix [34]. A matrix Q is a Seidel matrix provided it is self-adjoint with diagonal entries all 0 and off diagonal entries all of modulus 1. In the real case, the off diagonal entries must all be ± 1; these matrices may then be viewed as Seidel adjacency matrices of graphs. A similar graph-theoretic description and related combinatorial techniques have been used to examine the existence of complex Seidel matrices with entries that are cube roots of unity [12]. We study the existence of Seidel matrices with two eigenvalues and off-diagonal entries which are all pth roots of unity, where p is a prime, $p > 2$. The results presented here are a continuation of the efforts for the cube roots case and appeared in [9]. Essential for the derivation of necessary conditions is again the use of switching equivalence to put Seidel matrices in a standard form and thus impose additional rigidity on their structure. This allows us to rule out the existence of many Seidel matrices with two eigenvalues and thus the existence of certain complex equiangular Parseval frames, with an argument depending only on the choice of p, the number of frame vectors and the dimension of the Hilbert space. In addition to indicating the possible sizes of complex equiangular Parseval frames for $p = 5$ and $p = 7$, we confirm the existence of such frames with examples. After fixing notation and terminology in Section 1.3, we examine necessary conditions for the existence of complex Seidel matrices containing pth roots of unity and having only two eigenvalues in Section 2.1. The previously known consequences for $p = 3$ are summarized, and analogous results for $p = 5$ and $p = 7$ are developed in Section 2.2, which are complemented with examples in Section 2.3. A graph theoretic equivalency is developed and summarized in Section 2.4; this work appeared in [10] with Bodmann. In the remaining sections of Chapter 2 we explore connected results regarding an equivalency between signature matrices and complex Hadamard matrices and then investigate further existence conditions by examining higher order equations.

In Chapter 3 we pursue a completely different approach to the construction of equiangular Parseval frames. We search for these frames by applying an optimization procedure powered by gradient descent. Others [3, 15] have used frame potentials to show the existence of equal norm Parseval frames. There has also been success [8] in searching out equal norm frames inducing an ODE flow on the set of equal norm frames. Another optimization approach [46] uses an approximate geometric gradient descent powered by tangent space characterizations to construct Grassmanian frames. We take inspiration from all of this prior work and define a frame energy function in Section 3.1, and then seek out equiangular frames by feeding a Parseval frame into a function with equiangular minimizers. In Section 3.2 we explore a matrix manifold perspective and discuss the equivalencies between self-adjoint operators, one-parameter subgroups, and tangent vectors. These ideas together allow us to compute the gradient in Section 3.3 and then "step" closer to the equiangular frame.

1.3 Preliminaries

We begin by recalling some fundamental definitions and results in frame theory.

Definition 1.1. Given H, a real or complex Hilbert space, a finite family of vectors $\{f_1, f_2, ..., f_n\}$ in H is a *frame* for H if and only if there exist constants $A, B \in \mathbb{R}$ such that $A, B > 0$ and

$$A\|x\|^2 \leq \sum_{j=1}^{n} |\langle x, f_j \rangle|^2 \leq B\|x\|^2$$

for all $x \in H$. A and B are called *frame bounds* and are not unique. A frame is said to be an *A-tight frame* if we can choose $A = B$. A *normalized tight* frame, or *Parseval* frame, is a frame which admits $A = B = 1$. A frame $\{f_1, f_2, ..., f_n\}$ is called *equal norm* if there is $b > 0$ such that $\|f_j\| = b$. It is called *equiangular* if it is equal norm and if there is $c \geq 0$ such that $|\langle f_j, f_l \rangle| = c$ for all $j, l \in \{1, 2, \ldots n\}$ with $j \neq l$. Here, we are concerned mostly with equiangular Parseval frames for $\mathbb{C}^k$, equipped with the canonical inner product. We use the term (n, k)*-frame* to refer to a Parseval frame of n vectors for $\mathbb{C}^k$.

Our construction of equiangular Parseval frames makes use of an equivalence relation among frames [28] (see also [34, 11]).

Definition 1.2. Two frames, $\{f_1, f_2, ..., f_n\}$, and $\{g_1, g_2, ...g_n\}$ for a real or complex Hilbert space H are said to be *unitarily equivalent* if there exists a unitary operator U on H such that $g_j = Uf_j$ for all $1 \leq j \leq n$. Furthermore, we say that they are *switching equivalent* if there exists a unitary operator U on H, a permutation π on $\{1, 2, ..., n\}$ and a set of unimodular constants $\{\alpha_1, \alpha_2, ..., \alpha_n\}$ such that for each $j \in \{1, 2, \ldots n\}$, $g_j = \alpha_j U f_{\pi(j)}$.

It is well known [19] that the definition of a Parseval frame is equivalent to the reconstruction identity $x = \sum_{j=1} \langle x, f_j \rangle f_j$ for all $x \in H$.

Notice that switching a frame, meaning mapping all frame vectors with a unitary, permuting them and multiplying them with unimodular constants, leaves the reconstruction identity unchanged. From this point of view, it is very natural to identify two frames that can be obtained from each other by switching. We use switching equivalence to choose particular representatives of equivalence classes and derive essential properties of equiangular tight frames.

With a frame $F = \{f_1, f_2, \ldots, f_n\}$ for a real or complex Hilbert space H, we associate its analysis operator $V : H \to \ell^2(\{1, 2 \ldots n\})$ which maps $x \in H$ to its frame coefficients, $(Vx)_j = \langle x, f_j \rangle$ and its synthesis operator, the Hilbert adjoint V^*, with $V^*(y) = \Sigma_j^n f_j y_j$.

The synthesis operator is a left inverse to the analysis operator as

$$V^*V(x) = V^*((\langle x, f_j \rangle)) = \Sigma_j \langle x, f_j \rangle f_j = x.$$

This composition is called the frame operator while $VV^* = (f_i^* f_j)_{i,j} = (\langle f_j, f_i \rangle)_{i,j}$ is the Gram matrix. Essential properties of the frame F are encoded in the Grammian. For example, if F is equal norm then it is clear that the diagonal entries of the Grammian are identical and $(VV^*)_{j,j} = \|f_j\|^2 = b^2$ for some $b > 0$. Furthermore, if F is a Parseval frame, then V is an isometry as

$$||(Vx)||_2^2 = \Sigma_j^n |(Vx)_j|^2 = \Sigma_j^n |\langle x, f_j \rangle|^2 = ||x||^2$$

by the Parseval identity and so VV^* is an orthogonal projection. Consequently, for an equal-norm (n,k)-frame, $F = \{f_1, f_2, \ldots, f_n\}$, the trace of the Grammian is equal to its rank and thus $\|f_j\|^2 = k/n$ for all $1 \leq j \leq n$. Additionally, if F is an equiangular (n,k)-frame then the Frobenius norm of the Grammian is equal to the square root of its rank, and $|\langle f_j, f_l \rangle| = c_{n,k} := \sqrt{\frac{k(n-k)}{n^2(n-1)}}$, for all $j \neq l$ ([48], [34], see also [26]). This yields that the Grammian of an equiangular (n,k)-frame is of the form

$$VV^* = (\frac{k}{n})I_n + c_{n,k}Q,$$

where Q is a self-adjoint $n \times n$ matrix, with diagonal entries equal to 0, and off-diagonal entries all with modulus equal to 1. The matrix Q is called the *signature matrix* associated with the equiangular (n,k)-frame, $\{f_1, f_2, ..., f_n\}$.

Chapter 2

A Combinatorial Approach to the Design of Equiangular Parseval Frames

Our combinatorial chapter begins with a search for necessary conditions for the existence of complex Seidel matrices containing pth roots of unity. First, recall some important conclusions due to Seidel, and Holmes and Paulsen which characterizes the signature matrices of equiangular (n, k)-frames.

Theorem 2.1. *([45], and Proposition 3.2 and Theorem 3.3 of [34]) Let Q be a self-adjoint $n \times n$ matrix with $Q_{j,j} = 0$ and $|Q_{j,l}| = 1$ for all $j \neq l$, then the following three properties are equivalent:*

1. *Q is the signature matrix of an equiangular (n, k)-frame for some k;*
2. *$Q^2 = (n-1)I + \mu Q$ for some necessarily real number μ; and*
3. *Q has exactly two eigenvalues.*

Additionally, any matrix Q satisfying any of the three equivalent conditions has eigenvalues $\lambda_1 < 0 < \lambda_2$ for which the following five identities hold:

$$\mu = (n-2k)\sqrt{\frac{n-1}{k(n-k)}} = \lambda_1 + \lambda_2\,,$$
$$k = \frac{n}{2} - \frac{\mu n}{2\sqrt{4(n-1)+\mu^2}}\,,$$
$$\lambda_1 = -\sqrt{\frac{k(n-1)}{n-k}}, \quad \lambda_2 = \sqrt{\frac{(n-1)(n-k)}{k}}, \qquad \textit{and} \qquad n = 1 - \lambda_1\lambda_2\,.$$

When all of the entries of Q are real, Q must have diagonal entries equal to 0 and off-diagonal entries of ± 1. It has been shown (Theorem 3.10 of [34]) that in this case there is a one-to-one correspondence between the switching equivalence classes of real equiangular tight frames and regular two-graphs [45]. I'll refer to equation (2) from this result as the Seidel-Holmes-Paulsen equation from here on.

2.1 Design of signature matrices containing pth roots of unity

When switching from a frame $F = \{f_1, f_2, \ldots, f_n\}$ to $G = \{g_1, g_2, \ldots, g_n\}$ given by $g_j = \alpha_j U f_{\pi(j)}$, then the signature matrix associated with G is obtained by conjugating the signature matrix of F with a diagonal unitary and with a permutation matrix. This motivates the following definition of switching equivalence for Seidel matrices.

Definition 2.2. Two Seidel matrices Q and Q' are said to be *switching equivalent* if they can be obtained from each other by conjugating with a diagonal unitary and with a permutation matrix. Furthermore, we say that a Seidel matrix Q is in standard form provided its first row and column contains all 1s, except on the diagonal (which must be 0). We say that Q is *trivial* if its standard form has all off-diagonal entries equal to 1 and *nontrivial* if at least one off-diagonal entry is not equal to 1.

Two switching equivalent Seidel matrices have the same spectrum, since they are related by conjugation with a unitary. As the equivalence class of any Seidel matrix contains a matrix in standard form we may focus on examining matrices of this form with two eigenvalues. One goal is to find necessary conditions for the existence of certain Seidel matrices with two eigenvalues and hence for the existence of equiangular tight frames. In the real case Seidel and others [40, 45] established necessary and sufficient conditions in graph theoretic terms. A similar graph theoretic formulation was used to derive necessary conditions for complex equiangular tight frames when the the off-diagonal entries are cube roots of unity [12]. The fourth roots case was considered in [23]. Here we explore nontrivial standard Seidel matrices with off-diagonal entries which are pth roots of unity, for p prime, $p > 2$. These cases add to the description of families of complex equiangular tight frames, in analogy with the previous results. Much of the material in sections 2.1, 2.2 and 2.3 appeared as joint work with B. G. Bodmann in [9] and the results in section 2.4 have been published with B. G. Bodmann in [10].

To begin here, we consider nontrivial signature matrices whose off diagonal entries are pth roots of unity. Let $p \in \mathbb{N}$, $\omega = e^{2\pi i/p}$, and accordingly $\{1, \omega, \omega^2, ...\omega^{p-1}\}$ be the set of pth roots of unity. The overall strategy followed here mimics the treatment of $p = 3$ in [12], with some modifications that allow us to address the general pth roots case.

2.1.1 Signature matrices in standard form

Definition 2.3. For $p \in \mathbb{N}$, a matrix Q is a pth root Seidel matrix if it is self-adjoint, with diagonal entries all equal to 0 and off-diagonal entries which are all pth roots of unity. If, in addition, Q has exactly two eigenvalues, then Q is the pth root signature matrix of an equiangular tight frame.

The following lemma is verbatim as in the cube roots case.

Lemma 2.4. *(Lemma 3.2 in [9]) If Q' is an $n \times n$ pth root Seidel matrix, then it is switching equivalent to a pth root Seidel matrix of the form*

$$Q = \begin{pmatrix} 0 & 1 & \dots & \dots & 1 \\ 1 & 0 & * & \dots & * \\ \vdots & * & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & * \\ 1 & * & \dots & * & 0 \end{pmatrix}$$

where the entries marked with $*$ are pth roots of unity. Moreover, Q' is the signature matrix of an equiangular (n, k)-frame if and only if Q is the signature matrix of an equiangular (n, k)-frame.

Proof. Suppose that Q' is an $n \times n$ pth root Seidel matrix. So Q' is self-adjoint, $|Q_{j,l}| = 1$, for $j \neq l$, and $(Q')^2 = (n-1)I + \mu Q'$ for some $\mu \in \mathbb{R}$, by Theorem 2.1. Let U be the diagonal matrix with non-zero entries $U_{1,1} = 1$ and $U_{j,j} = Q'_{1,j}$, $j \in \{2, 3, \dots n\}$. Then, U is a unitary matrix, as $|Q'_{j,l}| = 1$ when $j \neq l$. Define $Q = UQ'U^*$ and note that Q is a self-adjoint $n \times n$ matrix with $Q_{j,j} = 0$ and $|Q_{j,l}| = 1$ when $j \neq l$. The off-diagonal entries of Q are pth roots of unity and as $Q'_{j,l} = \overline{Q'_{l,j}}$, the off-diagonal entries of the first row and first column are 1's. Therefore Q has the proposed form. So Q is a pth root Seidel matrix that is unitarily equivalent to Q'. As Q and Q' have the same eigenvalues, if one of them is the signature matrix of an equiangular (n, k)-frame , then so is the other. $\square$

Next we include a lemma concerning linear combinations of pth roots of unity with rational coefficients. This lemma is essential for deriving necessary conditions of pth root Seidel matrices having only two eigenvalues.

Lemma 2.5. *(Lemma 3.3 in [9]) Let* $\omega = e^{2\pi i/p}$*, where p is prime. If* $a_0, a_1, a_2, ..., a_{p-1} \in \mathbb{Q}$ *and* $a_0 1 + a_1\omega + a_2\omega^2 + ... + a_{p-1}\omega^{p-1} = 0$*, then* $a_0 = a_1 = a_2 = ... = a_{p-1}$.

Proof. Suppose that $a_0, a_1, a_2, ..., a_{p-1} \in \mathbb{Q}$ and

$$a_0 1 + a_1\omega + a_2\omega^2 + ... + a_{p-1}\omega^{p-1} = 0 \tag{2.1}$$

First we show that we can reduce Equation (2.1) to an equation in terms of $\{1, \omega, \omega^2, \cdots, \omega^{p-2}\}$. As $\omega^p = 1$, we know that $\omega^p - 1 = 0$, so $(\omega - 1)(\omega^{p-1} + \omega^{p-2} + \cdots + \omega^2 + \omega + 1) = 0$. Since $\omega \neq 1$, $1 + \omega + \omega^2 + \cdots + \omega^{p-1} = 0$, and therefore

$$a_{p-1} + a_{p-1}\omega + a_{p-1}\omega^2 + ... + a_{p-1}\omega^{p-1} = 0 \tag{2.2}$$

Subtracting Equation (2.2) from Equation (2.1), we see that

$$(a_0 - a_{p-1}) + (a_1 - a_{p-1})\omega + (a_2 - a_{p-1})\omega^2 + ... + (a_{p-2} - a_{p-1})\omega^{p-2} = 0 \tag{2.3}$$

But ω is a primitive pth root of unity, so the degree of $\mathbb{Q}(\omega)$ over $\mathbb{Q}$, that is $[\mathbb{Q}(\omega) : \mathbb{Q}] = \varphi(p) = p - 1$, where φ is the Euler function (see Proposition 8.3, p.299 in [35]). Therefore the minimal irreducible polynomial of ω over $\mathbb{Q}$ has degree $p - 1$. Specifically, this polynomial is the pth cyclotomic polynomial.

Since the degree of $f(x) = (a_0 - a_{p-1}) + (a_1 - a_{p-1})x + (a_2 - a_{p-1})x^2 + ... + (a_{p-2} - ap - 1)x^{p-2}$ is $p - 2$, which is smaller than the degree of the minimal irreducible polynomial of ω, we conclude from Equation (2.3) that $f(x)$ must be the zero polynomial. Therefore $(a_0 - a_{p-1}), (a_1 - a_{p-1}), (a_2 - a_{p-1}), ..., (a_{p-2} - a_{p-1}) = 0$, and so $a_0 = a_1 = a_2 = ... = a_{p-1}$. □

Notice that this result implies that $\{1, \omega, \omega^2, \ldots, \omega^{p-1}\}$ is a linearly dependent set over the rational numbers whereas $\{1, \omega, \omega^2, \ldots \omega^{p-2}\}$ is linearly independent. We can now apply these lemmas to further describe the structure and entries of Seidel matrices in standard form.

Theorem 2.6. *(Theorem 3.4 in [9]) Let Q be a nontrivial pth root Seidel matrix in standard form, where p is prime, and $Q^2 = (n-1)I + \mu Q$ for some $\mu \in \mathbb{R}$, then $e := \frac{n-\mu-2}{p}$ is an integer, and for any l with $2 \leq l \leq n$, the lth column of Q (and similarly the lth row) contains e entries equal to ω, e entries equal to ω^2 $\ldots$ and e entries equal to ω^{p-1}, and contains $e + \mu + 1 = \frac{n+(p-1)\mu+(p-2)}{p}$ entries equal to 1.*

Proof. For $2 \leq j \leq n$, p prime, define

$$
\begin{aligned}
x_{1,l} &:= \#\{i : Q_{j,l} = 1\}, \\
x_{2,l} &:= \#\{i : Q_{j,l} = \omega\}, \\
x_{3,l} &:= \#\{i : Q_{j,l} = \omega^2\}, \\
&\cdots \\
x_{p,l} &:= \#\{i : Q_{j,l} = \omega^{p-1}\}.
\end{aligned}
$$

Since the lth column of Q has $n-1$ non-zero entries (recall the zero on the diagonal), we have

$$x_{1,l} + x_{2,l} + ... + x_{p,l} = n - 1. \tag{2.4}$$

Also, since $Q^2 = (n-1)I + \mu Q$, and Q is in standard form, for $2 \leq l \leq n$,

$$\mu = \mu Q_{1,l} = [(n-1)I + \mu Q]_{1,l} = (Q^2)_{1,l} = (x_{1,l} - 1)1 + x_{2,l}\omega + x_{3,l}\omega^2 + ... + x_{p,l}\omega^{p-1}.$$

Therefore, $(x_{1,l} - \mu - 1) + x_{2,l}\omega + x_{3,l}\omega^2 + ... + x_{p,l}\omega^{p-1} = 0$, and so by Lemma 2.5,

$$x_{1,l} - \mu - 1 = x_{2,l} = x_{3,l} = ... = x_{p,l}. \tag{2.5}$$

Using these identities to eliminate $x_{j,l}$ with $j \geq 2$ in Equation (2.4) gives $x_{1,l} + (p - 1)(x_{1,l} - \mu - 1) = n - 1$, and we conclude

$$x_{1,l} = \frac{n + (p-1)\mu + (p-2)}{p}. \tag{2.6}$$

Equation (2.5) hence shows that for all $2 \leq j \leq p$,

$$x_{j,l} = \frac{n - \mu - 2}{p} \tag{2.7}$$

Since the quantities in (2.6) and (2.7) do not depend on l, they are valid for any column. In addition, since $Q = Q^*$ and $\overline{\omega^m} = \omega^{p-m}$ for all $1 \leq m \leq p$, the same equations hold for the rows of the Seidel matrix Q. □

2.1.2 The structure of nontrivial pth root signature matrices

Let Q be a pth root Seidel matrix, with p prime, and define

$$\alpha_{a,b} := \#\{k : Q_{j,k} = \omega^a \textit{ and } Q_{k,l} = \omega^{p-b}\},$$

for all $a, b \in \mathbb{Z}$. The implicit identity $\alpha_{a,b} = \alpha_{a,b\pm p}$ helps simplify notation in the computations below. We also define $R_t := \sum_{s=1}^{p} \alpha_{t,s}$ for $1 \leq t \leq p$, $C_t := \sum_{s=1}^{p} \alpha_{s,t}$ for $1 \leq t \leq p$, and $Z_t := \sum_{s=1}^{p} (\alpha_{s,s} - \alpha_{s,s-t})$ for $1 \leq t \leq p-1$.

From her on, we use modular arithmetic. If $q, r \in \mathbb{Z}$ and $p \in \mathbb{N}$, then $q \equiv r (mod\ p)$ means that $q - r$ is an integer multiple of p. On the other hand, when writing $q = r(mod\,p)$ it is implicit that $0 \leq q \leq p-1$.

Lemma 2.7. *(Lemma 3.5 in [9]) Suppose that Q is a nontrivial pth root Seidel matrix, with p prime. Additionally suppose that Q is in standard form and satisfies $Q^2 = (n-1)I + \mu Q$, then the following system of linear equations hold:*

1. *$R_1 = e - 1$, $R_t = e$ for $2 \leq t \leq p-1$, and $R_p = e + \mu + 1$,*

2. *$C_t = e$ for $1 \leq t \leq p-2$, $C_{p-1} = e - 1$, and $C_p = e + \mu + 1$,*

3. *$Z_1 = -\mu$, and $Z_t = 0$ for $2 \leq t \leq p-1$.*

These $(3p-1)$ equations fix the values of the row sums of Q, the column sums of Q, and the difference computed by subtracting the sum of cyclic off-diagonals of Q from the main diagonal.

Proof. As Q is nontrivial we know that $Q_{j,l} \neq 1$ for some $2 \leq j, l \leq n$ with $j \neq l$. Without loss of generality, let $Q_{j,l} = \omega$. Then the number of ωs in row j is $\alpha_{1,1} + \alpha_{1,2} + \cdots + \alpha_{1,p} + 1$ by the definition of α, with the $+1$ term coming from $\alpha_{j,l} = \omega$. We know that the number of ωs in row j is also equal to $e = \frac{n-\mu-2}{p}$ from Theorem 2.6. So $R_1 = \sum_{s=1}^{p} \alpha_{1,t} = e - 1$.

For $2 \leq t \leq p-1$, the number of ω^ts in row j is $\alpha_{t,1} + \alpha_{t,2} + \cdots + \alpha_{t,p}$, and by Theorem 2.6, the number of ω^ts in row j is e, so $R_t = \sum_{s=1}^{p} \alpha_{t,s} = e$.

Also, the number of 1s in row j is $\alpha_{p,1} + \alpha_{p,2} + \cdots + \alpha_{p,p}$ and by Theorem 2.6, the number of 1s in row j is $e + \mu + 1$, so $R_p = \sum_{s=1}^{p} \alpha_{p,s} = e + \mu + 1$.

As Theorem 2.6 holds for columns as well as for rows, we know that the number of $\omega^{(p-t)}$ in column l is $C_t = \sum_{s=1}^{p} \alpha_{s,t} = e$ for all $1 \leq l \leq p-2$.

The number of ωs in column l (remembering that $Q_{j,l} = \omega$) is $C_{(p-1)} = \sum_{s=1}^{p} \alpha_{s,(p-1)} = e - 1$, and the number of 1s in column l is $C_p = \sum_{s=1}^{p} \alpha_{s,p} = e + \mu + 1$.

Furthermore, since $Q^2 = (n-1)I + \mu Q$, we have that

$$\mu\omega = \mu Q_{i,j} = [(n-1)I + \mu Q]_{i,j} = (Q^2)_{i,j} = \sum_{k=1}^{n} Q_{i,k} Q_{k,j} = \sum_{j,l=1}^{p} (\alpha_{j,l})(\omega^j)(\omega^{p-l}).$$

Collecting powers of ω, we get $(\alpha_{1,1} + \alpha_{2,2} + \alpha_{3,3} + \cdots + \alpha_{p,p})1 + (\alpha_{1,p} + \alpha_{2,1} + \alpha_{3,2} + \cdots + \alpha_{p,(p-1)} - \mu)\omega + (\alpha_{1,(p-1)} + \alpha_{2,p} + \alpha_{3,1} + ... + \alpha_{p,(p-2)})\omega^2 + \cdots + (\alpha_{1,2} + \alpha_{2,3} + \alpha_{3,4} + \cdots + \alpha_{p,1})\omega^{p-1} = 0$.

That is, $\sum_{s=0}^{p-1} \sum_{t=1}^{p} (\alpha_{t,t-s}\omega^j) - \mu\omega = 0$.

It follows from Lemma 2.5 that

$$\begin{aligned}
\alpha_{1,1} + \alpha_{2,2} + \alpha_{3,3} + \cdots + \alpha_{p,p} &= \alpha_{1,p} + \alpha_{2,1} + \alpha_{3,2} + \cdots + \alpha_{p,(p-1)} - \mu \\
&= \alpha_{1,(p-1)} + \alpha_{2,p} + \alpha_{3,1} + \cdots + \alpha_{p,(p-2)} \\
&\vdots \\
&= \alpha_{1,2} + \alpha_{2,3} + \alpha_{3,4} + \cdots + \alpha_{p,1}
\end{aligned}$$

and therefore, the following $p-1$ equations hold:

$$\begin{aligned}
Z_1 &= \alpha_{1,1} - \alpha_{1,p} + \alpha_{2,2} - \alpha_{2,1} + \alpha_{3,3} - \alpha_{3,2} + \cdots + \alpha_{p,p} - \alpha_{p,(p-1)} = -\mu, \text{and} \\
Z_t &= \sum_{s=1}^{p} (\alpha_{s,s} - \alpha_{s,s-t}) = 0 \text{ for } 2 \leq t \leq p-1\,.
\end{aligned}$$

□

So when the hypotheses of Lemma 2.7 are satisfied, we have a total of $(3p-1)$ equations in p^2 unknowns which must also be satisfied. We state some results exploring the consequences of this lemma.

Theorem 2.8. *(Theorem 3.6 in [9]) For $p \in \mathbb{N}$ prime, $p > 2$, let Q be a nontrivial pth root signature matrix of an equiangular (n,k)-frame, satisfying $Q^2 = (n-1)I + \mu Q$, then the following assertions hold:*

1. *The value μ is an integer and $\mu \equiv (p-2)(mod\ p)$.*

2. *The integer n satisfies $n \equiv 0(mod\ p)$.*

3. *If $\lambda_1 < 0 < \lambda_2$ are the eigenvalues of Q, then λ_1 and λ_2 are integers with $\lambda_1 \equiv (p-1)(mod\ p)$ and $\lambda_2 \equiv (p-1)(mod\ p)$.*

4. *The integer $4(n-1)+\mu^2$ is a perfect square and $4(n-1)+\mu^2 \equiv 0(mod\ p^2)$.*

Proof. Using the same notation as in Lemma 2.7

$$R_p = \alpha_{p,1} + \alpha_{p,2} + \alpha_{p,3} + ... + \alpha_{p,p} = e + \mu + 1.$$

implies that $(\alpha_{p,1} + \alpha_{p,2} + \alpha_{p,3} + ... + \alpha_{p,p})$ is an integer and e is an integer, so μ must also be an integer.

To prove the first assertion, we define $q = \frac{p-1}{2}$ and introduce the following coefficients:

$$r_t = \begin{cases} \frac{2-t}{p} & \text{for } 1 \le t \le q+2, \\ \frac{p+2-t}{p} & \text{for } q+3 \le t \le p; \end{cases}$$

$$c_t = \begin{cases} \frac{t-1}{p} & \text{for } 1 \le t \le q+1, \\ \frac{t-1-p}{p} & \text{for } q+2 \le t \le p; \end{cases}$$

$$z_t = \begin{cases} \frac{1-t}{p} & \text{for } 1 \le t \le q+1, \\ \frac{p+1-t}{p} & \text{for } q+2 \le t \le p-1. \end{cases}$$

Applying Lemma 2.7, we see that

$$\sum_{t=1}^{p} r_t R_t + \sum_{t=1}^{p} c_t C_t + \sum_{t=1}^{p-1} z_t Z_t = [\frac{1}{p}(e-1) - \frac{1}{p}(e) - \frac{2}{p}(e) - \frac{3}{p}(e) - \cdots - \frac{q}{p}(e)$$
$$+ \frac{q}{p}(e) + \frac{q-1}{p}(e) + \frac{q-2}{p}(e) + \cdots + \frac{3}{p}(e) + \frac{2}{p}(e+\mu+1)]$$

$$
\begin{aligned}
&+ [\frac{1}{p}(e) + \frac{2}{p}(e) + \frac{3}{p}(e) + \cdots + \frac{q}{p}(e) - \frac{q}{p}(e) - \frac{q-1}{p}(e) \\
&- \frac{q-2}{p}(e) - \cdots - \frac{3}{p}(e) - \frac{2}{p}(e-1) - \frac{1}{p}(e+\mu+1)] \\
&+ [-\frac{1}{p}(0) - \frac{2}{p}(0) - \frac{3}{p}(0) - \cdots - \frac{q}{p}(0) \\
&+ \frac{q}{p}(0) + \frac{q-1}{p}(0) + \frac{q-2}{p}(0) + \cdots + \frac{2}{p}(0)] \\
&= \frac{\mu}{p} + \frac{2}{p}.
\end{aligned}
$$

Now we define $\{b_{j,l}\}_{j,l=1}^{p}$ to be the coefficients of $\{\alpha_{j,l}\}_{j,l=1}^{p}$ in the expression

$$\sum_{j,l=1}^{p} b_{j,l}\alpha_{j,l} = \sum_{t=1}^{p} r_t R_t + \sum_{t=1}^{p} c_t C_t + \sum_{t=1}^{p-1} z_t Z_t. \tag{2.8}$$

By the definition of R_t, C_t and Z_t, the expression (2.8) is a rational linear combination of $\{\alpha_{j,l}\}_{j,l=1}^{p}$. Our goal is to show that, in fact, it is a linear combination with integer coefficients $\{b_{j,l}\}_{j,l=1}^{p}$. As each $\alpha_{j,l}$ is an integer, this would show that the expression (2.8) is an integer.

To accomplish this, we consider five main cases of coefficients $\{b_{j,l}\}_{j,l=1}^{p}$; Case 1: $j = l$; Case 2: $j \neq l$, $1 \leq j \leq q+2$, $1 \leq l \leq q+1$; Case 3: $j \neq l$, $1 \leq j \leq q+2$, $p \geq l > q+1$; Case 4: $j \neq l$, $p \geq j > q+2$, $1 \leq l \leq q+1$; and Case 5: $j \neq l$, $p \geq j > q+2, p \geq l > q+1$. Distinguishing these cases is necessary because of the piecewise definition of r_t, c_t and z_t. To simplify notation when computing contributions from Z_t in expression (2.8), we define $s = (j-l)(mod\ p)$ and recall that our convention for modular arithmetic implies $0 \leq s \leq p-1$.

Case 1 We first consider the case when $j = l$. If $j = l = 1$, then as $\alpha_{1,1}$ appears in R_1, C_1, and in Z_t for all $1 \leq t \leq p-1$, the coefficient $b_{1,1}$ in expression (2.8) is $\frac{1}{p} + 0 + \sum_{t=2}^{q+1} \frac{1-t}{p} + \sum_{t=q+2}^{p-1} \frac{p+1-t}{p} = \frac{1}{p} + 0 - \frac{1}{p} = 0$.

If $j = l$ and $1 < j < q+2$, then $\alpha_{j,j}$ appears in R_j, C_j, and in Z_t for all $1 \leq t \leq p-1$, so the coefficient of $\alpha_{j,j}$ in expression (2.8) is $b_{j,j} = \frac{2-j}{p} + \frac{j-1}{p} + \frac{-1}{p} = 0$, whereas if $j \neq 1$ and $j = l > q+2$, then the coefficient is $b_{j,j} = \frac{p+2-j}{p} + \frac{j-p-1}{p} - \frac{1}{p} = 0$.

Finally, if $j = l = q+2$, then the coefficient of $\alpha_{j,l} = \alpha_{q+2,q+2}$ in expression (2.8) is $b_{q+2,q+2} = -\frac{q}{p} - \frac{q}{p} - \frac{1}{p} = -1$.

We conclude that if $j = l$, then the coefficient $b_{j,l}$ is an integer.

Case 2 For the remainder of the proof we focus on the coefficient of $\alpha_{j,l}$, where $j \neq l$. Note as $j \neq l$, $s \in \{1, 2, 3, ..., (p-1)\}$.

Let $1 \leq j \leq q+2, 1 \leq l \leq q+1$. Now suppose that $1 \leq s \leq q+1$, then the coefficient of $\alpha_{j,l}$, in expression (2.8) is $b_{j,l} = \frac{2-j}{p} + \frac{l-1}{p} + \frac{-(1-s)}{p} = \frac{l-j+s}{p} \in \mathbb{Z}$ as $s = (j-l)(mod\ p)$. If instead, $q+2 \leq s < p$, then $b_{j,l} = \frac{2-j}{p} + \frac{l-1}{p} + \frac{-(p+1-s)}{p} = \frac{j-l+s-p}{p} \in \mathbb{Z}$ as $s = (j-l)(mod\ p)$. Thus, for $1 \leq j \leq q+2, 1 \leq l \leq q+1$, the coefficient of $\alpha_{j,l}$, in expression (2.8) is an integer.

Case 3 Now let $1 \leq j \leq q+2, q+1 < l \leq p$. Suppose that $1 \leq s \leq (q+1)$, then $b_{j,l} = \frac{2-j}{p} + \frac{l-p-1}{p} + \frac{-(1-s)}{p} = \frac{l-j+s-p}{p} \in \mathbb{Z}$ as $s = (j-l)(mod\ p)$. If instead, $q+2 \leq s < p$, then $b_{j,l} = \frac{2-j}{p} + \frac{l-p-1}{p} + \frac{-(p+1-s)}{p} = \frac{l-j+s-2p}{p} \in \mathbb{Z}$ as $s = (j-l)(mod\ p)$. Therefore, when $1 \leq j \leq q+2$, $q+1 < l \leq p$, we have that the coefficient $b_{j,l}$ is an integer.

Case 4 Next, let $q+2 < j \leq p, 1 \leq l \leq q+1$ Suppose that $1 \leq s \leq q+1$, then $b_{j,l} = \frac{p+2-j}{p} + \frac{(l-1)}{p} + \frac{-(1-s)}{p} = \frac{l-j+s+p}{p} \in \mathbb{Z}$ as $s = (j-l)(mod\ p)$. If instead, $q+2 \leq s < p$, then the coefficient of $\alpha_{j,l}$ is $b_{j,l} = \frac{p+2-j}{p} + \frac{(l-1)}{p} + \frac{-(p+1-s)}{p} = \frac{l-j+s}{p} \in \mathbb{Z}$ as $s = (j-l)(mod\ p)$. Thus, if $q+2 < j \leq p$ and $1 \leq l \leq q+1$, then $b_{j,l}$ is an integer.

Case 5 Lastly, let $q+2 < j \leq p$ and $q+1 < l \leq p$. Now suppose that $1 \leq s \leq (q+1)$, then $b_{j,l} = \frac{p+2-j}{p} + \frac{l-1-p}{p} + \frac{-(1-s)}{p} = \frac{l-j+s}{p} \in \mathbb{Z}$ as $s = (j-l)(mod\ p)$. If instead,

$q+2 \leq s < p$, then the coefficient is $b_{j,l} = \frac{p+2-j}{p} + \frac{l-1-p}{p} + \frac{-(p+1-s)}{p} = \frac{l-j+s-p}{p} \in \mathbb{Z}$ as $s = (j-l)(mod\ p)$. Therefore, if $q+2 < j \leq p$ and $q+1 < l \leq p$, then the coefficient of $\alpha_{j,l}$, in expression (2.8) is an integer.

Having covered all cases, we conclude that the expression (2.8) is indeed an integer linear combination of $\{\alpha_{j,l}\}_{j,l=1}^{p}$, and as each $\alpha_{j,l}$ is an integer, so is the entire expression (2.8). Recalling that $\sum_{t=1}^{p}(r_tR_t + c_tC_t) + \sum_{t=1}^{p-1} z_tZ_t = (\mu+2)/p$, we see that $\mu+2 \equiv 0(mod\ p)$, and therefore, $\mu \equiv (p-2)(mod\ p)$.

To prove assertion (2) of this theorem, note that as $Q^2 = (n-1)I + \mu Q$, by Theorem 2.6, we have that $e = \frac{n-\mu-2}{p}$ is an integer. So $n-\mu-2 \equiv 0(mod\ p)$, and since $\mu \equiv (p-2)(mod\ p), n \equiv \mu+2(mod\ p) \equiv 0(mod\ p)$.

For assertion (3), we recall the equations in Theorem 2.1, $\mu = (n-2k)\sqrt{\frac{(n-1)}{k(n-k)}} = \lambda_1 + \lambda_2$. Since μ is an integer by assertion (1), we have $\sqrt{\frac{(n-1)}{k(n-k)}} \in \mathbb{Q}$ and $\lambda_1 = -\sqrt{\frac{k(n-1)}{(n-k)}} = -k\sqrt{\frac{(n-1)}{k(n-k)}} \in \mathbb{Q}$. In addition we know that $\lambda_2 = \frac{1-n}{\lambda_1} \in \mathbb{Q}$. Therefore, λ_1 and λ_2 are both rational. Since $Q^2 = (n-1)Q + \mu Q$, the polynomial $p(x) = x^2 - \mu x - (n-1)$ annihilates Q. So the minimal polynomial of Q divides $p(x)$ and λ_1 and λ_2 are rational roots of $p(x)$. However, the coefficients of $p(x)$ are all integers and the leading coefficient is 1, so by the Rational Root Theorem (see Lemma 6.11 in [35]), λ_1 and λ_2 are integers. Now, $\lambda_1 + \lambda_2 = \mu \equiv (p-2)(mod\ p)$ by part(1), and $\lambda_1\lambda_2 = 1-n \equiv 1(mod\ p)$, by part (2)) and the equations in Theorem 2.1, with $\lambda_1(mod\ p), \lambda_2(mod\ p) \in \{0,1,2,3,\ldots,(p-1)\}$. So $\lambda_2 = (p-2)-\lambda_1$, and $\lambda_1\lambda_2 = \lambda_1[(p-2)-\lambda_1] = 1$. Therefore, $\lambda_1 p - 2\lambda_1 - \lambda_1^2 = -2\lambda_1 - \lambda_1^2 = 1$. So, $\lambda_1^2 + 2\lambda_1 + 1 = 0$, that is, $\lambda_1^2 + 2\lambda_1 + 1 = mp$, for some $m \in \mathbb{Z}$. Using the quadratic formula, the roots of $\lambda_1^2 + 2\lambda_1 + (1-mp) = 0$, are $\lambda_1 = \frac{-2\pm\sqrt{4-4(1)(1-mp)}}{2} = -1 \pm \sqrt{mp}$. $\sqrt{mp}$ must be an integer, as $\lambda_1 \in \mathbb{Z}$. Since p is prime, m must therefore be a multiple of p, say $m = lp$, where l is a perfect square. So $\lambda_1 = -1 \pm \sqrt{mp} = -1 \pm \sqrt{l}p$ with $\sqrt{l} \in \mathbb{Z}$, and therefore $\lambda_1 \equiv (p-1)(mod\,p)$. Finally, $\lambda_2 = (p-2)-\lambda_1 \equiv ((p-2)-(p-1))(mod\,p) \equiv (p-1)(mod\ p)$.

To prove assertion(4) we use the fact that $k = \frac{n}{2} - \frac{\mu n}{2\sqrt{4(n-1)+\mu^2}}$ from the Theorem 2.1 equations. Therefore, $\sqrt{4(n-1)+\mu^2} = \frac{\mu n}{n-2k} \in \mathbb{Q}$ by part (1). $n, \mu \in \mathbb{Z}$ so $(4(n-1)+\mu^2) \in \mathbb{Z}$. Thus $\sqrt{4(n-1)+\mu^2} \in \mathbb{Q}$ if and only if $\sqrt{4(n-1)+\mu^2} \in \mathbb{Z}$. So $\sqrt{4(n-1)+\mu^2} = m \in \mathbb{Z}$ and therefore, $4(n-1)+\mu^2 = m^2$, that is, $4(n-1)+\mu^2$ is a perfect square. Furthermore, since $4(n-1)+\mu^2 = m^2$ and $\mu \equiv (p-2)(mod\, p)$, $n \equiv 0(mod\, p)$, by parts (1) and (2), we see that, $4(n-1)+\mu^2 \equiv 0(mod\, p)$. So $m^2 = 0(mod\, p)$. Therefore p divides m^2, but since p is prime, p must divide m, and therefore p^2 divides $m^2 = 4(n-1)+\mu^2$ and $4(n-1)+\mu^2 \equiv 0(mod\, p^2)$. □

A couple of immediate corollaries allow us to further characterize n, μ, λ_1, and λ_2.

Corollary 2.9. *(Corollary 3.7 in [9]) For p prime, $p > 2$, let Q be a nontrivial pth root signature matrix of an equiangular (n,k)-frame such that $Q^2 = (n-1)I + \mu Q$. Then there is $m \in \{0,1,2,\ldots,(p-1)\}$ such that $n \equiv mp\ (mod\ p^2)$, and $\mu \equiv (mp-2)\ (mod\ p^2)$.*

Proof. By Theorem 2.8(2), $n \equiv 0\ (mod\ p)$, so the equivalence class of $n(mod\ p^2)$ must have a representative in the set $\{0, p, 2p, 3p, \ldots, (p-1)p\}$. So $n \equiv mp\ (mod\ p^2)$ where $m \in \{0,1,2,3,...,(p-1)\}$. We also know by Theorem 2.8 (1) that $\mu \equiv (p-2)\ (mod\, p)$, so the equivalence class of $\mu(mod\ p^2)$ has a representative in the set $\{(p-2),(2p-2),(3p-2),...,(p^2-2)\}$, so $\mu \equiv (rp-2)\ (mod\ p^2)$, where $r \in \{0,1,2,...,(p-1)\}$. Additionally, by Theorem 3.1(d), we have that $4(n-1)+\mu^2 \equiv 0\ (mod\ p^2)$, so

$$4(n-1)+\mu^2 = 4(mp-1)+(rp-2)^2 \equiv 4p(m-r)(modp) \equiv 0\ (mod\ p^2)$$

and therefore $m \equiv r\ (mod\ p)$, as p is a prime with $p > 2$. But $m, r \in \{0,1,2,\ldots,(p-1)\}$ so $m = r$. That is, $n \equiv mp\ (mod\ p^2)$, and $\mu \equiv (mp-2)\ (mod\ p^2)$, where $m \in \{0,1,2,...,(p-1)\}$. □

Corollary 2.10. *For $p \in \mathbb{N}$ prime, $p > 2$, let Q be a nontrivial pth root signature matrix of an equiangular (n,k)-frame, satisfying $Q^2 = (n-1)I + \mu Q$, then $(\lambda_1 - \lambda_2)^2 \equiv 0\ mod\ p^2$,* *where λ_1, λ_2 are the eigenvalues of Q. Furthermore, $(\lambda_1 - \lambda_2)^2$ is the product of two perfect squares.*

Proof. By Theorem 2.8 that $4(n-1) + \mu^2 \equiv 0 (mod\ p^2)$. Using the H-P equations yields:

$$\begin{aligned} 4(n-1) + \mu^2 &= 4(1 - \lambda_1\lambda_2 - 1) + (\lambda_1 + \lambda_2)^2 \\ &= -4\lambda_1\lambda_2 + \lambda_1^2 - 2\lambda_1\lambda_2 + \lambda_2^2 \\ &= (\lambda_1 - \lambda_2)^2 \end{aligned}$$

and so $(\lambda_1 - \lambda_2)^2 \equiv 0\ mod\ p^2$. Furthermore, since $4(n-1) + \mu^2 = (\lambda_1 - \lambda_2)^2$ is a perfect square, and $(\lambda_1 - \lambda_2)^2 \equiv 0\ mod\ p^2$, it must be the case that $(\lambda_1 - \lambda_2)^2$ is the product of two perfect squares. □

The topic of complex equiangular tight frames with the maximal number of frame vectors is of special interest to quantum information theorists. Notice however, that the conditions derived in the preceding theorem rule out the "simplest" candidate for construction, the case of p^2 vectors in a p-dimensional Hilbert space when p is prime.

Corollary 2.11. *(Corollary 3.8 in [9]) Let $p > 3$ be prime. Then there exists no equiangular (p^2, p)-frame with a pth root signature matrix.*

Proof. By Theorem 2.8 (1), $\mu = (p-2)\sqrt{p+1}$ is an integer, and thus invoking the Rational Root Theorem, $p + 1$ is a perfect square; that is, $p = (r+1)(r-1)$ for some integer r, which contradicts the assumption that p is prime and $p > 3$. □

Remark 2.12. The previous theorem is a generalization of the cube root case established in Proposition 3.4 of [12]. That result stated that if a nontrivial cube root signature matrix Q of an equiangular frame (n, k)-frame satisfies $Q^2 = (n-1)I + \mu Q$ then either $n \equiv 0\ (mod\ 9)$ and $\mu \equiv 7\ (mod\ 9)$, or $n \equiv 3\ (mod\ 9)$ and $\mu \equiv 1\ (mod\ 9)$, or $n \equiv 6\ (mod\ 9)$ and $\mu \equiv 4\ (mod\ 9)$. This is the $p = 3$ case of Corollary 2.9. Here $m \in \{0, 1, 2\}$ producing three possibilities: $n = 0p \equiv 0\ (mod\ 9)$, and $\mu \equiv (0p-2)\ (mod\ 9) \equiv 7\ (mod\ 9)$, or $n = 1p \equiv 3\ (mod\ 9)$, and $\mu \equiv (1p-2)\ (mod\ 9) \equiv 1\ (mod\ 9)$, or $n = 2p \equiv 6\ (mod\ 9)$, and $\mu \equiv (2p-2)\ (mod\ 9) \equiv 4\ (mod\ 9)$.

2.2 pth root signature matrices

In Sections 2.1.1 and 2.1.2 we derived some conditions which the parameters of a non-trivial pth root Seidel matrix must satisfy in order to be the signature matrix of an equiangular (n,k)-frame. We now consider a few cases for small p values to illustrate the use of these conditions.

2.2.1 Cube root signature matrices

A search for possible cube root signature matrices was carried out in [12]. The calculations for possible cube root signature matrices yielded eight potential (n,k) pairs for $n < 100$: $(9,6),(33,11),(36,21),(45,12),(51,34),(81,45),(96,76)$, and $(99,33)$. Two of these pairs, $(9,6)$ and $(81,45)$, were confirmed to exist in Theorems 6.1 and 6.3 of that paper.

2.2.2 Fifth root signature matrices

Next we go through the calculations of possible (n,k) values for $2 \leq k < n \leq 50$ with $p = 5$. As $5e = n-\mu-2$ by Theorem 2.6, and $\mu = (n-2k)\sqrt{\frac{n-1}{k(n-k)}} \in \mathbb{Z}$ by the Theorem 2.1 equations, we have that $5e = n-2-\sqrt{q}(n-2k)$, where $q = \frac{n-1}{k(n-k)}$, and $\sqrt{q} \in \mathbb{Q}$. So, our strategy will be to begin with a multiple of 5 as our n value. Step 1 is to check for values of k where $\sqrt{q} = \sqrt{\frac{n-1}{k(n-k)}} \in \mathbb{Q}$. Step 2 is to calculate μ for any k satisfying step 1. We know that $\mu \in \mathbb{Z}$ and that for $n \equiv 5m\ (mod\ 25), m \in \{0,1,2,3,4\}$, we must have that $\mu \equiv 5m-2\ (mod\ 25)$.

$\boldsymbol{n=5}$ Step 1: Since $k=2,3$, or 4, $q=\frac{n-1}{k(n-k)}=\frac{4}{6},\frac{4}{9}$, or $\frac{4}{4}$, and $\sqrt{q}$ must be in $\mathbb{Q}$, $k\neq 2$. Step 2: $\mu=(n-2k)\sqrt{q}=\frac{-2}{3},-3$ for $k=2$ and 3 respectively. As neither of these yields $\mu\equiv 3(mod\ 25)$, there are no possible solutions for $n=5$.

$\boldsymbol{n=10}$ Step 1: For $2\leq k\leq 9$, we examine each $q=\frac{n-1}{k(n-k)}$, and see that $\sqrt{q}$ is not in $\mathbb{Q}$ when $k=3,4,6$, or 7. Step 2: Now, $\mu=\frac{9}{2},0,\frac{-9}{2},-8$ for $k=2,5,8$,and 9 respectively. As none of these yields $\mu\equiv 8\ (mod\ 25)$, there are no possible solutions for $n=10$.

$\boldsymbol{n=15}$ Step 1: Checking each q value for $2\leq k\leq 14$, $\sqrt{q}=\sqrt{\frac{n-1}{k(n-k)}}\in\mathbb{Q}$ for $k=7$ and 8 only. Step 2: $\mu=\frac{1}{2},\frac{-1}{2}$ for these values and as neither is equivalent to $13\ (mod\ 25)$, there are no possible solutions for $n=15$.

$\boldsymbol{n=20}$ Step 1: For $2\leq k\leq 19$, we check each $q=\frac{n-1}{k(n-k)}$, and note that $\sqrt{q}\in\mathbb{Q}$ only when $n=19$. Step 2: This yields a μ value of -18 which is not equivalent to $18\ (mod\ 25)$ so there are no possible solutions for $n=20$.

$\boldsymbol{n=25}$ Step 1: Looking at each q value for $2\leq k\leq 24$, we see that $\sqrt{q}=\sqrt{\frac{n-1}{k(n-k)}}\in\mathbb{Q}$ when $k=10,\ 15$, or 24. Step 2: These lead to μ values of $2,-2,-23$ respectively. $\mu=-2\equiv 23\ (mod\ 25)$, and neither 2 nor -23 has the same property. Therefore a $(25,15)$-frame is the only possible solution for $n=25$.

$\boldsymbol{n=30}$ Step 1: For $2\leq k\leq 29$, we can see that $\sqrt{q}\in\mathbb{Q}$, only when $k=29$. Step 2: When $k=29$, $\mu=-28$ which is not equivalent to $28\ (mod\ 49)$. Thus, there are no possible solutions for $n=30$.

$\boldsymbol{n=35}$ Step 1: Checking each q values for $2\leq k\leq 34$, $\sqrt{q}=\sqrt{\frac{n-1}{k(n-k)}}\in\mathbb{Q}$ for $k=17,18$, and 34 only. Step 2: These three k values correspond to $\mu=\frac{1}{3},\frac{-1}{3}$, and -33, none of which satisfies $\mu\equiv 10\ (mod\ 25)$, so there are no possible solutions for $n=35$.

$\boldsymbol{n=40}$ Step 1: For $2 \leq k \leq 39$, we examine each $q = \frac{n-1}{k(n-k)}$, and see that $\sqrt{q} \in \mathbb{Q}$ for $k = 39$ only. Step 2: When $k = 39$, then $\mu = -38 \equiv 12\ (mod\ 25) \not\equiv 13\ (mod\ 25)$. Therefore, there are no possible solutions for $n = 40$.

$\boldsymbol{n=45}$ Step 1: Looking at each q value for $2 \leq k \leq 44$, we see that $\sqrt{q} = \sqrt{\frac{n-1}{k(n-k)}} \in \mathbb{Q}$ when $k = 12,\ 33$, or 44. Step 2: These lead to μ values of $7, -7, -43$ respectively. Since $n \equiv 20\ (mod\ 25)$, we know that $\mu \equiv 18\ (mod\ 25)$. Neither 7 nor -43 has this property, but $\mu = -7$ does. Therefore a $(45, 33)$-frame is the only possible solution for $n = 45$.

$\boldsymbol{n=50}$ Step 1: Checking each q value for $2 \leq k \leq 49$, we note that $\sqrt{q} \in \mathbb{Q}$ for $k =$ $5, 10, 18, 25, 32, 40$, and 45. Step 2: These seven k values yield $\mu = \frac{56}{3}, \frac{21}{2}, \frac{49}{12}, 0, \frac{-49}{12}, \frac{-21}{2}$, and $\frac{-56}{3}$. Only 0 is an integer and as $\mu \equiv 0\ (mod\ 25) \not\equiv 23\ (mod\ 25)$, there are no possible solutions for $n = 50$.

This search has so far yielded two potential fifth root signature matrices, belonging to an equiangular $(25, 15)$-frame and a $(45, 33)$-frame, among the Parseval frames of $n \leq 50$ vectors.

2.2.3 Seventh root signature matrices

Now we go through the calculations of possible (n, k) values for $2 \leq k < n \leq 50$ with $p = 7$. Again, our strategy will be to begin with a multiple of 7 as our n value. Step 1 is to check for values of k where $\sqrt{q} = \sqrt{\frac{n-1}{k(n-k)}} \in \mathbb{Q}$. Step 2 is to calculate μ for any k satisfying step 1. We know that $\mu \in \mathbb{Z}$ and that for $n \equiv 7m\ (mod\ 49), m \in \{0, 1, 2, \cdots, 6\}$, we must have that $\mu \equiv 7m - 2\ (mod\ 49)$.

$\boldsymbol{n=7}$ Step 1: Since $2 \leq k \leq 6$, $q = \frac{n-1}{k(n-k)} = \frac{6}{10}, \frac{6}{12}, \frac{6}{12}, \frac{6}{10}$, or $\frac{4}{4}$, and $\sqrt{q}$ must be in $\mathbb{Q}$, so $k = 6$. Step 2: $\mu = (n - 2k)\sqrt{q} = -5 \equiv 44\ (mod\ 49)$ for $k = 6$. But $n = 7$ implies that $\mu \equiv 5\ (mod\ 49)$ so there are no possible solutions for $n = 7$.

$n = 14$ Step 1: Checking q values for $2 \le k \le 13$, we find that $\sqrt{q} = \sqrt{\frac{n-1}{k(n-k)}} \in \mathbb{Q}$ for $k = 13$ only. Step 2: As $k = 13$ implies that $\mu = -12 \equiv 37\ (mod\ 49) \not\equiv 12\ (mod\ 49)$ so there are no possible solutions for $n = 14$.

$n = 21$ Step 1: For $2 \le k \le 20$, we examine each $q = \frac{n-1}{k(n-k)}$, and see that $\sqrt{q} \in \mathbb{Q}$ for $k = 5, 16$, and 20. Step 2: These k values correspond to $\mu = \frac{11}{2}, \frac{-11}{2}$, and -19 respectively. However, as $n = 21$, we know that $\mu \equiv 19\ (mod\ 49)$, so there are no possible solutions for $n = 21$.

$n = 28$ Step 1: Now, for $2 \le k \le 27$, we examine each $q = \frac{n-1}{k(n-k)}$, and see that $\sqrt{q} \in \mathbb{Q}$ for $k = 3, 7, 12, 14, 16, 21$, and 27. Step 2: These k values correspond to $\mu = \frac{27}{2}, 6, \frac{3}{2}, 0, \frac{-3}{2}, -6, \frac{-27}{2}$, and -26 respectively. However, as $n = 28$, we know that $\mu \equiv 26\ (mod\ 49)$, so there are no possible solutions for $n = 28$.

$n = 35$ Step 1: Checking q values for $2 \le k \le 34$, we find that $\sqrt{q} = \sqrt{\frac{n-1}{k(n-k)}} \in \mathbb{Q}$ for $k = 34$ only. Step 2: As $k = 34$ implies that $\mu = -33 \equiv 16\ (mod\ 49) \not\equiv 33\ (mod\ 49)$ so there are no possible solutions for $n = 35$.

$n = 42$ Step 1: Looking at each q value for $2 \le k \le 41$, we see that $\sqrt{q} = \sqrt{\frac{n-1}{k(n-k)}} \in \mathbb{Q}$ when $k = 21$ or 41. Step 2: These lead to μ values of $0, -40$ respectively. Since $n \equiv 42\ (mod\ 49)$, we know that $\mu \equiv 40\ (mod\ 49)$. Neither 0 nor -40 has this property so there are no possible solutions for $n = 42$

$n = 49$ Step 1: For $2 \le k \le 48$, we examine each $q = \frac{n-1}{k(n-k)}$, and see that $\sqrt{q} \in \mathbb{Q}$ for $k = 21, 28$, and 48. Step 2: These k values correspond to $\mu = 2, -2, -19$ respectively. However, as $n \equiv 0\ (mod\ 49)$, we know that $\mu \equiv 47\ (mod\ 49)$, therefore a $(49, 28)$-frame is the only possible solution for $n = 49$.

Here the search has located one potential seventh root signature matrix belonging to an equiangular $(49, 28)$-frame, among the Parseval frames with $n \le 50$ vectors.

2.3 Examples of pth root signature matrices with two eigenvalues

As mentioned earlier, the existence of cube root signature matrices satisfying $Q^2 = (n-1)I - \mu Q$ was confirmed in [12]. The first example, corresponding to a $(9,6)$-frame is listed here in our notation. To facilitate the display of signature matrices, we only present the exponents of the pth root ω appearing in Q in a matrix A. This means, the entries of Q are $Q_{j,l} = \omega^{A_{j,l}} - \delta_{j,l}$ where $\delta_{j,l} = 0$ if $j \neq l$ and $\delta_{j,j} = 1$ for $j, l \in \{1, 2, \ldots n\}$.

Example 2.13. (Theorem 6.1 in [12]) The matrix

$$A := \begin{pmatrix} 000000000 \\ 000111222 \\ 000222111 \\ 021012012 \\ 021201120 \\ 021120201 \\ 012021021 \\ 012210102 \\ 012102210 \end{pmatrix}$$

gives rise to a 9×9 nontrivial cube root signature matrix Q belonging to an equiangular $(9,6)$-frame with entries $Q_{j,l} = \omega^{A_{j,l}} - \delta_{j,l}$. The fact that Q has two eigenvalues can be verified explicitly by confirming the matrix identity $Q^2 = 8I - 2Q$.

Based on our analysis of the necessary conditions in the previous section, a nontrivial fifth root Seidel matrix could exist for $n = 25$ and $k = 15$. This is indeed the case.

Example 2.14. Let

$$A := \begin{pmatrix} 0000000000000000000000000 \\ 0000011111222223333344444 \\ 0000044444333332222211111 \\ 0000022222444441111133333 \\ 0000033333111114444422222 \\ 0413201234012340123401234 \\ 0413240123123402340134012 \\ 0413234012234014012312340 \\ 0413223401340121234040123 \\ 0413212340401233401223401 \\ 0321404321043210432104321 \\ 0321443210104322104332104 \\ 0321432104210434321010432 \\ 0321421043321041043243210 \\ 0321410432432103210421043 \\ 0234103142031420314242031 \\ 0234142031142032031420314 \\ 0234131420203144203103142 \\ 0234120314314201420331420 \\ 0234114203420313142014203 \\ 0142302413024131302402413 \\ 0142341302130243024130241 \\ 0142330241241300241313024 \\ 0142324130302412413041302 \\ 0142313024413024130224130 \end{pmatrix},$$

let $\omega = e^{2\pi i/5}$, and, for $j, l \in \{1, 2, \ldots 25\}$, define the matrix Q by $Q_{j,l} := \omega^{A_{j,l}} - \delta_{j,l}$, then Q is a 25×25 nontrivial fifth root signature matrix of an equiangular $(25, 15)$-frame.

The matrix Q was found by performing an enumerative search in Matlab. To confirm that Q is a signature matrix, one needs only to check that $Q^2 = 24I - 2Q$. This has been verified using the symbolic computation package Mathematica.

The $p^2 \times p^2$ signature matrices in the above two examples have $\mu = -2$, and so $B = Q + I$ gives a corresponding Butson-type Hadamard matrix satisfying $B^2 = p^2 I$ ([13], [51], see also the online catalogue [56]) for $p \in \{3, 5\}$.

We construct such complex $p^2 \times p^2$ Hadamard matrices for any $p \geq 2$. First note that while Lemma 2.5 cannot be extended to values of p which are not prime, the converse holds for primes and non-primes alike.

Lemma 2.15. *(Lemma 5.3 in [9]) If $\omega \in \mathbb{C}$ such that $\omega \neq 1$, and $\omega^r = 1$ for some $r \in \mathbb{N}$, $r \geq 2$, then $\Sigma_{j=0}^{r-1} \omega^j = 0$.*

Proof. As $\omega^r = 1$ implies that $\omega^r - 1 = 0$, we see that $(\omega - 1)(\Sigma_{j=0}^{r-1} \omega^j) = 0$. Since, $\omega \neq 1$, it must be that $\Sigma_{j=0}^{r-1}\omega^j = 0$. □

Theorem 2.16. *(Theorem 5.4 in [9]) For any $p \in \mathbb{N}$, $p \geq 2$, let $\omega = e^{2\pi i/p}$. Define B to be a $p^2 \times p^2$ matrix composed of $p \times p$ blocks where for $1 \leq j \leq p$, $1 \leq l \leq p$, $B_{j,l} = (\omega^{(1-l)(x-1)+(j-1)(y-1)})_{x,y=1}^{p}$, where x and y denote the row and column within the $p \times p$ block $B_{j,l}$. This matrix satisfies $B = B^*$ and $B^2 = p^2 I$.*

Proof. To begin with, we define the diagonal unitary $p \times p$ matrix D with non-zero entries $D_{j,j} = \omega^{j-1}$. The definition of the blocks in B is then simply expressed by

$$B_{j,l} = D^{1-l} J D^{j-1}$$

where J is the $p \times p$ matrix containing only 1's.

With the unitarity of D it is straightforward to verify that $B_{j,l}^* = B_{l,j}$ and thus B is self-adjoint.

Next, we notice that for $x \in \mathbb{Z}_p$ such that $x \neq 0$, $\omega^x \neq 1$, and $(\omega^x)^p = 1$. Thus by Lemma 2.15, $\sum_{j=0}^{p-1} \omega^{jx} = \sum_{j=0}^{p-1} (\omega^x)^j = 0$. Consequently, $JD^x J = 0$ if $x \neq 0$. This simplifies computing the $p \times p$ blocks of the square $S := B^2$,

$$\begin{aligned} S_{j,l} &= \Sigma_{k=1}^{p} B_{j,k} B_{k,l} \\ &= \Sigma_{k=1}^{p} D^{1-k} J D^{j-1} D^{1-l} J D^{k-1} \\ &= \Sigma_{k=1}^{p} D^{1-k} J D^{j-l} J D^{k-1} \\ &= \begin{cases} 0 & \text{for } j \neq l \\ p^2 I & \text{for } j = l \end{cases} \end{aligned}$$

In the last step we use that when $j = l$, each (a, b)-entry of

$$\Sigma_{k=1}^{p} D_{1-k} J J D_{k-1} = p \Sigma_{k=1}^{p} D_{1-k} J D_{k-1} = p \Sigma_{k=1}^{p} B_{k,k}$$

is

$$\omega^0 + \omega^{a-b} + \omega^{2(a-b)} + ... + \omega^{(p-2)(a-b)} + \omega^{(p-1)(a-b)} = \begin{cases} 0 & \text{for } a \neq b \\ p & \text{for } a = b \end{cases}$$

as $(a-b)\ (mod\ p) \neq 0$ implies that $\omega^{a-b} \neq 1$. This together with the fact that $\omega^p = 1$ by definition allows us to apply Lemma 2.15 to obtain the desired result. Thus $S_{j,l} = p^2 I$ for $j = l$, and as $S_{j,l} = 0$ for $j \neq l$, we then have that $S = B^2 = p^2 I$. □

If $B = Q + I$ and $B^2 = (Q+I)^2 = p^2 I$, then $Q^2 = (p^2 - 1)I - 2Q$. The matrix Q is by the definition of B in standard form and nontrivial. It is the signature matrix of an equiangular (p^2, k)-frame, with $k = p(p+1)/2$ following from $\mu = -2$ and Theorem 2.1. We summarize this consequence.

Corollary 2.17. *(Corollary 5.5 in [9]) Let $p \in \mathbb{N}$, $p \geq 2$ and let B be as in the preceding theorem, then $Q = B - I$ is a $p^2 \times p^2$ nontrivial pth root signature matrix belonging to an equiangular $(p^2, \frac{p(p+1)}{2})$-frame.*

Another consequence of the identity $(Q + I)^2 = nI$ implicit in this construction is that the above examples can be used to obtain signature matrices for $n = p^{2m}$, $m \in \mathbb{N}$, by a tensorization argument as in the cube-roots case [12]. Moreover, one can take tensor products of Butson-type Hadamard matrices $Q_1 + I$ and $Q_2 + I$ belonging to different values p_1, p_2. This gives a signature matrix $Q = (Q_1 + I) \otimes (Q_2 + I) - I \otimes I$ containing roots of unity belonging to $p = p_1 p_2$ which is not prime and thus the necessary conditions derived here do not apply without appropriate modifications.

2.4 Graph-theoretic view

While the theoretical and constructive aspects of the cube root case were successfully generalized in 2009 [12] an intriguing graph-theoretic interpretation in the cube root case remained unexploited in the general setting. After developing an appropriate system of definitions, we can investigate a more general directed graph formulation. In the real case, much of what is known about constructing equiangular Parseval frames was developed by Seidel using the correspondence between self-adjoint matrices and (undirected) graphs. Here we develop a formulation with directed graphs beginning with the following definitions.

Definition 2.18. The labeled directed graph $G = (V, E, \Omega)$ associated with a $n \times n$ Seidel matrix $Q = (Q_{j,l})_{j,l=1}^{n}$ containing pth roots of unity is a complete directed graph (V, E) with vertices $V = \{1, 2, \ldots n\}$ and directed edges $E = V \times V$ together with a function Ω which assigns to each edge $(j, l) \in E$ the value $\Omega(j, l) = Q_{j,l}$. When speaking of such a labeled graph for a given Seidel matrix Q, we denote it by $G(Q)$.

Seidel matrix switching equivalency can be reformulated in terms of graphs. Multiplying a matrix by a unimodular constant is equivalent to multiplying the label of each edge by the same constant. Meanwhile, switching via conjugation of with a permutation matrix, corresponds to a permutation of the vertices and edges of the graph. Notice that without a permutation, edge labels are generally affected by switching, but the product of all labels along any closed loop does not change.

Starting with loops of length 3 which form the boundary of an oriented face.

If A is an oriented face in the graph, then we write $A = (j, l, m)$, with vertices $j \neq l \neq m \neq j$, and identify such sequences which are obtained by cyclic permutation from one another. We abbreviate the set of oriented faces by Δ.

Definition 2.19. Given a labeled graph $G = (V, E, \Omega)$ with $\Omega : E \to \{0\} \cup \{\omega^j\}_{j=1}^{p}$, with ω a primitive pth root of unity, then we associate with each oriented face $A = (j, l, m)$, $j \neq l \neq m \neq j$ the flux $\Phi(A) = q$ where $0 \leq q \leq p - 1$ and $\omega^q = \Omega(j, l)\Omega(l, m)\Omega(m, j)$.

When taking the sum of all outward fluxes of a tetrahedron, we obtain a multiple of

p.

Proposition 2.20. *(Proposition 3.3 in [10]) Given a labeled directed graph associated with a Seidel matrix containing pth roots of unity, and a tetrahedron (j, l, m, q) (no two vertices are equal) of the graph with oriented faces $A_1 = (j, l, m)$, $A_2 = (m, l, q)$, $A_3 = (q, l, j)$, $A_4 = (j, m, q)$, then $\Phi(A_1) + \Phi(A_2) + \Phi(A_3) + \Phi(A_4) \equiv 0 \,(mod\, p)$.*

Proof. Without loss of generality we can consider a graph with 4 vertices. The result is a restatement of the fact that taking the product of the labels of the directed edges gives unity, because for each pair (j, l), $j \neq l$, the factors $\Omega(j, l)$ and $\Omega(l, j) = \overline{\Omega(j, l)}$ appear in the product $\omega^{\Phi(A_1)}\omega^{\Phi(A_2)}\omega^{\Phi(A_3)}\omega^{\Phi(A_4)}$ exactly once. □

We call a directed edge $\epsilon \in E$ adjacent to an oriented face $A \in \Delta$ if ϵ appears in the periodically extended sequence of vertices defining A. In an abuse of notation we abbreviate this by $\epsilon \in A$.

Proposition 2.21. *(Proposition 3.4 in [10]) A labeled directed graph $G = (V, E, \Omega)$ with $\Omega : E \to \{0\} \cup \{\omega^j\}_{j=1}^p$, ω a pth root of unity, is associated with a Seidel matrix Q having two eigenvalues, $Q^2 = (n-1)I + \mu Q$ for some $\mu \in \mathbb{R}$, if and only if the following criteria are satisfied:*

1. *$\Omega(j, l) = \overline{\Omega(l, j)}$ for all $(j, l) \in E$;*

2. *$\Omega(j, j) = 0$ for all $j \in V$;*

3. *$\sum_{A:(j,l)\in A} \omega^{\Phi(A)} = \mu$, where the sum is over all triangles A adjacent to any given edge (j, l).*

Proof. The criteria are simply graph-theoretic restatements of the properties of a Seidel matrix Q with two eigenvalues. □

Using Lemma 2.5 again, we deduce the graph-theoretic analogue of Theorem 2.6.

Corollary 2.22. *(Proposition 3.6 in [10]) Let Q be the pth root Seidel matrix, with p prime, and $Q^2 = (n-1)I + \mu Q$, and let $G = (V, E, \Omega)$ be the labeled directed graph associated with Q, then for every fixed edge $(j, l) \in E$, $j \neq l$, the set of adjacent, oriented triangles $\{A \in \Delta : A \ni (j, l)\}$ is partitioned into $J_q = \{A : (j, l) \in A \in \Delta, \Phi(A) = q\}$ and the sizes of the sets in this partition are given by $|J_q| = e$ for $q \in \{1, 2, \ldots, p-1\}$ and $|J_0| = e + \mu$, with $e = (n - \mu - 2)/p$ an integer.*

Proof. For any edge j, l, $j \neq l$, we have $\Sigma_m Q_{j,m} Q_{m,l} Q_{l,j} = \mu$. For each m, $Q_{j,m} Q_{m,l} Q_{l,j} = \omega^{\Phi}((j, m, l))$ which is a pth root of unity with p prime. Lemma2.5 then implies that the faces can be partitioned into sets with the prescribed sizes, depending on their flux. The total number of faces is $n - 2$, which gives the integrality condition for $e = (n - \mu - 2)/p$. $\square$

This corollary fixes the number of fluxes of faces adjacent to one edge. We can also deduce a relationship between the fluxes of certain adjacent faces. To this end, we denote the set of closed, non-degenerate loops of length 4 by $\square$. Each $T \in \square$ is given by a sequence of 4 pairwise different vertices, $T = (j, l, m, q)$, which can be interpreted as a Hamiltonian circuit in a tetrahedron. With $T = (j, l, m, q)$ we associated the flux $\Phi(T) = r$ where $0 \leq r \leq p - 1$ and $\omega^r = \Omega(j, l)\Omega(l, m)\Omega(m, q)\Omega(q, j)$. Moreover, $\epsilon \in T$ means that the edge ϵ appears as a subsequence in the periodic extension of the sequence (j, l, m, q).

Corollary 2.23. *(Proposition 3.7 in [10]) Let Q be the pth root Seidel matrix, with p prime, and $Q^2 = (n-1)I + \mu Q$, and let $G = (V, E, \Omega)$ be the labeled directed graph associated with Q, then for every fixed edge $(j, l) \in E$, $j \neq l$, summing the fluxes over the set of closed, non-degenerate loops of length 4 adjacent to (j, l) gives*

$$\sum_{T:(j,l) \in T} \omega^{\Phi(T)} = \mu^2 - n + 2 \,.$$

Proof. When iteration the equation for Q we obtain $Q^3 = \mu(n-1)I + (n-1+\mu^2)Q$. Inspecting the matrix product $(Q^3)_{l,j}Q_{j,l} = \sum_{m,q} Q_{l,m}Q_{m,q}Q_{q,j}Q_{j,l}$ shows that we can restrict the sum to $l \neq m \neq q \neq j$ because the diagonal of Q vanishes. There are then $n-1$ remaining terms for which $q = l$ which each contribute unity. Among the $q \neq l$ terms, $n-2$ have $m = j$ and contribute unity. Removing these terms amounts to restricting to non-degenerate loops which leaves $\sum_{T:(j,l)\in T} \omega^{\Phi(T)} = (Q^3)_{l,j}Q_{j,l} - 2n + 3 = \mu^2 - n + 2$. $\square$

Corollary 2.24. *Let Q be an $n \times n$, pth root Seidel matrix, with p prime, and $Q^2 = (n-1)I + \mu Q$, $\mu \in \mathbb{R}$, then $g := ((n-2)(n-3) - \mu^2 + 1)/p$ is an integer.*

Proof. We know that $\sum_{T:(j,l)\in T} \omega^{\Phi(T)} = \mu^2 - 1$, so the same argument as before gives that the sets $J_r^4 = \{T \in \Box : (j,l) \in T, \Phi(T) = r\}$ have the cardinality $|J_r^4| = g$ when $1 \leq r \leq p-1$ and $|J_0^4| = g + \mu^2$ for some fixed non-negative integer g. Any fixed edge $(j,l) \in E$ has a total number of $(n-2)(n-3)$ Hamiltonian circuits of length 4 originating in it, so $pg + \mu^2 - 1 = (n-2)(n-3)$. $\square$

We can combine this result with our earlier integrality conditions.

Corollary 2.25. *If Q is an $n \times n$, pth root Seidel matrix, p prime, and $Q^2 = (n-1)I + \mu Q$, $\mu \in \mathbb{R}$, then $7 - \mu^2 \equiv 0(mod\ p)$.*

Proof. We know from Theorem 2.8 that $n \equiv 0(mod\ p)$, so $(n-2)(n-3)+1 = n^2-5n+7 \equiv 7(mod\ p)$. Now inserting this in the preceding corollary gives the desired result. $\square$

2.5 Consequences of the Seidel-Holmes-Paulsen equation

2.5.1 Relation to Hadamard matrices

One straightforward, yet interesting connection is between the signature matrices of equiangular (n,k) frames and complex Hadamard matrices.

Definition 2.26. A *complex Hadamard matrix* is an $n \times n$ matrix, H, all of whose entries lie on the unit circle, such that $HH^* = nI_n$. Furthermore, a *Hadamard design*, or matrix, U, is said to be skew if $U + U^T + I = J$ holds where J is the matrix of all 1's.

As noted earlier, B in Theorem 2.16 is a complex Hadamard matrix.

Szöllösi generalized this result [50] by characterizing special signature matrices as complex Hadamard matrices as below.

Theorem 2.27. *(Theorem 2.2 in [50]) Let Q be a self-adjoint $n \times n$ matrix with $Q_{i,i} = 0$ and $|Q_{i,j}| = 1$ for all $i \neq j$. Then the following are equivalent:*

1. $Q^2 = (n-1)I + \mu Q$ *for some necessarily real* $2 \leq \mu \leq 2$*; and*
2. $H = Q + \mu$ *is a complex Hadamard matrix for* $\mu = -\frac{\mu}{2} \pm i\sqrt{1 - \frac{|\mu|^2}{4}}$.

Now one can choose any complex Hadamard matrix to obtain an n'th root signature matrix and corresponding frame. In particular, Szöllösi demonstrated the following using a result by Butson [13] which implies existence for complex Hadamards of order $2^a p^b$.

Corollary 2.28. *(Corollary 2.5 in [50]) For every prime p there is a nontrivial pth root signature matrix of order $4^a p^{2b}$ for all $0 \leq a \leq b$ corresponding to an equiangular $(4^a p^{2b}, 2^a p^b (2^a p^b + 1)/2)$ frame.*

The Paley Hadamard matrices were found (by Paley) using the quadratic residues in finite fields of odd order [33]. Moreover, it has been shown [37] that the designs of prime orders are skew. Szöllösi makes the observation [50] that this can be used to generate infinitely many equiangular Parseval frames.

Corollary 2.29. *(Corollary 2.11 of [50]) Suppose that we have a skew Hadamard design of order $n \geq 3$. Then there are equiangular $(n, (n-1)/2)$ and $(n, (n+1)/2)$ frames.*

Returning to the signature matrices, notice that we can characterize the complex Hadamard in terms of the signature matrix, Q.

Corollary 2.30. *Let Q be a self-adjoint $n \times n$ matrix with $Q_{j,j} = 0$ and $|Q_{j,l}| = 1$ for all $j \neq l$, such that Q is the signature matrix of an equiangular (n, k)-frame for some k. Then $BB^* = (\frac{4n-4+\mu^2}{4})I$, where $Q^2 = (n-1)Q + \mu Q$, and $B = Q - \frac{\mu}{2}I$.*

Proof. Since Q is the signature matrix of an equiangular (n, k)-frame for some k, $Q^2 = (n-1)Q + \mu Q$ by Theorem 2.1, and $\mu \in \mathbb{R}$ Rearranging this result, we see that

$$Q^2 - \mu Q = (n-1)I$$

or equivalently that,

$$(Q - \frac{\mu}{2}I)^2 = (n - 1 + \frac{\mu^2}{4})I$$

Now, since $Q = Q^*$ and $\mu \in \mathbb{R}$, in must be the case that $(Q - \frac{\mu}{2}I) = (Q - \frac{\mu}{2}I)^*$, so

$$(Q - \frac{\mu}{2}I)(Q - \frac{\mu}{2}I)^* = (\frac{4n-4+\mu^2}{4})I$$

and the result holds. □

When $\mu = \pm 2$ in the above corollary, we have exactly that $BB^* = nI$, and $B = Q \pm I$.

2.5.2 Higher order Seidel-Holmes-Paulsen equations

While Theorem 2.8 provided necessary conditions for us to check in Section 2.2, necessary and sufficient conditions for μ, n, λ_1, and λ_2 have been elusive. One avenue of exploration is to derive a closed form for higher order Seidel-Holmes-Paulsen equations. Here, we present a characterizations of these sets of equations. We leave it as an open problem to study combinatorial properties implied by these equations.

Proposition 2.31. *Let Q be a self-adjoint $n \times n$ matrix with $Q_{j,j} = 0$ and $|Q_{j,l}| = 1$ for all $j \neq l$ such that Q is the signature matrix of an equiangular (n, k)-frame.*

Then $Q^k = a_k I + b_k Q$, *where* $a_k = \frac{d^k}{dt^k}|_{t=0}\ (e^{\frac{\mu t}{2}}(cosh(t\lambda) - \frac{\mu}{2\lambda}sinh(t\lambda))$, *and* $b_k = \frac{d^k}{dt^k}|_{t=0}\ (e^{\frac{\mu t}{2}}\frac{1}{\lambda}sinh(t\lambda))$. *Here,* $\mu = \lambda_1 + \lambda_2$, *and* $\lambda = \frac{\lambda_1 - \lambda_2}{2}$, *where* λ_1, λ_2 *are the eigenvalues of* Q.

Proof. By the Holmes-Paulsen result, we know that $Q^2 = (n-1)I + \mu Q$ and we'd like to derive equations of the form $Q^k = a_k I + b_k Q$. To this end, let's find functions of t, $a(t), b(t)$, such that $e^{tQ} = a(t)I + b(t)Q$. Then

$$\frac{d^k}{dt^k}|_{t=0}\ e^{tQ} = Q^k\ ,\ \frac{d^k}{dt^k}|_{t=0}\ a(t) = a_k\ ,\ \frac{d^k}{dt^k}|_{t=0}\ b(t) = b_k$$

To find $a(t), b(t)$, define $\overline{Q} = Q - \frac{\mu}{2}I$. Q has eigenvalues $\lambda_1 < 0 < \lambda_2$, with $\mu = \lambda_1 + \lambda_2$. So by Weyl's Theorem, $\overline{Q}$ has eigenvalues $\frac{\lambda_1 - \lambda_2}{2}, \frac{\lambda_2 - \lambda_1}{2}$. Let us rename these eigenvalues (of $\overline{Q}$) to be λ and $-\lambda$, respectively.

Define $\overline{a}(t)$, $\overline{b}(t)$ as $e^{t\overline{Q}} = \overline{a}(t)I + \overline{b}(t)\overline{Q}$, and now,

$$\begin{aligned}
e^{tQ} &= (e^{\frac{\mu t}{2}})e^{t\overline{Q}} \\
&= (e^{\frac{\mu t}{2}})(\overline{a}(t)I + \overline{b}(t)\overline{Q}) \\
&= (e^{\frac{\mu t}{2}})((\overline{a}(t) - \frac{\mu}{2}\overline{b}(t))I + \overline{b}(t)Q
\end{aligned}$$

Therefore, $a(t) = e^{\frac{\mu t}{2}}(\overline{a}(t) - \frac{\mu}{2}\overline{b}(t))$, and $b(t) = e^{\frac{\mu t}{2}}\overline{b}(t)$, and we are now left to look for $\overline{a}, \overline{b}$.

To accomplish this, let's first look at the powers of $\overline{Q}$.

$$\begin{aligned}
(\overline{Q})^2 &= (Q - \frac{\mu}{2}I)^2 \\
&= Q^2 - \mu Q + \frac{\mu^2}{4}I \\
&= (n - 1 + \frac{\mu^2}{4})I \\
&= (1 - \lambda_1\lambda_2 - 1 + \frac{(\lambda_1 + \lambda_2)^2}{4}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{(\lambda_1 - \lambda_2)^2}{4} I \\
&= \lambda^2 I
\end{aligned}
$$

So $\overline{Q}^n = \lambda^n I$, when n is even, and $\overline{Q}^n = \lambda^{n-1}\overline{Q}$, when n is odd.

Therefore

$$
\begin{aligned}
e^{t\overline{Q}} &= \Sigma_{n=0}^{\infty} \frac{t^n}{n!}\overline{Q}^n \\
&= \Sigma_{even}^{\infty} \frac{t^n}{n!}\lambda^n I + \Sigma_{odd}^{\infty} \frac{t^n}{n!}\lambda^{n-1}\overline{Q} \\
&= cosh(t\lambda) I + \frac{1}{\lambda} sinh(t\lambda)\overline{Q}
\end{aligned}
$$

and $\overline{a}(t) = cosh(t\lambda)$, and $\overline{b}(t) = \frac{1}{\lambda} sinh(t\lambda)$. Now, we can solve for $a(t) = e^{\frac{\mu t}{2}}(cosh(t\lambda) - \frac{\mu}{2\lambda} sinh(t\lambda))$, and $b(t) = e^{\frac{\mu t}{2}} \frac{1}{\lambda} sinh(t\lambda)$ and the result follows. $\square$

Chapter 3

Equiangular Parseval Frames as the Solutions of an Optimization Problem

In this chapter, we will explore a gradient descent method for finding equiangular Parseval frames. The idea is to characterize equiangular Parseval frames as optimizers for an energy function, which could also be called a frame potential. Similar approaches have been used to locate equiangular frames over other sets of frames. Benedetto and Fickus [3] used frame potentials to show the existence of an abundance of equal-norm Parseval frames. Bodmann and Casazza [8] successfully searched for an equal-norm frame in the vicinity of almost equal-norm Parseval frames by constructing a system of ODEs which generated a flow on the set of Parseval frames converging to an equal-norm frame. In other work, Casazza, Fickus et al. [14] developed an iterative method for increasing the tightness of an unit norm frame via gradient descent using a frame potential function. Their search over the set of unit norm frames converged to a Parseval frame under certain conditions. In their paper, Bodmann and Casazza use a dilation, switching equivalence, and energy distance estimates. Here, we will follow this strategy to the extent possible with the goal of approaching equiangular Parseval frames. In contrast however, we will define a single function dependent on the Gram matrix of a frame rather than the frame vectors themselves. Minimizing this function induces a flow on the set of Parseval frames to an equiangular frame. In order to compute the gradient function we will discuss and use a matrix manifold viewpoint. This approach is also used by Strawn [46] to construct Grassmanian frames. Recall that the gradient of a smooth scalar field f on a Riemannian manifold M denoted by $grad\ f(x)$ is the unique element of T_xM that satisfies $\langle grad\ f(x), \chi\rangle_x = Df(x)[\chi]$ for all $\chi \in T_xM$ (p.46 [1]). We will in turn use the following optimization property to pursue a minimum.

Proposition 3.1. *(p.46, [1]) The norm of grad $f(x)$ gives the steepest slope of f at x:*

$$||grad\ f(x)|| = Df(x)[\frac{grad\ f(x)}{||grad\ f(x)||}]$$

3.1 Frame energy and a gradient descent

We begin however, with a dilation, in the same manner as the equal-norm case [8]. If $\{f_j\}_{j=1}^n$ is a Parseval frame for a real or complex Hilbert space, then in a slight abuse of notation we can write the Gram matrix as $G = (\langle f_k, f_j\rangle)_{j,k=1}^n$ is an orthogonal projection matrix and we have the expression $G_{k,j} = \langle Ge_k, Ge_j\rangle = \langle V^*e_k, V^*e_j\rangle$ with the canonical orthonormal basis $\{e_j\}_{j=1}^n$ on $l^2(\{1, 2, ..., n\})$ and V^*, the adjoint of the analysis operator of $\{f_j\}_{j=1}^n$.

Definition 3.2. Let $M = \{G :\ G$ is the Grammian for an (n, k) Parseval frame$\}$, a submanifold of the Hermitians. TM will denote the tangent space to M. We also define the frame energy of an (n, k) Parseval frame $F = \{f_j\}_{j=1}^n$ with Grammian $G = (g_{j,l})_{j,l=1}^n$

by $U_C : M \to \mathbb{R}$, where

$$U_C(G) = (\Sigma_{j,l}^n |g_{j,l}|^4) - \frac{k^2(k^2+n-2k)}{n^2(n-1)}$$

For notational convenience, we'll also define, $U(G) = \Sigma_{j,l}^n |g_{j,l}|^4$

Lemma 3.3. *For $G \in M$, where G is associated with an equiangular (n,k) frame, $U_C(G) = 0$.*

Proof. For $G \in M$, where G is associated with an equiangular (n,k) frame, we have

$$\begin{aligned}
U_C(G) &= \Sigma_{j,l}^n |g_{j,l}|^4 - \frac{k^2(k^2+n-2k)}{n^2(n-1)} \\
&= \Sigma_j^n |g_{j,j}|^4 + \Sigma_{j \neq l}^n |g_{j,l}|^4 - \frac{k^2(k^2+n-2k)}{n^2(n-1)} \\
&= n\frac{k^4}{n^4} + n(n-1)\frac{k^2(n-k)^2}{n^4(n-1)^2} - \frac{k^2(k^2+n-2k)}{n^2(n-1)} \\
&= \frac{k^2(k^2+n-2k)}{n^2(n-1)} - \frac{k^2(k^2+n-2k)}{n^2(n-1)} \\
&= 0
\end{aligned}$$

and so $U_C(G) = 0$ for G equiangular. □

Now we characterize $U_C(G_0) = 0$. We want to minimize U_C as a strategy for searching for the nearest equiangular Parseval frame to any given starting point G_0, where $G_0 \in M$ is the Grammian of any Parseval frame.

Proposition 3.4. *An alternate expression for the frame energy of an (n,k) Parseval frame $F = \{f_j\}_{j=1}^n$ with Grammian $G = (g_{j,l})_{j,l=1}^n$ is*

$$W(G) = \Sigma_{j \neq l}^n (|g_{j,l}|^2 - \frac{k(n-k)}{n^2(n-1)})^2 + \Sigma_j^n (|g_{j,j}|^2 - \frac{k^2}{n^2})^2 + \frac{2k(k-1)}{n-1} \Sigma_j^n (|g_{j,j}| - \frac{k}{n})^2$$

Proof.

$$
\begin{aligned}
W(G) &= \Sigma_{j\neq l}^{n}(|g_{j,l}|^2 - \frac{k(n-k)}{n^2(n-1)})^2 + \Sigma_j^n(|g_{j,j}|^2 - \frac{k^2}{n^2})^2 + \frac{2k(k-1)}{n-1}\Sigma_j^n(|g_{j,j}| - \frac{k}{n})^2 \\
&= \Sigma_{j\neq l}^{n}(|g_{j,l}|^4 - \frac{2k(n-k)}{n^2(n-1)}|g_{j,l}|^2) + \frac{k^2(n-k)^2}{n^3(n-1)} + \Sigma_j^n(|g_{j,j}|^4 - \frac{2k^2}{n^2}|g_{j,j}|^2) + \frac{k^4}{n^3} \\
&+ \frac{2k(k-1)}{n-1}\Sigma_j^n(|g_{j,j}|^2 - \frac{2k}{n}|g_{j,j}|) + \frac{2k^3(k-1)}{n^2(n-1)} \\
&= U(G) - \frac{2k(n-k}{n^2(n-1)}\Sigma_{j,k}^n|g_{j,k}|^2 + \frac{2k^3n(k-1) + k^2(n-k)^2 + k^4(n-1)}{n^3(n-1)} \\
&= U(G) + \frac{k^2(2k-k^2-n)}{n^2(n-1)} \\
&= U_C(G)
\end{aligned}
$$

Thus, we conclude that $W(G) = U_C(G) = 0$ if and only if G is associated with an equiangular Parseval frame. □

This expression more clearly illustrates the measurement that the frame energy is providing. Any (n,k) equiangular Parseval frame has a Grammian with diagonal entries equal to k/n and off-diagonal entries equal to $\sqrt{\frac{k(n-k)}{n^2(n-1)}}$. So the energy function is a measure of deviation from these values.

Notice that it must be the case that *grad* U = *grad* W as they differ only by a constant. However, we will compute each time derivative directly, extracting information from each calculation.

3.2 An isometry for the tangent space

One can compute *grad* U by using a basis. That is,

$$grad\ U = \Sigma_1^n B_j(U)B_j$$

where $\{B_j\}_1^n$ form an orthonormal basis for the tangent space of the domain of U. We will compute the gradient norm of the frame energy in two different ways. One is by constructing an orthonormal basis for TM, for use in the above formula. The other is to map TM to $TSU(n)$ with an isometry and then compute the gradient in the tangent space of the group of special unitaries. To this end, we first consider a map from the special unitaries onto a subset of the Grammians. We will show that this map is a Riemannian submersion allowing us to lift the frame energy to $SU(n)$ and use the pushforward of our map to compute the gradient of the frame energy. This approach relies on the equivalences between self-adjoint operators and one-parameter subgroups established by Stone and between one-parameter subgroups and tangent vectors shown by Lie.

We will characterize TM in our next proposition, in preparation for this calculation. First we recall some definitions and results from [18] and [24].

Definition 3.5. The special unitary group of degree n, denoted $SU(n)$ is the group of $n \times n$ unitaries with determinant 1. A strongly continuous one-parameter unitary group is a function $U : \mathbb{R} \to B(H)$ such that for all $s, t \in \mathbb{R}$: (a) $U(t)$ is a unitary operator, (b) $U(s+t) = U(s)U(t)$, and (c) if $h \in H$ and $t_0 \in \mathbb{R}$, then $U(t)h \to U(t_0)h$ as $t \to t_0$

Stone's Theorem gives us another way to view this information.

Theorem 3.6. *(p. 330, Theorem 5.6 [18]) Let U be a strongly continuous 1-parameter unitary group, then there exists a unique self-adjoint operator A such that*

$$U_t = e^{itA} \quad t \in \mathbb{R}.$$

Conversely, let A be a self-adjoint operator on a Hilbert space H. Then

$$U_t := e^{itA} \quad t \in \mathbb{R}$$

is a strongly continuous one-parameter family of unitary operators.

In our setting, Stone's Theorem establishes that $A \in TSU(n)$ if and only if $e^{tA} \in SU(n)$ for all $t \in \mathbb{R}$ producing the following familiar result.

Lemma 3.7. *The tangent space to $SU(n)$ at the identity is denoted $TSU(n)$ and is the set of traceless $n \times n$ skew-Hermitians.*

Proof. By Stone's Theorem, $A \in TSU(n)$ if and only if $e^{tA} \in SU(n)$ for all $t \in \mathbb{R}$. For any $e^{tA} \in SU(n)$ for all $t \in \mathbb{R}$ it must be the case that $det(e^{tA}) = 1$. This implies that $e^{t\ tr(A)} = 1$ for all values of t. So $tr(A) = 0$. Furthermore, notice that if A is skew-Hermitian, then $(e^{tA})^* = e^{tA^*} = e^{-tA}$, and $(e^{tA})^* e^{tA} = e_{tA}(e^{tA})^* = I$. Conversely, if $(e^{tA})^* e^{tA} = I$, then differentiating each side of this equation yields $A^* e^{tA^*} e^{tA} + e^{tA^*} A e^{tA} = 0$. Evaluating at $t = 0$ shows us that $A^* + A = 0$ and therefore $A^* = -A$. So $TSU(n) = su(n)$, the skew-Hermitians. $\square$

We now recall a classic result regarding the exponential map.

Proposition 3.8. *(p. 116, Proposition 8.33 in [24]) The exponential map is the unique map from a Lie algebra g to a Lie group G taking 0 to e and whose differential at the origin*

$$(exp_*)_0 = T_0 g = g \to T_e G = g$$

is the identity and whose restrictions to the lines through the origin in g are one-parameter subgroups of G.

In order to combine and exploit these equivalencies we now define a function from $SU(n)$ to M. Once the function is confirmed to be a Riemannian submersion we can lift our frame energy function to $SU(n)$ and compute the gradient in that setting. The Riemannian metric on the tangent space TM originates from the embedding in $M_n(\mathbb{C})$; so it is the familiar Hilbert-Schmidt inner product.

Definition 3.9. Given a Gram matrix G of an equiangular Parseval frame, we define the function $P_G : SU(n) \to M$, where $P_G(U) = U^*GU$, for each $G \in M$. Then the pushforward of P_G is defined as $(P_G)_*(X)(f) = X(f(P_G))$ for all $X \in TSU(n)$ and all $f \in \mathbb{C}^\infty(M)$.

Recalling the definition of a Riemannian submersion [25] we seek to confirm that P is a smooth submersion and that for each $V \in SU(n)$, $T_V P$ is an isometry between the orthogonal complement of $(T_V P)^{-1}(0)$ in $T_V SU(n)$ and $T_{P(V)}M$.

Lemma 3.10. *The function $P_G : SU(n) \to M$, where $P_G(U) = U^*GU$ is surjective.*

Proof. For any $G \in M$, we have $P_G(U) = U^*GU$. Notice that $(P_G(U))^* = U^*G^*U = U^*GU = P_G(U)$, and $(P_G(U))^2 = U^*GUU^*GU = U^*GU = P_G(U)$ as U is a unitary and G is an orthogonal projection. So, $P_G(U)$ is also an orthogonal projection and therefore P is surjective onto M. □

Lemma 3.11. *$(P_G)_*$ is a partial isometry, taking $TSU(n)$ to TM.*

Proof. For any $Y \in TSU(n)$ is in the kernel of $(P_G)_*$ provided that $Y(U_f(U^*GU)) = 0$ for all $U_f \in \mathbb{C}^\infty(M)\}$. Curves in M have the form $U^*(t)G_0U(t)$ and $TSU(n)$ is unitary, so using Stone's Theorem, we can rewrite the condition $Y(U_f(U^*GU)) = 0$ as $\frac{d}{dt}|_{t=0}e^{tY}G_0e^{-tY} = 0$.

Since G_0 is a projection, there exists a basis where $G_0 = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$ and I is the $k \times k$ identity block. We can write $Y \in TSU(n)$ as $Y = \begin{pmatrix} Y_1 & Y_2 \\ -Y_2^* & Y_3 \end{pmatrix}$, where Y_1, Y_3 are skew-Hermitian. Now, $YG_0 - G_0Y = 0$ implies that

$$\begin{pmatrix} Y_1 & Y_2 \\ -Y_2^* & Y_3 \end{pmatrix}\begin{pmatrix} I & 0 \\ 0* & 0 \end{pmatrix} - \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} Y_1 & Y_2 \\ -Y_2^* & Y_3 \end{pmatrix} = \begin{pmatrix} Y_1 & 0 \\ -Y_2^* & 0 \end{pmatrix} - \begin{pmatrix} Y_1 & Y_2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -Y_2 \\ -Y_2^* & 0 \end{pmatrix}$$

and so $Y_2 = 0$. That is, $Y = \begin{pmatrix} Y_1 & 0 \\ 0 & Y_3 \end{pmatrix}$ where Y_1 is a $k \times k$ skew-Hermitian, and Y_3 is a $(n-k) \times (n-k)$ skew-Hermitian. We can view these matrices as Hermitians, by factoring out an i; i.e. $Y = iZ$, where $Z = \begin{pmatrix} Z_1 & 0 \\ 0 & Z_3 \end{pmatrix}$, with Z_1, a $k \times k$ Hermitian, and Z_3 an $(n-k) \times (n-k)$ Hermitian each with 0-diagonals.

Denoting the set of such Z by Z_S, we now have that $ker(P_G)_* = \{Z \in TSU(n) : Z \in Z_S\}$, and $(ker(P_G)_*)^{\perp} = \{X \in TSU(n) : tr(XZ) = 0 \text{ for all } Z \in Z_S\}$.

For any $X = \begin{pmatrix} X_1 & X_2 \\ -X_2^* & X_3 \end{pmatrix}$, with X_1, X_3 skew-Hermitian, $XZ = \begin{pmatrix} X_1 Z_1 & X_2 Z_3 \\ -X_2^* Z_1 & X_3 Z_3 \end{pmatrix}$. In order for $tr(XZ) = 0$ for all such Z, it must be the case that $X_1, X_3 = 0$ and so $X = \begin{pmatrix} 0 & X_2 \\ -X_2^* & 0 \end{pmatrix} = i \begin{pmatrix} 0 & X_4 \\ X_4^* & 0 \end{pmatrix}$

where X_4 is a $k \times (n-k)$ matrix. So, $(ker(P_G)_*)^{\perp} = \{X \in TSU(n) : X = i \begin{pmatrix} 0 & X_4 \\ X_4^* & 0 \end{pmatrix}\}$

Now evaluating $(P_G)_*$ on it's orthogonal complement, we see that for any $X \in (ker(P_G)_*)^{\perp}$, $(P_G)_*(X)(f) = X(f(P_G)) = \frac{d}{dt}|_{t=0} e^{tX} G_0 e^{-tX} = XG_0 - G_0 X$

Again, working in the basis where $G_0 = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$, $XG_0 - G_0X = X$, and so $||X|| = ||(P_G)_*(X)(f)||$ on $(ker(P_G)_*)^{\perp}$. Now, since $(P_G)_*$ is a partial isometry on $(ker(P_G)_*)^{\perp}$, $(P_G)_*$ is a partial isometry between $TSU(n)$ and TM. $\square$

So P is confirmed by the above lemmas to be a smooth Riemannian submersion. Now consider the lifted energy function on $SU(n)$ where $\widetilde{U}(V) = U(V^* G V)$ for all $V \in SU(n)$.

The submersion property gives us that

$$\frac{d}{dt}|_{t=0}\widetilde{U}(\gamma(t)) = \frac{d}{dt}|_{t=0}U((\gamma(t))^{*}G(\gamma(t)))$$

as in $p.48-49$ of [1]. Further, given a tangent vector $Y \in TSU(n)$, then the one-parameter subgroup generated by I is a curve, γ, of unitaries such that $\gamma(0) = I$, $\gamma'(0) = iH$ with $H = H^{*}$. with $(YU)(I) = \frac{d}{dt}|_{t=0}\widetilde{U}(\gamma(t)) = \frac{d}{dt}|_{t=0}U((\gamma(t))^{*}G(\gamma(t)))$, as in $p.39-43$ of [1]. We are now ready to use these ideas to characterize the tangent space to M.

Theorem 3.12. *For $M = \{G :$ G is the Grammian for an (n,k) Parseval frame$\}$ then the tangent space to M at any $G_0 \in M$ is characterized by*

$$T_{G_0}(M) = \{Z \in M_n(\mathbb{C}) : G_0 Z G_0 = 0 \text{ and } (I - G_0)Z(I - G_0) = 0.$$

Proof. Consider any curve γ in M that passes through G_0 at time 0. $\gamma(t) = G(t) \in M$ for all $t \geq 0$ Now, since all $G(t) \in M$ are Gram matrices of Parseval frames, they are projections and so $G(t)G(t) = G(t)$. Differentiating both sides of this equation yields,

$$\dot{G(t)}G(t) + G(t)\dot{G(t)} = \dot{G(t)}$$

so $\dot{G(0)}$ belongs to the set $\{Z \in \mathbb{C}^{n\times n} : ZG_0 + G_0 Z = Z\}$. Again, since all $G(t)$ are projections, $(G(t))^3 = G(t)$, and differentiating this equation gives,

$$\dot{G(t)}G(t)^2 + G(t)\dot{G(t)}G(t) + G(t)^2\dot{G(t)} = \dot{G(t)}$$

This can be rewritten as $\dot{G(t)}G(t) + G(t)\dot{G(t)}G(t) + G(t)\dot{G(t)} = \dot{G(t)}$, so $\dot{G(0)}$ belongs to the set $\{Z \in \mathbb{C}^{n\times n} : ZG_0 + G_0 Z G_0 + G_0 Z = Z\}$. So, for $Z \in \mathbb{C}^{n\times n}$ to be in both sets, it must be the case that $G_0 Z G_0 = 0$. Combining these characterizations of tangent matrices we have that $T_{G_0}(M) = \{Z \in M_n(\mathbb{C}) : G_0 Z G_0 = 0$ and $(I - G_0)Z(I - G_0) = 0$. □

Proposition 3.13. *Let $M = \{G : \ G$ is the Grammian for an (n,k) Parseval frame$\}$ and TM denote the tangent space to M. Then an orthonormal basis for TM is the set $\{H_{j,l}, K_{j,l} : H_{j,l} = E_{j,l} + E_{l,j},$ and $K_{j,l} = iE_{j,l} - iE_{l,j},$ where $1 \leq j < l \leq n,$ and $E_{j,l}$ denotes an $n \times n$ matrix with a 1 in the (j,l) position and 0s elsewhere$\}$.*

Proof. By Theorem 3.12, we know that $T_{G_0}(M) = \{Z \in M_n(\mathbb{C}) : G_0 Z G_0 = 0 \text{and} (I - G_0)Z(I - G_0) = 0$. These equations are satisfied regardless of the basis used and since G_0 is a projection, there exists a basis where $G_0 = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$

Now viewing Z with the appropriate blocks (i.e. those which match the dimensions for multiplication by the blocks of G_0), we have

$$G_0 Z G_0 = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} Z_1 & Z_2 \\ Z_2^* & Z_3 \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} Z_1 & 0 \\ 0 & 0 \end{pmatrix} = 0_{n \times n}$$

but also that,

$$(I - G_0)Z(I - G_0) = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} A_1 & A_2 \\ A_2^* & A_3 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & A_3 \end{pmatrix} = 0_{n \times n}$$

Since the block Z_1 is $k \times k$ where $k = rank\ G_0$, and the block A_3 is an $(n-k) \times (n-k)$ matrix, it must be the case that

$$Z = \begin{pmatrix} 0 & Z_2 \\ Z_2^* & 0 \end{pmatrix}$$

So $TM = \{Z \in \mathbb{C}^{n \times n} : Z = Z^*$ and Z contains 2 zero-blocks including the diagonal$\}$. Therefore, TM is spanned by $H_{j,k}, j < k$, and $K_{j,l}, j < l$ as claimed. □

3.3 Gradient computations and bounds

Now that we have characterized a constructed a basis for TM, we are able to compute the gradient of our energy function. In the next two propositions we will compute both *grad* U and *grad* W, recalling that these are in fact equal. The *grad* U expression is far more concise formula allowing for ease of investigation into the minima. Meanwhile, the *grad* W formulation will illustrate that the gradient is bounded by the energy itself.

Proposition 3.14. *For $F = \{f_j\}_{j=1}^n$, an (n,k) Parseval frame with Grammian $G = (g_{j,l})_{j,l=1}^n$ and $U(G) = (\Sigma_{j,l}^n |g_{j,l}|^4)$, then the gradient of U is an $n \times n$ skew-Hermitian matrix with 0s on the diagonal where*

$$[grad\ U]_{a,b} = [8i\Sigma_{j=1}^n (|g_{j,a}|^2 - |g_{j,b}|^2) g_{a,j} g_{j,b}]_{a,b}$$

Proof. We begin with any $G_0 \in M$ and consider the curves in M that pass through G_0 at $t = 0$. We can chose among the equivalence class belonging to each curve a representative γ, of the form $\gamma(t) = (U(t))^* G_0 U(t)$, so by Stone's Theorem $U(t) = e^{itA}$. So applying this idea to the basis elements, we start with $E_{a,a}$. and consider $\gamma_{a,a}(t) = e^{-itE_{a,a}} G_0 e^{itE_{a,a}}$, with $\gamma_{a,a}(0) = G_0$. Using the fact that $E_{a,a}^m = E_{a,a}$ for all m, we can rewrite our function as:

$$\gamma_{a,a}(t) = [(e^{-it} - 1)E_{a,a} + I] G_0 [(e^{it} - 1)E_{a,a} + I]$$

So $\gamma_{a,a}(t)$ is an $n \times n$ matrix whose entries match those of G excluding the off-diagonal entries of the first row and the first column.

$$\gamma_{a,a}(t) = \begin{pmatrix} g_{1,1} & \cdots & g_{1,a-1} & e^{it}g_{1,a} & g_{1,a+1} & \cdots & g_{1,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ g_{a-1,1} & \cdots & g_{a-1,a-1} & e^{it}g_{a-1,a} & g_{a-1,a+1} & \cdots & g_{a-1,n} \\ e^{-it}g_{a,1} & \cdots & e^{-it}g_{a,a-1} & g_{a,a} & e^{-it}g_{a,a+1} & \cdots & e^{-it}g_{a,n} \\ g_{a+1,1} & \cdots & g_{a+1,a+1} & e^{it}g_{a+1,a} & g_{a+1,a+1} & \cdots & g_{a+1,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ g_{n,1} & \cdots & g_{n,a-1} & e^{it}g_{n,a} & g_{n,a+1} & \cdots & g_{n,n} \end{pmatrix}$$

$$U(\gamma_{a,a}(t)) = \Sigma_{j,l}^{n}|e^{-itE_{a,a}}G_0e^{itE_{a,a}}|^4 = \Sigma_{j,l}^{n}|g_{j,l}|^4 = U(G_0)$$

since $|e^{-it}g_{a,m}|^4 = (e^{-it}g_{a,m}e^{it}\overline{g_{a,m}})^2 = |g_{a,m}|^4$, and $|e^{it}g_{m,a}|^4 = (e^{it}g_{m,a}e^{-it}\overline{g_{m,a}})^2 = |g_{m,a}|^4$. So $\frac{dU(\gamma_{a,a}(t))}{dt}|_{t=0} = 0$ and therefore the $E_{j,j}$ components do not contribute to the gradient of U.

Now we consider $\gamma_{a,b}(t) = e^{-itH_{a,b}}G_0e^{itH_{a,b}}$, with $\gamma_{a,b}(0) = G_0$. U extends to all $n \times n$ matrices and so we can embed the tangent space of G_0 into all $n \times n$ matrices. Now the curve γ can be linearized, $\gamma \to \widetilde{\gamma}$ and $\widetilde{\gamma}$ rewritten as

$$\begin{aligned} \widetilde{\gamma_{a,b}}(t) &= G_0 + [\frac{d}{dt}|_{t=0}e^{-itH_{a,b}}G_0e^{itH_{a,b}}]t \\ &= G_0 + (-iH_{a,b}e^{-itH_{a,b}}G_0e^{itH_{a,b}} + e^{-itH_{a,b}}G_0iH_{a,b}e^{itH_{a,b}})t \\ &= G_0 + it(G_0H_{a,b} - H_{a,b}G_0) \end{aligned}$$

For example, $G_0 + it(G_0H_{1,2} - H_{1,2}G_0)$ has the form

$$\begin{pmatrix} (g_{1,1} + it(g_{1,2} - g_{2,1})) & (g_{1,2} + it(g_{1,1} - g_{2,2})) & (g_{1,3} - itg_{2,3}) & \dots & (g_{1,n} - itg_{2,n}) \\ (g_{2,1} + it(g_{2,2} - g_{1,1})) & (g_{2,2} + it(g_{2,1} - g_{1,2})) & (g_{2,3} - itg_{1,3}) & \dots & (g_{2,n} - itg_{1,n}) \\ (g_{3,1} + itg_{3,2}) & (g_{3,2} + itg_{3,1}) & g_{3,3} & \dots & g_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (g_{n,1} + itg_{n,2}) & (g_{n,2} + itg_{n,1}) & g_{n,3} & \dots & g_{n,n} \end{pmatrix}$$

In order to compute $\frac{dU(\widetilde{\gamma_{a,b}(t)})}{dt}|_{t=0}$, we will take each entry of $G_0 + it(G_0H_{a,b} - H_{a,b}G_0)$ in absolute value to the fourth power, differentiate with respect to t, and evaluate at $t = 0$. These terms must then all be summed. For this reason we will only concern ourselves with the linear terms of the fourth powers of the absolute value of our entries. Also, notice that only the a'th and b'th rows and columns will contribute to this sum.

Looking first at the $(a, a), (b, b), (a, b)$ and (b, a) locations and contributions to this sum:

$$\begin{aligned} \frac{d}{dt}(|g_{a,a} + it(g_{a,b} - g_{b,a})|^4)|_{t=0} &= \frac{d}{dt}((g_{a,a} + it(g_{a,b} - g_{b,a})(\overline{g_{a,a}} - it(\overline{g_{a,b}} - \overline{g_{b,a}})))^2|_{t=0} \\ &= \frac{d}{dt}(|g_{a,a}|^2 + 2g_{a,a}it(g_{a,b} - g_{b,a}) + t^2|g_{a,b} - g_{b,a}|^2)^2|_{t=0} \\ &= \frac{d}{dt}(4g_{a,a}^3 it(g_{a,b} - g_{b,a}))|_{t=0} \\ &= 4g_{a,a}^3 i(g_{a,b} - g_{b,a}) \end{aligned}$$

and

$$\begin{aligned}\frac{d}{dt}(|g_{a,b}+it(g_{a,a}-g_{b,b})|^4)|_{t=0} &= \frac{d}{dt}((g_{a,b}+it(g_{a,a}-g_{b,b})(\overline{g_{a,b}}-it(\overline{g_{a,a}}-\overline{g_{b,b}})))^2|_{t=0}\\ &= \frac{d}{dt}(|g_{a,b}|^2+it(g_{a,a}-g_{b,b})(g_{b,a}-g_{a,b})+t^2(g_{a,a}-g_{b,b})^2)^2|_{t=0}\\ &= 2ig_{a,b}^2(g_{a,a}-g_{b,b})(g_{b,a}-g_{a,b}).\end{aligned}$$

Similarly $\frac{d}{dt}(|g_{b,b}+it(g_{b,a}-g_{a,b})|^4)|_{t=0} = 4ig_{b,b}^3(g_{b,a}-g_{a,b})$, and $\frac{d}{dt}(|g_{b,a}+it(g_{b,b}-g_{a,a})|^4)|_{t=0} = 2ig_{a,b}^2(g_{b,b}-g_{a,a})(g_{a,b}-g_{b,a})$

So the sum of these four locations of the $\frac{dU(\widetilde{\gamma_{a,b}(t)})}{dt}|_{t=0}$ matrix is

$$4i(g_{a,b}-g_{b,a})(g_{a,a}^3-g_{b,b}^3+|g_{a,b}|^2(g_{b,b}-g_{a,a}))$$

Turning now to the rest of the a'th column, we see that

$$\begin{aligned}\frac{d}{dt}(|g_{c,a}+itg_{c,b}|^4)|_{t=0} &= \frac{d}{dt}((g_{c,a}+itg_{c,b})(g_{a,c}-itg_{b,c}))^2|_{t=0}\\ &= \frac{d}{dt}(|g_{c,a}|^2+it(g_{c,b}g_{a,c}-g_{c,a}g_{b,c})+t^2|g_{c,b}|^2)^2|_{t=0}\\ &= 2i|g_{c,a}|^2(g_{c,b}g_{a,c}-g_{c,a}g_{b,c})\end{aligned}$$

and so, $\Sigma_{j\neq a,b}^n 2i|g_{j,a}|^2(g_{j,b}g_{a,j}-g_{j,a}g_{b,j})$ sums the remaining contributions of the a'th column. Notice that the a'th row generates the same sum, while the b'th row and column each generate $\Sigma_{j\neq a,b}^n 2i|g_{j,b}|^2(g_{j,a}g_{b,j}-g_{j,b}g_{a,j})$. These four sum to $4i\Sigma_{j\neq a,b}^n(|g_{j,a}|^2-|g_{j,b}|^2)(g_{j,b}g_{a,j}-g_{j,a}g_{b,j})$.

Letting $j = a, b$ in the above sum yields

$$4i(|g_{a,a}|^2 - |g_{a,b}|^2)(g_{a,b}g_{a,a} - g_{a,a}g_{b,a}) + 4i(|g_{b,a}|^2 - |g_{b,b}|^2)(g_{b,b}g_{a,b} - g_{b,a}g_{b,b}) =$$

$$4i(g_{a,a}^3 - g_{b,b}^3 + |g_{a,b}|^2(g_{b,b} - g_{a,a})(g_{a,b} - g_{b,a})).$$

These are exactly the terms computed prior to the row and column sums.

Therefore, $\frac{dU(\widetilde{\gamma_{a,b}(t)})}{dt}|_{t=0} = 4i\Sigma_{j=1}^{n}(|g_{j,a}|^2 - |g_{j,b}|^2)(g_{j,b}g_{a,j} - g_{j,a}g_{b,j})$. This is the component of the gradient of U in the direction of $H_{a,b}$.

A similar analysis for the coefficient of the $K_{a,b}$ component of the gradient begins by considering $\gamma_{a,b}(t) = e^{-itK_{a,b}}G_0e^{itK_{a,b}}$, with $\gamma_{a,b}(0) = G_0$. Again, U extends to all $n \times n$ matrices and so we can embed the tangent space of G_0 into all $n \times n$ matrices. Then the curve can be linearized to $\widetilde{\gamma}$ and rewritten as

$$\begin{aligned}
\widetilde{\gamma_{a,b}}(t) &= G_0 + [\frac{d}{dt}|_{t=0}e^{-itK_{a,b}}G_0e^{itK_{a,b}}]t \\
&= G_0 + (-iK_{a,b}e^{-itK_{a,b}}G_0e^{itK_{a,b}} + e^{-itK_{a,b}}G_0iK_{a,b}e^{itK_{a,b}})t \\
&= G_0 + it(G_0K_{a,b} - K_{a,b}G_0)
\end{aligned}$$

For example, $G_0 + it(G_0K_{1,2} - K_{1,2}G_0)$ has the form

$$\begin{pmatrix} (g_{1,1}+t(g_{1,2}+g_{2,1})) & (g_{1,2}+t(g_{2,2}-g_{1,1})) & (g_{1,3}+tg_{2,3}) & \cdots & (g_{1,n}+tg_{2,n}) \\ (g_{2,1}+t(g_{2,2}-g_{1,1})) & (g_{2,2}+t(-g_{2,1}-g_{1,2})) & (g_{2,3}-tg_{1,3}) & \cdots & (g_{2,n}-ig_{1,n}) \\ (g_{3,1}+tg_{3,2}) & (g_{3,2}-tg_{3,1}) & g_{3,3} & \cdots & g_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (g_{n,1}+tg_{n,2}) & (g_{n,2}-tg_{n,1}) & g_{n,3} & \cdots & g_{n,n} \end{pmatrix}$$

Again, to compute $\frac{dU(\widetilde{\gamma_{a,b}(t)})}{dt}|_{t=0}$, we take each entry of $G_0 + it(G_0K_{a,b} - K_{a,b}G_0)$ in absolute value to the fourth power, differentiate with respect to t, and then evaluate at $t = 0$. Again, these contributions must then all be summed.

Looking first at the (a, a) location and contributions to this sum we have

$$\begin{aligned} \frac{d}{dt}(|g_{a,a}+t(g_{a,b}+g_{b,a})|^4)|_{t=0} &= \frac{d}{dt}((g_{a,a}+t(g_{a,b}+g_{b,a})(g_{a,a}+t(g_{b,a}+g_{a,b})))^2|_{t=0} \\ &= \frac{d}{dt}(g_{a,a}^2+2g_{a,a}t((g_{a,b}+g_{b,a}))+t^2(g_{a,b}+g_{b,a})^2)^2|_{t=0} \\ &= 4g_{a,a}^3(g_{a,b}+g_{b,a}) \end{aligned}$$

and

$$\begin{aligned} \frac{d}{dt}(|g_{a,b}+t(g_{b,b}-g_{a,a})|^4)|_{t=0} &= \frac{d}{dt}((g_{a,b}+t(g_{b,b}-g_{a,a})(g_{b,a}+t(g_{b,b}-g_{a,a})))^2|_{t=0} \\ &= \frac{d}{dt}(|g_{a,b}|^2+t(g_{b,b}-g_{a,a})(g_{b,a}-g_{a,b})+t^2(g_{b,b}-g_{a,a})^2)^2|_{t=0} \\ &= 2|g_{a,b}|^2(g_{b,b}-g_{a,a})(g_{b,a}-g_{a,b}). \end{aligned}$$

Similarly $\frac{d}{dt}(|g_{b,b} + t(-g_{b,a} - g_{a,b})|^4)|_{t=0} = -4g_{b,b}^3(g_{a,b} + g_{b,a})$, and $\frac{d}{dt}(|g_{b,a} + t(g_{b,b} - g_{a,a})|^4)|_{t=0} = 2|g_{a,b}|^2(g_{b,b} - g_{a,a})(g_{b,a} - g_{a,b})$

So the sum of these four locations of the $\frac{dU(\widetilde{\gamma_{a,b}(t)})}{dt}|_{t=0}$ matrix is

$$4(g_{a,b} + g_{b,a})(g_{a,a}^3 - g_{b,b}^3 + |g_{a,b}|^2(g_{b,b} - g_{a,a}))$$

Turning now to the rest of the a'th column, we see that

$$\begin{aligned}
\frac{d}{dt}(|g_{c,a} + tg_{c,b}|^4)|_{t=0} &= \frac{d}{dt}((g_{c,a} + tg_{c,b})(g_{a,c} + tg_{b,c}))^2|_{t=0} \\
&= \frac{d}{dt}(|g_{c,a}|^2 + t(g_{c,b}g_{a,c} - g_{c,a}g_{b,c}) + t^2|g_{c,b}|^2)^2|_{t=0} \\
&= 2|g_{c,a}|^2(g_{c,b}g_{a,c} + g_{c,a}g_{b,c})
\end{aligned}$$

and so, $\Sigma_{j\neq a,b}^n 2|g_{j,a}|^2(g_{j,b}g_{a,j} + g_{j,a}g_{b,j})$ sums the remaining contributions of the a'th column. The a'th row generates the same sum, while the b'th row and column each generate $\Sigma_{j\neq a,b}^n -2|g_{j,b}|^2(g_{j,a}g_{b,j} + g_{j,b}g_{a,j})$. These four sum to $4\Sigma_{j\neq a,b}^n(|g_{j,a}|^2 - |g_{j,b}|^2)(g_{j,a}g_{b,j} + g_{j,b}g_{a,j})$.

Letting $j = a, b$ in the above sum yields

$$4(|g_{a,a}|^2 - |g_{a,b}|^2)(g_{a,a}g_{b,a} + g_{a,b}g_{a,a}) + 4(|g_{b,a}|^2 - |g_{b,b}|^2)(g_{b,a}g_{b,b} + g_{b,b}g_{a,b}) =$$

$$4(g_{a,a}^3 - g_{b,b}^3 + |g_{a,b}|^2(g_{b,b} - g_{a,a})(g_{a,b} + g_{b,a})).$$

These are exactly the terms computed prior to the row and column sums.

Therefore, $\frac{dU(\widetilde{\gamma_{a,b}(t)})}{dt}|_{t=0} = 4\Sigma_{j=1}^n(|g_{j,a}|^2 - |g_{j,b}|^2)(g_{j,a}g_{b,j} + g_{j,b}g_{a,j})$. This is the component of the gradient of U in the direction of $K_{a,b}$.

Now the gradient of U is an $n \times n$ Hermitian matrix with 0s on the diagonal and (a,b) entries for $a < b$ written as

$$\begin{aligned}[grad\ U]_{a,b} &= [4i(\Sigma_{j=1}^{n}(|g_{j,a}|^2 - |g_{j,b}|^2)(g_{j,b}g_{a,j} - g_{j,a}g_{b,j} + g_{b,j}g_{j,a} + g_{a,j}g_{j,b})]_{a,b} \\ &= [8i\Sigma_{j=1}^{n}(|g_{j,a}|^2 - |g_{j,b}|^2)g_{a,j}g_{j,b}]_{a,b}\end{aligned}$$

□

Similar calculations provide us with $grad\ W$, as shown below.

Proposition 3.15. *For an (n,k) Parseval frame, $F = \{f_j\}_{j=1}^{n}$ with Grammian $G = (g_{j,l})_{j,l=1}^{n}$ and $W(G)$ defined as above, then the gradient of W is an $n \times n$ matrix with 0s on the diagonal where*

$$\begin{aligned}&\frac{dW(\gamma_{a,b}(t))}{dt}|_{t=0} = \\ &\Sigma_{j\neq a}^{n} 2i(|g_{j,a}|^2 - \frac{k(n-k)}{n^2(n-1)})[(1-2g_{a,a})(g_{b,j}g_{j,a} - g_{j,b}g_{a,j}) + 2|g_{j,a}|^2(g_{a,b} - g_{b,a})] \\ &+ \Sigma_{j\neq b}^{n} 2i(|g_{j,b}|^2 - \frac{k(n-k)}{n^2(n-1)})[(1-2g_{b,b})(g_{j,a}g_{b,j} - g_{a,j}g_{j,b}) + 2|g_{b,j}|^2(g_{a,b} - g_{b,a})] \\ &+ \Sigma_{j}^{n} 2i(g_{a,j}g_{j,b} - g_{j,a}g_{b,j})(2g_{j,j}(g_{j,j}^2 - \frac{k^2}{n^2}) + \frac{2k(k-1)}{n-1}(g_{j,j} - \frac{k}{n})) \\ &+ 2i(|g_{b,a}|^2 - \frac{k(n-k)}{n^2(n-1)})(g_{a,b} - g_{b,a})(2|g_{a,b}|^2 + 2g_{a,a}g_{b,b}) \\ &+ i(2g_{a,a}(|g_{a,a}|^2 - \frac{k^2}{n^2}) + \frac{2k(k-1)}{n-1}(g_{a,a} - \frac{k}{n})) \\ &+ i(2g_{b,b}(|g_{b,b}|^2 - \frac{k^2}{n^2}) + \frac{2k(k-1)}{n-1}(g_{b,b} - \frac{k}{n}))\end{aligned}$$

Proof. We begin with any $G_0 \in M$ and consider the curves in M that pass through G_0 at $t = 0$. Define $\gamma_Z(t) = G_0 + it((I - G_0)ZG_0 + G_0Z(I - G_0))$ and note that each $\gamma_Z(0) = G_0$ for all $Z \in \mathbb{C}^{n\times n}$. The curve is in M as $(I - G_0)ZG_0 + G_0Z(I - G_0) \in M$ for all $Z \in \mathbb{C}^{n\times n}$. Let's evaluate γ at the basis elements of $\mathbb{C}^{n\times n}$, namely $E_{a,b}$. We can then compute the gradient of W component wise. Let $\gamma_{a,b}(t) = G_0 + it((I - G_0)E_{a,b}G_0 + G_0E_{a,b}(I - G_0))$, so $\gamma_{a,b}(t) =$

$$\begin{pmatrix} g_{1,1} - 2itg_{1,a}g_{b,1} & \dots & g_{1,b} + itg_{1,a}(1 - 2g_{b,b}) & \dots & g_{1,n} - 2itg_{1,a}g_{b,n} \\ g_{2,1} - 2itg_{2,a}g_{b,1} & \dots & g_{2,b} + itg_{2,a}(1 - 2g_{b,b}) & \dots & g_{2,n} - 2itg_{2,a}g_{b,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ g_{a,1} + it(1 - 2g_{a,a})g_{b,1} & \dots & g_{a,b} + it(g_{b,b} + g_{a,a} - 2g_{a,a}g_{b,b}) & \dots & g_{a,n} = it(1 - 2g_{a,a})g_{b,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ g_{n,1} - 2itg_{n,a}g_{b,1} & \dots & g_{n,b} + itg_{n,a}(1 - 2g_{b,b}) & \dots & g_{n,n} - 2itg_{n,a}g_{b,n} \end{pmatrix}$$

So all the entries are $g_{j,l} - 2itg_{j,a}g_{b,l}$ with the exception of the ath row and bth column. Recall that

$$W(G) = \Sigma_{j\neq l}^n(|g_{j,l}|^2 - \frac{k(n-k)}{n^2(n-1)})^2 + \Sigma_j^n(|g_{j,j}|^2 - \frac{k^2}{n^2})^2 + \frac{2k(k-1)}{n-1}\Sigma_j^n(|g_{j,j}| - \frac{k}{n})^2$$

so to compute $\frac{dW(\gamma_{a,b}(t))}{dt}|_{t=0}$, we'll consider various cases.

When $a = b$, with $j \neq l$, $j \neq a$, and $k \neq b$, the (j,l) contribution to $\frac{dW(\gamma_{a,b}(t))}{dt}|_{t=0}$ is

$$\begin{aligned} &\frac{d}{dt}(|g_{j,l} - 2itg_{j,a}g_{b,l}|^2 - \frac{k(n-k)}{n^2(n-1)})^2|_{t=0} \\ &\qquad = \frac{d}{dt}((g_{j,l} - 2itg_{j,a}g_{a,l})(g_{l,j} + 2itg_{a,j}g_{l,a}) - \frac{k(n-k)}{n^2(n-1)})^2|_{t=0} \end{aligned}$$

$$= \frac{d}{dt}((|g_{j,l}|^2 - \frac{k(n-k)}{n^2(n-1)}) + 2it(g_{j,l}g_{a,j}g_{l,a} - g_{l,j}g_{j,a}g_{a,l}) + 4t^2|g_{j,a}|^2|g_{b,l}|)^2|_{t=0}$$
$$= 4i(|g_{j,l}|^2 - \frac{k(n-k)}{n^2(n-1)})(g_{j,l}g_{a,j}g_{l,a} - g_{l,j}g_{j,a}g_{a,l}).$$

However, notice that the (l, j) contribution is exactly the negative of this term so the sum over all such contributions is 0.

When $a = b$, with $j \neq l$, and $j = a$, the (j, l) entry yields

$$\frac{d}{dt}(|g_{a,l} + it(1-2g_{a,a})g_{b,l}|^2 - \frac{k(n-k)}{n^2(n-1)})^2|_{t=0}$$
$$= \frac{d}{dt}((g_{a,l} + it(1-2g_{a,a})g_{b,l})(g_{l,a} + it(1-2g_{a,a})g_{l,b}) - \frac{k(n-k)}{n^2(n-1)})^2|_{t=0}$$
$$= \frac{d}{dt}((|g_{a,l}|^2 - \frac{k(n-k)}{n^2(n-1)}) + it((1-2g_{a,a)}g_{b,l}g_{l,a} - (1-2g_{a,a})g_{l,b}g_{a,l})$$
$$+ 4t^2(1-2g_{a,a})^2|g_{b,l}|)^2|_{t=0}$$
$$= 2i(|g_{a,l}|^2 - \frac{k(n-k)}{n^2(n-1)})(1-2g_{a,a})(g_{a,l}g_{l,a} - g_{l,a}g_{a,l})$$
$$= 0.$$

Similarly, when $a = b$, with $j \neq l$, and $l = a$, the (j, l) contribution to $\frac{dW(\gamma_{a,b}(t))}{dt}|_{t=0}$ is 0.

Now, computing the contribution of the diagonal excluding the (a, a) location, where $\delta = \frac{2k(k-1)}{n-1}$, we see that when $a = b$, with $j = l$

$$\frac{d}{dt}((|g_{j,j} - 2itg_{j,a}g_{a,j}|^2 - \frac{k^2}{n^2})^2 + \delta(|g_{j,j} - 2itg_{j,a}g_{a,j}| - \frac{k}{n})^2)|_{t=0}$$
$$= \frac{d}{dt}((g_{j,j} - 2itg_{j,a}g_{a,j})(g_{j,j} + 2itg_{a,j}g_{j,a}) - \frac{k^2}{n^2})^2|_{t=0}$$
$$+ \frac{d}{dt}\delta((g_{j,j} - 2itg_{j,a}g_{a,j})(g_{j,j} + 2itg_{a,j}g_{j,a}) - \frac{2k}{n}\sqrt{g_{j,j}^2 + 4t^2g_{j,a}^2g_{a,j}^2} + \frac{k^2}{n^2})|_{t=0}$$
$$= \frac{d}{dt}((g_{j,j}^2 - \frac{k^2}{n^2}) + 2it(g_{j,j}g_{a,j}g_{j,a} - g_{j,j}g_{j,a}g_{a,j}) + 4t^2g_{j,a}^2g_{a,j}^2)^2|_{t=0}$$

$$
\begin{aligned}
&+ \frac{d}{dt}\delta((g_{j,j}^2 + \frac{k^2}{n^2}) - \frac{2k}{n}\sqrt{g_{j,j}^2 + 4t^2 g_{j,a}^2 g_{a,j}^2} + 4t^2 g_{j,a}^2 g_{a,j}^2)|_{t=0} \\
&= 0
\end{aligned}
$$

as the only linear term includes the expression $(g_{j,j}g_{a,j}g_{j,a} - g_{j,j}g_{j,a}g_{a,j}) = 0$.

Meanwhile the (a, a) location for $a = b$, with $j = l$ also does not contribute to the sum as

$$
\begin{aligned}
&\frac{d}{dt}((|g_{a,a} + it(2g_{a,a} - 2g_{a,a}^2)|^2 - \frac{k^2}{n^2})^2 + \delta(|g_{a,a} + it(2g_{a,a} - 2g_{a,a}^2)| - \frac{k}{n})^2)|_{t=0} \\
&\quad = \frac{d}{dt}((g_{a,a}^2 - \frac{k^2}{n^2}) + t^2(2g_{a,a} - 2g_{a,a}^2)^2)^2|_{t=0} \\
&\quad + \frac{d}{dt}\delta(\sqrt{g_{a,a}^2 + t^2(2g_{a,a} - 2g_{a,a}^2)^2)} - \frac{k}{n})^2|_{t=0} \\
&\quad = 0
\end{aligned}
$$

as there are no linear terms prior to differentiation. So the $E_{j,j}$ components do not contribute to the gradient of W.

Now we turn our attention to the component of the gradient corresponding to $E_{a,b}$, where $a \neq b$. When $j \neq l$, $j \neq a, b$, and $k \neq a, b$, the (j, l) contribution to $\frac{dW(\gamma_{a,b}(t))}{dt}|_{t=0}$ cancels with the (l, j) contribution just as in the $a = b$ case.

When $j = a$, and $l \neq a, b$ the (j, l) entry yields

$$
\begin{aligned}
&\frac{d}{dt}(|g_{a,l} + it(1 - 2g_{a,a})g_{b,l}|^2 - \frac{k(n-k)}{n^2(n-1)})^2|_{t=0} \\
&\qquad = \frac{d}{dt}((g_{a,l} + it(1 - 2g_{a,a})g_{b,l})(g_{l,a} + it(1 - 2g_{a,a})g_{l,b}) - \frac{k(n-k)}{n^2(n-1)})^2|_{t=0} \\
&\qquad = \frac{d}{dt}(((|g_{a,l}|^2 - \frac{k(n-k)}{n^2(n-1)}) + it((1 - 2g_{a,a})g_{b,l}g_{l,a} - (1 - 2g_{a,a})g_{l,b}g_{a,l}) \\
&\qquad + 4t^2(1 - 2g_{a,a})^2|g_{b,l}|)^2|_{t=0}
\end{aligned}
$$

$$= 2i(|g_{a,l}|^2 - \frac{k(n-k)}{n^2(n-1)})(1 - 2g_{a,a})(g_{b,l}g_{l,a} - g_{l,b}g_{a,l}).$$

Summing across the ath row, excluding the (a,a), and (a,b) locations gives a contribution of $\Sigma_{l\neq a,b}^{n} 2i(|g_{a,l}|^2 - \frac{k(n-k)}{n^2(n-1)})(1-2g_{a,a})(g_{b,l}g_{l,a} - g_{l,b}g_{a,l})$. Meanwhile the contribution of the ath column excluding the (a,a), and (b,a) locations is $\Sigma_{j\neq a,b}^{n} 4i(|g_{j,a}|^2 - \frac{k(n-k)}{n^2(n-1)})(g_{j,a}g_{a,j}g_{a,b} - g_{a,j}g_{j,a}g_{b,a})$ and so combining these sums yields

$$\Sigma_{j\neq a,b}^{n} 2i(|g_{j,a}|^2 - \frac{k(n-k)}{n^2(n-1)})[(1-2g_{a,a})(g_{b,j}g_{j,a} - g_{j,b}g_{a,j}) + 2|g_{j,a}|^2(g_{a,b} - g_{b,a})]$$

.

Similarly, summing the contributions to $\frac{dW(\gamma_{a,b}(t))}{dt}|_{t=0}$ of the bth row and column (this time excluding the (b,b) and (b,a) locations) gives us

$$\Sigma_{j\neq a,b}^{n} 2i(|g_{j,b}|^2 - \frac{k(n-k)}{n^2(n-1)})[(1-2g_{b,b})(g_{j,a}g_{b,j} - g_{a,j}g_{j,b}) + 2|g_{b,j}|^2(g_{a,b} - g_{b,a})]$$

Now, computing the contribution of the diagonal excluding the (a,a) and (b,b) locations, where $\delta = \frac{2k(k-1)}{n-1}$, we have

$$\begin{aligned}
&\frac{d}{dt}((|g_{j,j} - 2itg_{j,a}g_{b,j}|^2 - \frac{k^2}{n^2})^2 + \delta(|g_{j,j} - 2itg_{j,a}g_{b,j}| - \frac{k}{n})^2)|_{t=0} \\
&\quad = \frac{d}{dt}((g_{j,j} - 2itg_{j,a}g_{b,j})(g_{j,j} + 2itg_{a,j}g_{j,b}) - \frac{k^2}{n^2})^2|_{t=0} \\
&\quad + \frac{d}{dt}\delta((g_{j,j} - 2itg_{j,a}g_{b,j})(g_{j,j} + 2itg_{a,j}g_{j,b}) \\
&\quad - \frac{2k}{n}\sqrt{g_{j,j}^2 + 2itg_{j,j}(g_{a,j}g_{j,b} - g_{j,a}g_{b,j}) + 4t^2|g_{j,a}|^2|g_{b,j}|^2} + \frac{k^2}{n^2})|_{t=0} \\
&\quad = \frac{d}{dt}((g_{j,j}^2 - \frac{k^2}{n^2}) + 2itg_{j,j}(g_{a,j}g_{j,b} - g_{j,a}g_{b,j}) + 4t^2|g_{j,a}|^2|g_{b,j}|^2)^2|_{t=0}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{d}{dt}\delta((g_{j,j}^2+\frac{k^2}{n^2})+2itg_{j,j}(g_{a,j}g_{j,b}-g_{j,a}g_{b,j})\\
&-\frac{2k}{n}\sqrt{g_{j,j}^2+2it(g_{j,j}g_{a,j}g_{j,b}-g_{j,j}g_{j,a}g_{b,j})+4t^2|g_{j,a}|^2|g_{b,j}|^2+4t^2g_{j,a}^2g_{a,j}^2})|_{t=0}\\
&=2i(g_{a,j}g_{j,b}-g_{j,a}g_{b,j})(2g_{j,j}(g_{j,j}^2-\frac{k^2}{n^2})+\delta g_{j,j}-\frac{k\delta}{n})
\end{aligned}
$$

and so the diagonal contributes

$$\Sigma_{j\neq a,b}^{n}2i(g_{a,j}g_{j,b}-g_{j,a}g_{b,j})(2g_{j,j}(g_{j,j}^2-\frac{k^2}{n^2})+\frac{2k(k-1)}{n-1}(g_{j,j}-\frac{k}{n}))$$

Four points remain for our investigation before we sum everything; the (a,a), (b,b), (a,b), and (b,a) locations. Notice that the (a,b) location provides a contribution of

$$\frac{d}{dt}(|g_{a,b}+it(g_{b,b}+g_{a,a}-2g_{a,a}g_{b,b})|^2-\frac{k(n-k)}{n^2(n-1)})^2|_{t=0}$$

$$=2i(|g_{a,b}|^2-\frac{k(n-k)}{n^2(n-1)})(g_{b,a}-g_{a,b})(g_{b,b}+g_{a,a}-2g_{a,a}g_{b,b})$$

while the (b,a) entry yields $4i(|g_{b,a}|^2-\frac{k(n-k)}{n^2(n-1)})(g_{a,b}-g_{b,a})|g_{a,b}|^2$. These sum to

$$2i(|g_{b,a}|^2-\frac{k(n-k)}{n^2(n-1)})(g_{a,b}-g_{b,a})(2|g_{a,b}|^2+2g_{a,a}g_{b,b}-g_{b,b}-g_{a,a})$$

.

Now, again letting $\delta=\frac{2k(k-1)}{n-1}$, we evaluate the (a,a) location and note that

$$
\begin{aligned}
&\frac{d}{dt}((|g_{a,a}+it(1-2g_{a,a})g_{b,a}|^2-\frac{k^2}{n^2})^2+\delta(|g_{a,a}+it(1-2g_{a,a})g_{b,a}|-\frac{k}{n})^2)|_{t=0}\\
&=\frac{d}{dt}((g_{a,a}+it(1-2g_{a,a})g_{b,a})(g_{a,a}-it(1-2g_{a,a}g_{a,b})-\frac{k^2}{n^2})^2|_{t=0}
\end{aligned}
$$

$$+\frac{d}{dt}\delta((g_{a,a}^2+itg_{a,a}(1-2g_{a,a})(g_{b,a}-g_{a,b})+t^2(1-2g_{a,a})^2|g_{a,b}|^2$$
$$-\frac{2k}{n}\sqrt{(g_{a,a}^2+itg_{a,a}(1-2g_{a,a})(g_{b,a}-g_{a,b})+t^2(1-2g_{a,a})^2|g_{a,b}|^2}+\frac{k^2}{n^2}))|_{t=0}$$
$$=\frac{d}{dt}((g_{a,a}^2-\frac{k^2}{n^2})+itg_{a,a}(1-2g_{a,a})(g_{b,a}-g_{a,b})+t^2(1-2g_{a,a})^2|g_{a,b}|^2)^2|_{t=0}$$
$$+\delta(ig_{a,a}(1-2g_{a,a})(g_{b,a}-g_{a,b})-\frac{ik}{n}(1-2g_{a,a})(g_{b,a}-g_{a,b})$$
$$=i(1-2g_{a,a})(g_{b,a}-g_{a,b})[2g_{a,a}(g_{a,a}^2-\frac{k^2}{n^2})+\delta(g_{a,a}-\frac{k}{n})].$$

Similarly, (b,b) location contributes $i(1-2g_{b,b})(g_{b,a}-g_{a,b})[2g_{b,b}(g_{b,b}^2-\frac{k^2}{n^2})+\frac{2k(k-1)}{n-1}(g_{b,b}-\frac{k}{n})]$.

We are finally ready to sum all of the components of the gradient of W. Combining the sums and terms above gives us

$$\frac{dW(\gamma_{a,b}(t))}{dt}|_{t=0}=$$
$$\Sigma_{j\neq a,b}^n 2i(|g_{j,a}|^2-\frac{k(n-k)}{n^2(n-1)})[(1-2g_{a,a})(g_{b,j}g_{j,a}-g_{j,b}g_{a,j})+2|g_{j,a}|^2(g_{a,b}-g_{b,a})]$$
$$+\Sigma_{j\neq a,b}^n 2i(|g_{j,b}|^2-\frac{k(n-k)}{n^2(n-1)})[(1-2g_{b,b})(g_{j,a}g_{b,j}-g_{a,j}g_{j,b})+2|g_{b,j}|^2(g_{a,b}-g_{b,a})]$$
$$+\Sigma_{j\neq a,b}^n 2i(g_{a,j}g_{j,b}-g_{j,a}g_{b,j})(2g_{j,j}(g_{j,j}^2-\frac{k^2}{n^2})+\frac{2k(k-1)}{n-1}(g_{j,j}-\frac{k}{n}))$$
$$+2i(|g_{b,a}|^2-\frac{k(n-k)}{n^2(n-1)})(g_{a,b}-g_{b,a})(2|g_{a,b}|^2+2g_{a,a}g_{b,b}-g_{b,b}-g_{a,a})$$
$$+i(2g_{a,a}(|g_{a,a}|^2-\frac{k^2}{n^2})+\frac{2k(k-1)}{n-1}(g_{a,a}-\frac{k}{n}))(1-2g_{a,a})(g_{b,a}-g_{a,b})$$
$$+i(2g_{b,b}(|g_{b,b}|^2-\frac{k^2}{n^2})+\frac{2k(k-1)}{n-1}(g_{b,b}-\frac{k}{n}))(1-2g_{b,b})(g_{b,a}-g_{a,b}).$$

Letting $j=b$ in the first sum, $j=a$ in the second, and $j=a,b$ in third sum and subtracting these from the remaining terms allows us to rewrite this expression as below.

$$\frac{dW(\gamma_{a,b}(t))}{dt}|_{t=0} =$$
$$\Sigma_{j\neq a}^{n} 2i(|g_{j,a}|^2 - \frac{k(n-k)}{n^2(n-1)})[(1-2g_{a,a})(g_{b,j}g_{j,a} - g_{j,b}g_{a,j}) + 2|g_{j,a}|^2(g_{a,b} - g_{b,a})]$$
$$+ \Sigma_{j\neq b}^{n} 2i(|g_{j,b}|^2 - \frac{k(n-k)}{n^2(n-1)})[(1-2g_{b,b})(g_{j,a}g_{b,j} - g_{a,j}g_{j,b}) + 2|g_{b,j}|^2(g_{a,b} - g_{b,a})]$$
$$+ \Sigma_{j}^{n} 2i(g_{a,j}g_{j,b} - g_{j,a}g_{b,j})(2g_{j,j}(g_{j,j}^2 - \frac{k^2}{n^2}) + \frac{2k(k-1)}{n-1}(g_{j,j} - \frac{k}{n}))$$
$$+ 2i(|g_{b,a}|^2 - \frac{k(n-k)}{n^2(n-1)})(g_{a,b} - g_{b,a})(2|g_{a,b}|^2 + 2g_{a,a}g_{b,b})$$
$$+ i(2g_{a,a}(|g_{a,a}|^2 - \frac{k^2}{n^2}) + \frac{2k(k-1)}{n-1}(g_{a,a} - \frac{k}{n}))$$
$$+ i(2g_{b,b}(|g_{b,b}|^2 - \frac{k^2}{n^2}) + \frac{2k(k-1)}{n-1}(g_{b,b} - \frac{k}{n}))$$

So the gradient of W is an $n \times n$ matrix with 0s on the diagonal and (a, b) entries written as above. □

Now we can use this expression to bound $|grad\ W|$ by the frame energy. This result is similar to Lemma 3.19 in [8].

Lemma 3.16. *For a fixed Parseval frame F, the gradient of W is bounded by W. Specifically,*

$$|grad\ W| \leq \begin{cases} ((8n^2+40n+16)^2 \frac{3n(n-1)^2}{2k(k-1)} W(G))^{1/2} & 1 > \frac{2k(k-1)}{n-1}, \\ ((8n^2+40n+16)^2 \frac{12nk^2(k-1)^2}{n-1} W(G))^{1/2} & 1 \leq \frac{2k(k-1)}{n-1} \end{cases}$$

Proof. Using Minkowski's inequality and the fact that each $|g_{j,l}| \leq 1$ as G is an orthonormal projection, we can obtain a bound on 1-norm of *grad W* as follows:

$$
\begin{aligned}
||grad\ W||_1 &= \Sigma_{a,b}^n \left|\frac{dW(\gamma_{a,b}(t))}{dt}|_{t=0}\right| \\
&\leq \Sigma_{a,b}^n |2i\Sigma_{j\neq a}^n (|g_{j,a}|^2 - \frac{k(n-k)}{n^2(n-1)})((1-2g_{a,a})(g_{b,j}g_{j,a} - g_{j,b}g_{a,j}) + 2|g_{j,a}|^2(g_{a,b} - g_{b,a}))| \\
&+ \Sigma_{a,b}^n |2i\Sigma_{j\neq b}^n (|g_{j,b}|^2 - \frac{k(n-k)}{n^2(n-1)})((1-2g_{b,b})(g_{j,a}g_{b,j} - g_{a,j}g_{j,b}) + 2|g_{j,b}|^2(g_{a,b} - g_{b,a}))| \\
&+ \Sigma_{a,b}^n |2i\Sigma_j^n (2g_{j,j}(g_{j,j}^2 - \frac{k^2}{n^2}) + \frac{2k(k-1)}{n-1}(g_{j,j} - \frac{k}{n}))(g_{a,j}g_{j,b} - g_{j,a}g_{b,j})| \\
&+ \Sigma_{a,b}^n |2i(|g_{a,b}|^2 - \frac{k(n-k)}{n^2(n-1)})(2|g_{a,b}|^2 + 2g_{a,a}g_{b,b}(g_{b,a} - g_{a,b})| \\
&+ \Sigma_{a,b}^n |i(2g_{a,a}(|g_{a,a}|^2 - \frac{k^2}{n^2}) + \frac{2k(k-1)}{n-1}(g_{a,a} - \frac{k}{n}))| \\
&+ \Sigma_{a,b}^n |i(2g_{b,b}(|g_{b,b}|^2 - \frac{k^2}{n^2}) + \frac{2k(k-1)}{n-1}(g_{b,b} - \frac{k}{n}))| \\
&\leq 40n\Sigma_{j\neq l}^n ||g_{j,l}|^2 - \frac{k(n-k)}{n^2(n-1)}| \\
&+ 8n^2\Sigma_j^n |g_{j,j}^2 - \frac{k^2}{n^2}| + 8n^2\frac{2k(k-1)}{n-1}\Sigma_j^n |g_{j,j} - \frac{k}{n}| \\
&+ 16\Sigma_{a\neq b}^n ||g_{a,b}|^2 - \frac{k(n-k)}{n^2(n-1)}| \\
&+ 2n\Sigma_a^n |g_{a,a}^2 - \frac{k^2}{n^2}| + n\frac{2k(k-1)}{n-1}\Sigma_a^n |g_{a,a} - \frac{k}{n}| \\
&+ 2n\Sigma_b^n |g_{b,b}^2 - \frac{k^2}{n^2}| + n\frac{2k(k-1)}{n-1}\Sigma_b^n |g_{b,b} - \frac{k}{n}| \\
&\leq (40n+16)\Sigma_{j\neq l}^n ||g_{j,l}|^2 - \frac{k(n-k)}{n^2(n-1)}| \\
&+ (8n^2+4n)\Sigma_j^n |g_{j,j}^2 - \frac{k^2}{n^2}| + (4n^2+2n)\frac{2k(k-1)}{n-1}\Sigma_j^n |g_{j,j} - \frac{k}{n}| \\
&+ 16\Sigma_{a\neq b}^n ||g_{a,b}|^2 - \frac{k(n-k)}{n^2(n-1)}|
\end{aligned}
$$

Converting our comparison now to the l^2 norm where $\alpha_m = max\{1, \frac{2k(k-1)}{n-1}\}$, we see that

$$
\begin{aligned}
||grad\ W||_1 \leq & (8n^2+40n+16)\alpha_m[\Sigma_{j\neq k}^n|(|g_{j,k}|^2 - \frac{k(n-k)}{n^2(n-1)})| \\
& + \Sigma_j^n|(g_{j,j}^2 - \frac{k^2}{n^2})| + \Sigma_j^n|(g_{j,j} - \frac{k}{n})|] \\
& \leq (8n^2+40n+16)\alpha_m\sqrt{n(n-1)}[(\Sigma_{j\neq k}^n|(|g_{j,k}|^2 - \frac{k(n-k)}{n^2(n-1)})|^2)^{1/2} \\
& + (\Sigma_j^n|(g_{j,j}^2 - \frac{k^2}{n^2})|^2)^{1/2} + (\Sigma_j^n|(g_{j,j} - \frac{k}{n})|^2)^{1/2}] \\
& \leq (8n^2+40n+16)\alpha_m\sqrt{n(n-1)}\sqrt{3}[\Sigma_{j\neq k}^n|(|g_{j,k}|^2 - \frac{k(n-k)}{n^2(n-1)})|^2 \\
& + \Sigma_j^n|(g_{j,j}^2 - \frac{k^2}{n^2})|^2 + \Sigma_j^n|(g_{j,j} - \frac{k}{n})|^2]^{1/2}
\end{aligned}
$$

So for $\alpha_m = 1$, we have

$$
\begin{aligned}
||grad\ W||_2^2 \leq & ||grad\ W||_1^2 \\
& \leq (8n^2+40n+16)^2\alpha_m^2 3n(n-1)[\Sigma_{j\neq k}^n|(|g_{j,k}|^2 - \frac{k(n-k)}{n^2(n-1)})|^2 \\
& + \Sigma_j^n|(g_{j,j}^2 - \frac{k^2}{n^2})|^2 + \Sigma_j^n|(g_{j,j} - \frac{k}{n})|^2] \\
& \leq (8n^2+40n+16)^2\frac{3n(n-1)^2}{2k(k-1)}[\Sigma_{j\neq k}^n|(|g_{j,k}|^2 - \frac{k(n-k)}{n^2(n-1)})|^2 \\
& + \Sigma_j^n|(g_{j,j}^2 - \frac{k^2}{n^2})|^2 + \frac{2k(k-1)}{n-1}\Sigma_j^n|(g_{j,j} - \frac{k}{n})|^2] \\
& \leq (8n^2+40n+16)^2\frac{3n(n-1)^2}{2k(k-1)}[W(G)]
\end{aligned}
$$

Meanwhile, for $\alpha_m = \frac{2k(k-1)}{n-1} > 1$,

$$
||grad\ W||_2^2 \leq ||grad\ W||_1^2
$$

$$\le (8n^2+40n+16)^2\alpha_m^2 3n(n-1)[\Sigma_{j\neq k}^n|(|g_{j,k}|^2-\frac{k(n-k)}{n^2(n-1)})|^2$$
$$+\Sigma_j^n|(g_{j,j}^2-\frac{k^2}{n^2})|^2+\Sigma_j^n|(g_{j,j}-\frac{k}{n})|^2]$$
$$\le (8n^2+40n+16)^2\alpha_m^2[\Sigma_{j\neq k}^n|(|g_{j,k}|^2-\frac{k(n-k)}{n^2(n-1)})|^2$$
$$+\Sigma_j^n|(g_{j,j}^2-\frac{k^2}{n^2})|^2+\frac{2k(k-1)}{n-1}\Sigma_j^n|(g_{j,j}-\frac{k}{n})|^2]$$
$$\le (8n^2+40n+16)^2\frac{12nk^2(k-1)^2}{n-1}[W(G)]$$

□

It is straightforward to verify that for the Grammian, G of any equiangular Parseval frame, $grad\, U(G)=0$. We include this as a quick lemma. As a consequence, if $U(G)\to 0$ then convergence is at most a linear rate as

$$\frac{d}{dt}U_C(G(t))=-||grad\ U_C||^2\ge -cU_C(G(t)).$$

Lemma 3.17. *If $F=\{f_j\}_{j=1}^n$ is an (n,k) equiangular Parseval frame with Grammian $G=g_{j,l}$ then grad U is the 0-matrix.*

Proof. By Proposition 3.14 we can write,

$$\begin{aligned}
[grad\ U]_{a,b} &= 8i\Sigma_{j=1}^n(|g_{j,a}|^2-|g_{j,b}|^2)g_{a,j}g_{j,b}\\
&= 8i\Sigma_{j\neq a,b}^n(|g_{j,a}|^2-|g_{j,b}|^2)g_{a,j}g_{j,b}+(|g_{a,a}|^2-|g_{a,b}|^2)g_{a,a}g_{a,b}+(|g_{b,a}|^2-|g_{b,b}|^2)g_{a,b}g_{b,b}\\
&= 8i(0+0)\\
&= 0
\end{aligned}$$

The sum is 0 due to equiangularity and the other two terms sum to 0 due to the equal-norm property of F. So every entry of *grad* $U(F)$ is zero. □

In our next result, we characterize the summing conditions required of the frame vectors of any fixed point.

Theorem 3.18. *If F is a Parseval frame, with Gram matrix G such that grad $U(G)$ is the zero matrix then the following summing conditions hold for all values of a and b:*

$$\Sigma_j |\langle f_j, f_a \rangle|^2 \langle f_a, f_j \rangle \langle f_j, f_b \rangle = \Sigma_j |\langle f_j, f_b \rangle|^2 \langle f_a, f_j \rangle \langle f_j, f_b \rangle$$

Proof. Omitting the explicit time dependence of the vectors, we can see that

$$[grad\ U]_{a,b} = 8i \Sigma_{j=1}^{n} (|g_{j,a}|^2 - |g_{j,b}|^2) g_{a,j} g_{j,b} = 0$$

when $\Sigma_{j=1}^{n}(|g_{j,a}|^2 - |g_{j,b}|^2) g_{a,j} g_{j,b} = 0$ for all a, b. Equivalently,

$$\Sigma_{j=1}^{n} |g_{j,a}|^2 g_{a,j} g_{j,b} = \Sigma_{j=1}^{n} |g_{j,b}|^2 g_{a,j} g_{j,b}$$

or in terms of the frame vectors, we can write

$$\Sigma_j |\langle f_j, f_a \rangle|^2 \langle f_a, f_j \rangle \langle f_j, f_b \rangle = \Sigma_j |\langle f_j, f_b \rangle|^2 \langle f_a, f_j \rangle \langle f_j, f_b \rangle.$$

□

The above condition is satisfied even in cases where F is not equiangular.

Example 3.19. Given a real or complex Hilbert space H of dimension d and an orthonormal basis $\{e_1, e_2, ...e_d\}$ for H, we can construct a Parseval frame $\{f_j\}_{j=1}^{2d+1}$ by

$$f_j = \begin{cases} \frac{1}{\sqrt{2}} e_j & 1 \leq j \leq d, \\ \frac{-1}{\sqrt{2}} e_{j-d} & d+1 \leq j \leq 2d, \\ 0 & j = 2d+1 \end{cases}$$

It is fairly straightforward to confirm that this frame satisfies the summing conditions in the previous proposition and is therefore a fixed point for our system. However, it is not an equiangular Parseval frame.

We recall the following definition.

Definition 3.20. A frame F is said to be *orthodecomposable* if it can be split into two nontrivial subcollections F_1, and F_2 satisfying $F_1^* F_2 = 0$. That is, span F_1 and span F_2 are nontrivial orthogonal subspaces.

Notice that our non-equiangular example is such a frame. Orthodecomposable frames generally occur with complications of local structure. However, we will see that while frame energy decreases we converge to a fixed point.

Definition 3.21. We define σ_n to be the uniform probability measure on the n-torus $\mathbb{T}^n = \{c \in \mathbb{F}^n : |c_j| = 1 \ for \ all \ j\}$, where $\mathbb{F}$ is $\mathbb{R}$, or $\mathbb{C}$.

Recall the definition of two frames being switching equivalent and notice that if $F = \{f_j\}_{j=1}^n$, and $G = \{g_j\}_{j=1}^n$ are switching equivalent, then $U(F) = U(G)$. However, while $U(F)$ is switching invariant, *grad* $U(F)$ depends on which representative of its switching-equivalence class is chosen.

Proposition 3.22. *Given a Parseval frame $F = \{f_j\}_{j=1}^n$, then either F is a fixed point of U, or there is a choice $c \in \mathbb{T}^n$ such that grad $U(F^{(c)})_{a,b} < 0$ for some (a, b).*

Proof. Let G denote the Grammian of F. For the switched frame $F^{(c)}$, we know that each (a,b) entry of the gradient matrix of U is

$$grad\ U(F^{(c)})_{a,b} = 8i\Sigma_j^n(|g_{j,a}|^2 - |g_{j,b}|^2)c_a c_j^* c_j c_b^* g_{a,j} g_{j,b}$$

Integrating over the torus $\mathbb{T}^n$ with respect to the switching invariant measure σ_n gives us that

$$\int_{\mathbb{T}^n} c_a c_j^* c_j c_b^*\ d\sigma_n(c) = \delta_{a,j}\delta_{j,b} + \delta_{a,b}\delta_{j,j}$$

Since terms with $a = b$ do not contribute to the gradient,

$$\int_{\mathbb{T}^n} grad\ U(F^{(c)})_{a,b}\ d\sigma_n(c) = 0.$$

That is, every matrix entry of the gradient averages to 0. Since the average is equal to zero, there must be a choice of c which gives $grad\ U(F^{(c)})_{a,b} < 0$ for some (a,b). □

Therefore there exists a curve $\gamma_{a,b}$ such that

$$\frac{d}{dt}U(\gamma_{a,b}(t)) = grad\ U(F^{(c)}) \cdot \frac{d}{dt}\widetilde{\gamma}_{a,b}(t) \leq 0.$$

Corollary 3.23. *For any Parseval frame $F = \{f_j\}_{j=1}^n$, with Grammian $G(0)$ applying a gradient descent as described before provides a trajectory $\{G(t)\}_{t\geq 0}$ under which accumulation points are fixed points.*

Proof. By the previous result we know that $(grad\ U(G(0))_{a,b} < 0$ for some (a,b), and so

$$\frac{d}{dt}|_{t=0}\ U(\gamma_{a,b}(t)) = grad\ U \cdot \frac{d\gamma_{a,b}}{dt} < 0$$

where $\gamma_{a,b}(t) = G_0 + it((I - G_0)E_{a,b}G_0 + G_0E_{a,b}(I - G_0))$. By the Cauchy-Schwarz inequality,

$$-grad\ U \cdot \frac{grad\ U}{||grad\ U||}||\frac{d\gamma}{dt}|| \leq grad\ U \cdot \frac{d\gamma}{dt}$$

and so the direction of steepest descent must be negative. Now, since the gradient is negative and the gradient norm of U is bounded we can apply Theorem 4.6 of [17] to conclude that when when the limit of $G(t)$ exists, it must converge to a fixed point of U. □

Several open problems remain including the following:

1. Can the stationary points of the frame energy function be further characterized?
2. What is the rate of convergence to a fixed point?
3. Can we compute distance from a Parseval frame to a stationary point of the frame energy?
4. Can we rule out the existence of non-equiangular Parseval frames as stationary points if the energy is sufficiently low?

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