# APPLICATIONS OF FINITE GROUPS TO PARSEVAL FRAMES

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Presented to

the Faculty of the Department of Mathematics

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In Partial Fulfillment

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Doctor of Philosophy

By

Preeti Singh

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*Dedicated to my late grandmother Mrs. Swaran Kaur*

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# Abstract

Frames are fundamental tools that are robust to quantization, resilient to additive noise, give stable reconstruction after erasures, and give greater freedom to capture important signal characteristics. Constructing tight frames using various mathematical techniques has been an important area of investigation. In this dissertation we study combinatorial techniques to construct uniform tight frames.

We begin by constructing uniform Parseval frames using group representations. We examine conditions on a representation of a group to form a frame representation. In addition, we give an explicit construction of Parseval frame vectors for these frame representations. For insight into applications, we measure the correlation between the frame elements using characters of group representations and give necessary and sufficient conditions for the maximum correlation to be as small as possible. We also derive bounds on the maximum correlation between the frame elements of a tight frame constructed using our techniques. This enables us to differentiate between the behavior of two tight frames in applications.

Equiangular tight frames are an important class of finite dimensional frames because of their superior performance and numerous applications. We present a new tool to construct equiangular tight frames using groups and the left regular representation of a group. We prove that many equiangular tight frames arise from subsets of groups which we will call “signature sets”.

Subsequently, we define “quasi-signature sets” and examine real equiangular tight frames associated to these subsets of groups. This approach yields further results and establishes new correspondences. We are able to show many examples of equiangular

tight frames arising from quasi-signature sets.

Difference sets are another subsets in groups that are seen to be associated with equiangular frames. We will look at the relationship between difference sets and signature sets. We observe a correspondence between reversible Hadamard difference sets and signature sets.

We extend these results to complex equiangular tight frames where the inner product between any pair of vectors is a common multiple of a cube root of unity and exhibit equiangular tight frames that arise from groups in this manner.

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# Chapter 1

# Introduction

Frames are redundant signal representations having many applications in wavelet theory [1, 38, 40], signal and image processing [5, 6, 7, 8, 10], data transmission with erasures [11, 18, 35], quantum computing [9, 29], CDMA systems [39, 57] and more. The theory of frames was initiated by Duffin and Schaeffer [28] in 1952 as a part of an ongoing development of non-harmonic Fourier series. The role of frames in signal processing was initiated by Daubechies, Grossman, and Meyer [24]. In recent years, great progress has been made in the understanding and implementation of frames.

Of particular interest are tight frames as they are closest in behavior to orthonormal bases. Naimark [2] and Han and Larson [37] showed that all tight frames are projections of orthonormal bases from a larger space. If all the vectors in a tight frame have the same norm, then the frame is called a **uniform tight frame**. In [35], it was shown that uniform tight frames optimize robustness to quantization noise. It was also shown that one erasure from a uniform tight frame cannot destroy the property of being a frame.

A broad spectrum of researchers have worked on the construction of tight frames. It has been been approached by mathematicians having backgrounds in functional analysis [16, 19, 21], operator theory [33, 37, 38], graph theory [11, 41, 55], number theory [44, 59] etc. In this thesis we employ combinatorial techniques to show a construction of uniform tight frames using finite groups.

## 1.1 Outline

This dissertation includes work on two projects, both involving construction of uniform tight frames using combinatorial techniques. In what follows, we outline the organization of this thesis, and briefly discuss the chapters that are presented. We begin by giving some background on frame theory, and group representation theory in Chapter 2. Chapters 3 and 4 focus on the first project of this dissertation. Here we present an operator theoretic construction of tight frames using group representations.

In [37], an operator theoretic approach to discrete frame theory has been presented. Frames have a natural geometric interpretation as sequences of vectors which dilate (geometrically ) to bases. A major advantage of this approach is presentation of the proofs of some of the existing results on frames, in a simplified, and direct manner. Our motivation has been to use group representations to generalize the examples of the frames of the form $\{M^x C^y v : x, y \in \mathbb{Z}_k\}$ referred as chirps in [17, 60] and $\{T^x M^y u : x, y \in \mathbb{Z}_k\}$ known as finite Gabor frames [31, 34, 36] where

- $M^x$ is the modulation operator;

- $C^y$ is the chirp modulation operator;

- $T^x$ is the translation operator.

In Chapter 3, we will look at group representations ranging from the irreducible representations known as the building blocks, to subrepresentations of the left regular representation. We shall call a vector $v \in \mathbb{C}^k$ a **Parseval frame vector** for $\pi$, if the collection $\{\pi(g)v\}_{g\in G}$ is a Parseval frame for $\mathbb{C}^k$. We present new proofs of some of the results proved in [33], and [37] and extend them to applications in Chapter 4. Since our approach has been to employ frames for the Hilbert space of $k \times k$ matrices, our procedures are more direct. Moreover, we easily derive verifiable conditions for a vector to be a Parseval frame vector.

This approach has led to new results, and has simplified some of the existing results. For example a comparison between the behavior in applications of two tight frames constructed using group representations has been presented. We have also succeeded in our endeavor to classify finite Gabor frames [31, 34, 36], and frames corresponding to the modulation of chirps [17, 60] using group representations. Gabor (or Weyl-Heisenberg) frames provide the fundamental tool for modern day signal/image processing. These frames are known under various names: oversampled DFT filter banks, complex modulated filter banks, short-time Fourier filter banks, and Gabor filter banks, and have been studied in [14, 13, 15]. In a recent work [48], the authors study finite dimensional Gabor systems with $n^2$ vectors, in $\mathbb{C}^n$ where $n$-prime which are optimal for $n^2 - n$ erasures.

This is followed by some results on the frame correlation using character theory of groups in Chapter 4. Since all tight frames do not behave the same in applications, we intend to look at those tight frames for which the frame elements are designed to be as uncorrelated as possible. The degree to which frame elements are uncorrelated is measured by the quantity $\max_{i \neq j}\{|\langle f_i, f_j \rangle|\}$. Subsequently, we show that this quantity is directly related to the characters of group representations, and estimate bounds on the maximum correlation using character theory of groups.

In the second project, we have worked on generating equiangular tight frames using finite groups, and the left regular representation of a group [54]. Chapters 5, 6, and 7 give details of the work done on this project for the real case whereas in Chapters 8 and 9, we present an extension of our work to the complex case.

Equiangular tight frames play an important role in several areas of mathematics, ranging from signal processing (see, e.g. [3], [16], [45], [46], and references therein) to quantum computing. A detailed study of this class of frames was initiated by Strohmer and Heath [55], and Holmes and Paulsen [41]. Holmes and Paulsen have shown that equiangular tight frames give error correction codes that are robust against two erasures. Bodmann and Paulsen [11] analyze arbitrary numbers of erasures for equiangular tight frames. Sustik, Tropp, Dhillon and Heath [56] derive necessary conditions on the existence of equiangular tight frames. Casazza, Redmond and Tremain [20] gave a classification of equiangular tight frames for real Hilbert spaces of dimension less than or equal to 50. Equiangular tight frames potentially have many practical, and theoretical applications, see for example in [45], [46], and [47].

The problem of the existence of equiangular tight frames is known to be equivalent to the existence of a certain type of matrix called a Seidel matrix [49] or signature matrix [41] with two eigenvalues. A matrix $Q$ is a Seidel matrix provided that it is self-adjoint, its diagonal entries are 0, and its off-diagonal entries are all of modulus one. In the real case, these off-diagonal entries must all be $\pm 1$; such matrices can then be interpreted as (Seidel) adjacency matrices of graphs.

A wide group of researchers have worked on the construction of equiangular frames. This has resulted in the cross fertilization between different areas of mathematics such as graph theory, operator theory and linear algebra. In Chapter 5, we present a new approach to construct signature matrices [41] by using subsets of groups which we call **signature sets**. Using basic facts from group theory, we develop necessary, and sufficient conditions for the existence of signature sets. The beauty of group theory has resulted in the construction of signature sets in a clear and concise manner thereby providing numerous examples of real equiangular tight frames associated with them.

Another class of subsets of groups known as difference sets are seen to be associated with equiangular tight frames as shown in [44] and [59]. In Chapter 6, we aim to establish a relation between signature sets and difference sets. We will show that the existence of signature sets for $(n, \frac{n-\sqrt{n}}{2})$-equiangular frames is equivalent to the existence of certain reversible Hadamard difference sets [25]. An active research area is the determination of those groups that support a reversible Hadamard difference set, (see [25, 30]).

If $Q$ is a Seidel matrix, we say that $Q$ is in a standard form if its first row and column contains only 1's except on the diagonal. It has been shown in [49] that with an appropriate diagonal unitary, the switching equivalence class of any Seidel matrix contains a matrix in standard form. In Chapter 7, we investigate the case of signature matrices in standard form. Subsequently, we will define **quasi-signature sets** to construct signature matrices in the standard form. As a result, we present two algorithms to generate equiangular frames of the type $(2k, k)$ arising from quasi-signature sets.

We will also look at an important relationship between **Artin's conjecture** [4] (1927), and one of the algorithms based on constructing equiangular frames containing $2k$ vectors in a $k$ dimensional Hilbert space. Artin's conjecture states that: "Every integer $a$, not equal to $-1$ or to a square, is a primitive root mod $p$ of infinitely many primes". In the nineteenth century, several mathematicians proved (see chapter VII in [26] for references) that whenever $p$ is of the form $4q + 1$, $q$-odd prime, 2 is a primitive root (mod $p$).

In Chapters 8 and 9, an extension of the results in the case of real equiangular frames to the case when the entries of $Q$ are cube roots of are presented. In [12], it was shown that the existence of such matrices is equivalent to the existence of certain highly regular directed graphs. Using group theory and combinatorics, we will show an analogy of the results between the approach we have used, and shown in [12]. Using our techniques, we are able to recover the $(9, 6)$-cube root equiangular frame from the group of quaternions.

In Chapter 10 we give a comprehensive list of the examples of equiangular tight frames obtained using groups. Finally, in Chapter 11, we state implications and some of the future projects related to the work presented in this thesis.

# Chapter 2

# Background

For the reader's convenience, this chapter catalogs the fundamentals needed to understand and comprehend the observations and results in the following chapters. We begin by giving an elementary exposition of frames, representation theory for finite groups, basics of group theory, and number theory suitable for our work in subsequent chapters.

## 2.1 Basic frame theory

**Definition 2.1.** *A finite family of vectors* $\{f_1, \ldots, f_n\}$ *in a Hilbert space* $\mathcal{H}$ *of dimension* $k$ *is called an* $(n, k)$*-frame provided that there exist strictly positive real numbers* $A$ *and* $B$ *such that*

$$A\|x\|^2 \leq \sum_{j=1}^{n} |\langle x, f_j \rangle|^2 \leq B\|x\|^2 \qquad \textit{for all } x \in \mathcal{H}. \tag{2.1}$$

*$A$, $B$ are called frame bounds.*

A frame $\{f_j\}_{j=1}^n$ is called a **tight frame**, if $A = B$. In this case we have

$$x = \frac{1}{A}\sum_{j=1}^{n}\langle x, f_j\rangle f_j, \qquad \text{for all } x \in \mathcal{H}. \tag{2.2}$$

A frame $\{f_1, \dots, f_n\}$ for $\mathbb{C}^k$ is called a **uniform frame** if there is a constant $u > 0$ such that $\|f_i\| = u$ for all $i$.

If all the vectors in a tight frame have unit norm, then the frame is called a **normalized tight** frame.

A **Parseval frame** is a tight frame with frame bound $A = 1$. The equation (2.2), by pulling $\frac{1}{A}$ into the sum, is equivalent to

$$x = \sum_{j=1}^{n}\langle x, \frac{1}{\sqrt{A}} f_j\rangle \frac{1}{\sqrt{A}} f_j, \qquad \text{for all } x \in \mathcal{H}. \tag{2.3}$$

In other words, any tight frame can be rescaled to be a tight frame with frame bound 1, i.e., a Parseval tight frame. Given a frame $\{f_i\}_{i\in I}$, consider the map $V : \mathcal{H} \to l^2(I)$ defined by

$$(Vx)_i = \langle x, f_i\rangle, \quad i \in I, \quad x \in \mathcal{H}.$$

The operator $V$ is called the *analysis operator*. We now show that if $\{f_i\}_{i\in I}$ is a Parseval frame, then $V$ is an isometry.

Let $\{f_i\}_{i\in I}$ be a Parseval frame. Thus, by Parseval's identity (2.3),

$$\|(Vx)\|_2^2 = \sum_{i\in I} |(Vx)_i|^2 = \sum_{i\in I} |\langle x, f_i\rangle|^2 = \|x\|_{\mathcal{H}}^2$$

where $\|\cdot\|_2$ denotes the $l^2$ norm, and $\|\cdot\|_{\mathcal{H}}$ denotes the norm in $\mathcal{H}$. Hence, $V$ is an isometry.

The adjoint $V^* : l^2(I) \to \mathcal{H}$ of $V$ is given by

$$V^*(y) = \sum_{i \in I} f_i y_i, \qquad \text{for all } y = (y_i)_{i \in I} \in l^2(I).$$

We will now show that the adjoint $V^*$ acts a left inverse to $V$. Considering $V^*V$, we have

$$V^*V(x) = V^*((\langle x, f_i \rangle)_{i \in I}) = \sum_{i \in I} \langle x, f_i \rangle f_i = x, \qquad \text{for all } x \in \mathcal{H}.$$

Thus $V^*$ acts as a left inverse to $V$. If we identify the analysis operator $V$ of an $(n, k)$-frame with an $n \times k$ matrix, using the standard basis, then the columns of $V^*$ are the frame vectors. Thus,

$$V^* = [f_1 \quad f_2 \quad \dots \quad f_n]$$

and consequently

$$V = \begin{bmatrix} f_1^* \\ f_2^* \\ \vdots \\ f_n^* \end{bmatrix}$$

Therefore,

$$VV^* = (f_i^* f_j)_{i,j} = (\langle f_j, f_i \rangle)_{i,j}$$

Thus, $VV^*$ is called the *Grammian (or correlation)* matrix corresponding to the frame $\{f_i\}_{i=1}^n$.

*We shall let $\mathcal{F}(n,k)$ denote the collection of all Parseval frames for $\mathbb{F}^k$ consisting of $n$ vectors, and refer to such a frame as either a real or complex $(n,k)$-frame, depending on whether or not the field $\mathbb{F}$ is the real numbers or the complex numbers.*

Using some basic operator theory, $\mathcal{F}$ is an $(n,k)$-Parseval frame if and only if the Grammian (or correlation) matrix $VV^* = (\langle f_j, f_i\rangle)_{i,j=1}^n$ of the frame vectors is a self-adjoint $n \times n$ projection of rank $k$. Moreover, the rank of a projection is equal to its trace, so $tr(VV^*) = k$. Thus, when $\mathcal{F}$ is a uniform $(n,k)$-frame, each of the diagonal entries of $VV^*$ must be equal to $\frac{k}{n}$, and hence each frame vector $f_j$ must be of length $\|f_j\| = \frac{k}{n}$.

Conversely, given an $n \times n$ self-adjoint projection $P$ of rank $k$, we can always factor it as $P = VV^*$ with an $n \times k$ matrix $V$, by choosing an orthonormal basis for the range of $P$ as the column vectors of $V$. It follows that $V^*V = I_k$ and hence $V$ is the matrix of an isometry and so corresponds to an $(n,k)$-frame. Moreover, if $P = WW^*$ is another factorization of $P$, then necessarily $W^*W = I_k$ and there exists a unitary $U$ such that $W^* = UV^*$ and hence the two corresponding frames differ by multiplication by this unitary. Thus, $P$ determines a unique unitary equivalence class of frames.

### 2.1.1 Equivalence of frames

In the following, we wish to identify certain frames as being equivalent [41].

**Definition 2.2.** *[41] Given frames $\mathcal{F} = \{f_1, \dots, f_n\}$ and $\mathcal{G} = \{g_1, \dots, g_n\}$, we say that they are type-I equivalent if there exists a unitary (orthogonal, in the real case)* *matrix $U$ such that $g_i = Uf_i$ for all $i$.*

If $V$ and $W$ are the analysis operators for $\mathcal{F}$ and $\mathcal{G}$, respectively, then it is clear that $\mathcal{F}$ and $\mathcal{G}$ are type-I equivalent if and only if $V = WU$ or equivalently, if and only if

$$VV^* = WUU^*W^*$$

Thus, there is a one-to-one correspondence between the $n \times n$ projections of rank $k$ and type-I equivalence classes of $(n, k)$-frames.

**Definition 2.3.** *[41] Two frames $\mathcal{F} = \{f_1, \ldots, f_n\}$ and $\mathcal{G} = \{g_1, \ldots, g_n\}$ are said to be type-II equivalent if $\mathcal{G}$ can be obtained by permuting the vectors of $\mathcal{F}$.*

**Definition 2.4.** *[41] Two frames $\mathcal{F} = \{f_1, \ldots, f_n\}$ and $\mathcal{G} = \{g_1, \ldots, g_n\}$ are said to be type-III equivalent if the vectors in $\mathcal{G}$ differ from the vectors of $\mathcal{F}$ by multiplication with unimodular complex numbers $(\pm 1)$ in the complex (real) case.*

Finally, we say that two frames are equivalent if they belong to the same equivalence class in the equivalence relation generated by these three equivalence relations.

## 2.2 Equiangular tight frames

**Definition 2.5.** *An $(n, k)$-tight frame is called equiangular tight if all of the frame vectors are non-zero and the angle between the lines generated by any pair of frame vectors is a constant; that is, provided that there is a constant $b$ such that*

$$|\langle f_i/\|f_i\|, f_j/\|f_j\|\rangle| = b \quad \textit{for all} \quad i \neq j.$$

Many places in the literature define equiangular tight to mean that the $(n,k)$-frame is uniform and that there is a constant $c$ so that $|\langle f_i, f_j\rangle| = c$ for all $i \neq j$. However, the assumption that the frame is uniform is not needed in our definition as the following result shows.

**Proposition 2.6.** *[12, Proposition 1.2] Let $\{f_1, \dots, f_n\}$ be a tight frame for $\mathbb{C}^k$. If all frame vectors are non-zero and if there is a constant $b$ so that $|\langle f_i/\|f_i\|, f_j/\|f_j\|\rangle| = b$ for all $i \neq j$, then $\|f_i\| = \|f_j\|$ for every $i$ and $j$.*

*Proof.* Without loss of generality, we may assume that the frame is a Parseval frame, so that $P = (\langle f_j, f_i\rangle)_{i,j=1}^n$ is a projection of rank $k$. Hence, $P = P^2$ and so upon equating the $(i,i)$-th entry and using the fact that the trace of $P$ is $k$, we see that $\|f_i\|^2 = \langle f_i, f_i\rangle = \sum_{j=1}^n \langle f_j, f_i\rangle\langle f_i, f_j\rangle = \|f_i^4\| + \sum_{j\neq i}^n b^2\|f_i\|^2\|f_j\|^2 = \|f_i\|^4 + b^2\|f_i\|^2(k - \|f_i\|^2)$, which shows that $\|f_i\|^2$ is a (non-zero) constant independent of $i$.

□

In [41], a family of $(n,k)$-frames was introduced that was called 2-uniform frames. It was then proved that a Parseval frame is 2-uniform if and only if it is equiangular tight. Thus, these terminologies are interchangeable in the literature, but the equiangular tight terminology has become more prevalent.

**Definition 2.7.** *A matrix $Q$ is called a Seidel matrix provided that it is self-adjoint, its diagonal entries are $0$, and its off-diagonal entries are all of modulus $1$.*

The previous section shows that an $(n, k)$-frame is determined up to unitary equivalence by its Grammian matrix. This reduces the problem of constructing an $(n, k)$-frame to constructing an $n \times n$ self-adjoint projection $P$ of rank $k$. If an $(n, k)$-frame $\{f_1, f_2, \ldots, f_n\}$ is uniform, then it is known that $\|f_i\|^2 = \frac{k}{n}$ for all $i = \{1, 2, \ldots, n\}$.

It is shown in [41, Theorem 2.5] that if $\{f_1, \ldots, f_n\}$ is an $(n, k)$-equiangular tight tight frame, then for all $i \neq j$,

$$|\langle f_j, f_i \rangle| = c_{n,k} = \sqrt{\frac{k(n-k)}{n^2(n-1)}}.$$

Thus we may write

$$VV^* = (\frac{k}{n})I_n + c_{n,k}Q$$

where $Q$ is a self-adjoint $n \times n$ matrix satisfying $Q_{ii} = 0$ for all $i$ and $|Q_{ij}| = 1$ for all $i \neq j$. This matrix $Q$ is called the *signature matrix* associated with the $(n, k)$-equiangular tight frame.

The following theorem characterizes the signature matrices of equiangular tight $(n, k)$- frames.

**Theorem 2.8.** *[41, Theorem 3.3] Let $Q$ be a self-adjoint $n \times n$ matrix with $Q_{ii} = 0$ and $|Q_{ij}| = 1$ for all $i \neq j$. Then the following are equivalent:*

*(a) $Q$ is the signature matrix of an equiangular tight $(n, k)$-frame for some $k$;*

*(b) $Q^2 = (n-1)I + \mu Q$ for some necessarily real number $\mu$; and*

*(c) $Q$ has exactly two eigenvalues.*

This result reduces the problem of constructing equiangular tight $(n, k)$-frames to the problem of constructing Seidel matrices with two eigenvalues. In particular, condition (b) in Theorem 2.8 is particularly useful since it gives an easy-to-check condition to verify that a matrix $Q$ is the signature matrix of an equiangular tight tight frame. Furthermore, if $Q$ is a matrix satisfying any of the three equivalent conditions in Theorem 2.8, and if $\lambda_1 < 0 < \lambda_2$ are its two eigenvalues, then the parameters $n$, $k$, $\mu$, $\lambda_1$, and $\lambda_2$ satisfy the following properties:

$$\mu = (n-2k)\sqrt{\frac{n-1}{k(n-k)}} = \lambda_1 + \lambda_2, \qquad k = \frac{n}{2} - \frac{\mu n}{2\sqrt{4(n-1)+\mu^2}} \tag{2.4}$$

$$\lambda_1 = -\sqrt{\frac{k(n-1)}{n-k}}, \qquad \lambda_2 = \sqrt{\frac{(n-1)(n-k)}{k}}, \qquad n = 1 - \lambda_1\lambda_2. \tag{2.5}$$

These equations follow from the results in [41, Proposition 3.2] and [41, Theorem 3.3], and by solving for $\lambda_1$ and $\lambda_2$ from the given equations. In the case when the entries of $Q$ are all real, we have that the diagonal entries of $Q$ are 0 and the off-diagonal entries of $Q$ are $\pm 1$. In [41] and [55], it has been noted that there is a one-to-one correspondence between frame equivalence classes of real equiangular tight frames and regular two-graphs [49].

**Definition 2.9.** *Two Seidel matrices $Q$ and $Q'$ are switching equivalent if they can be obtained from each other by conjugating with a diagonal unitary and a permutation matrix.*

**Proposition 2.10.** *Let $Q$ be a signature matrix of an $(n, k)$-equiangular tight frame.*

*If $Q$ is switching equivalent to a Seidel matrix $Q'$, then $Q'$ is also a signature matrix of an $(n, k)$-equiangular tight frame.*

*Proof.* Since $Q$ and $Q'$ are switching equivalent, there exists a diagonal unitary $U$ and a permutation $P$ such that $Q' = UPQP^tU^*$. Then

$$
\begin{aligned}
Q'^2 &= (UPQP^tU^*)^2 \\
&= UPQ^2P^tU^* \\
&= UP((n-1)I + \mu Q)P^tU^* \qquad\qquad \text{(from Theorem 2.8)} \\
&= (n-1)I + \mu UPQP^tU^* \\
&= (n-1)I + \mu Q'.
\end{aligned}
$$

Again by using Theorem 2.8, $Q'$ forms a signature matrix for an $(n, k)$-equiangular tight frame. □

## 2.3 Representation theory of groups

*The following material on representation theory for finite groups is taken from the course notes [51].*

Let $V$ be a vector space over the field $\mathbb{F}$ where $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$. Let $\mathcal{L}(V)$ denote the linear transformations of $V$ into $V$. We let $GL(V)$ denote the group of invertible linear maps.

**Definition 2.11.** *[51, Definition 3.1] If $G$ is a group, then a representation of $G$ on $V$, is a homomorphism, $\pi : G \to GL(V)$.*

A representation is called **faithful** if $\pi$ is one-to-one.

When $V = \mathbb{F}^n$, we have that $\mathcal{L}(V) = M_n(\mathbb{F})$ the set of $n \times n$ matrices with entries from $\mathbb{F}$, and $GL(V) = GL(n, \mathbb{F})$.

### 2.3.1 Subrepresentations

Let $\pi : G \to GL(V)$ be a representation. A vector subspace, $W \subseteq V$ is called **invariant** or sometimes, $\pi(G)$-invariant, provided that $\pi(g)W \subseteq W$; i.e., for any $w \in W$ and any $g \in G$, we have that $\pi(g)w \in W$.

In this case we define $\pi_W(g) : W \to W$ to be the restriction of the map $\pi(g)$ to $W$. It is easy to see that $\pi_W(g_1)\pi_W(g_2) = \pi_W(g_1 g_2)$ and that $\pi_W(e) = I_W$. From these facts it follows that $\pi_W(g) \in GL(W)$ and that $\pi_W : G \to GL(W)$ is a representation. This representation is called a **subrepresentation** of $\pi$.

**Definition 2.12.** *[51, Definition 3.37] A representation, $\pi : G \to GL(V)$ is irreducible if the only $\pi(G)$-invariant subspaces of $V$ are $V$ and $(0)$.*

**Definition 2.13.** *[51, Definition 3.35] Let $G$ be a group and let $\pi_i : G \to GL(W_i)$, $i = 1, 2$ be representations. If there exists an invertible linear map, $T : W_1 \to W_2$ such that $T^1\pi_2(g)T = \pi_1(g)$ for all $g \in G$, then we say that $\pi_1$ and $\pi_2$ are equivalent representations and we write, $\pi_1 \sim \pi_2$ to denote that $\pi_1$ and $\pi_2$ are equivalent.*

**Definition 2.14.** *[51, Definition 3.48] Let $\pi : G \to GL(V_i)$, $i = 1, 2$ be representations. The set $\mathcal{I}(\pi_1, \pi_2) = \{T \in \mathcal{L}(V_1, V_2) : \pi_2(g)T = T\pi_1(g)$, for every $g \in G\}$ is called the space of intertwining maps between $\pi_1$ and $\pi_2$.*

Note that when $\pi_1 = \pi_2 = \pi$, then $\mathcal{I}(\pi, \pi) = \pi(G)'$.

**Proposition 2.15.** *[51, Proposition 3.49] Let $G$ be a group and let $\pi_i : G \to GL(V_i)$, $i = 1, 2$ be representations. Then $\mathcal{I}(\pi_1, \pi_2)$ is a vector subspace of $\mathcal{L}(V_1, V_2)$, and $\pi_1 \sim \pi_2$ if and only if there exists an invertible linear transformation in $\mathcal{I}(\pi_1, \pi_2)$.*

**Theorem 2.16.** *[51, Theorem 3.50][Schur's Lemma] Let $G$ be a finite group and let $\pi_i : G \to GL(V_i)$, $i = 1, 2$ be irreducible representations. Then $\pi_1 \sim \pi_2$ if and only if $\mathcal{I}(\pi_1, \pi_2) \neq (0)$. In the case that $V_i$, $i = 1, 2$ are vector spaces over $\mathbb{C}$, then $dim(\mathcal{I}(\pi_1, \pi_2))$ is either $0$ or $1$.*

**Proposition 2.17.** *[51, Proposition 3.36] Let $G$ be a group, let $\pi : G \to GL(V)$ be a representation, and let $W_i \subseteq V$, $i = 1, 2$ be a complementary pair of $\pi(G)$-invariant subspaces. Then $\pi \sim \pi_{W_1} \oplus \pi_{W_2}$.*

**Theorem 2.18.** *[51, Theorem 3.39] Let $G$ be a finite group and let $\pi : G \to GL(V)$ be a finite dimensional representation of $G$. Then there exists an integer $k$ and $\pi(G)$-invariant subspaces, $W_1, \ldots, W_k$ of $V$ , such that:*

*(i)* $V = W_1 + \cdots + W_k$, *and* $W_i \cap \sum_{j \neq i} W_j = (0)$, *for all* $i \neq j$;

*(ii) the subrepresentations,* $\pi_{W_i} : G \to GL(W_i)$, $1 \leq i \leq k$, *are irreducible;*

*(iii)* $\pi \sim \pi_{W_1} \oplus \cdots \oplus \pi_{W_k}$ .

### 2.3.2 Character theory of finite groups

**Definition 2.19.** *[51, Definition 4.1] Let $A = (a_{i,j}) \in M_n(\mathbb{F})$, then the trace of $A$ is the quantity $Tr(A) = \sum_{i=1}^{n} a_{i,i}$.*

**Remark 2.20.** *[51, Remark 4.4] If $V$ is any $n$-dimensional space, then by choosing a basis for $V$ we may identify $\mathcal{L}(V)$ with $M_n(\mathbb{F})$ and in this way define the trace of a linear map on $V$. If we choose a different basis for $V$, then the two matrix representations for a linear map that we obtain in this fashion will differ by conjugation by an invertible matrix. Thus, the value of the trace that one obtains in this way is independent of the particular basis, and by the above corollary will always be equal to the sum of the eigenvalues of the linear transformation. Hence, there is a well-defined trace functional on $\mathcal{L}(V)$.*

**Definition 2.21.** *[51, Definition 4.5] Let $G$ be a group and let $\pi : G \to GL(V)$ be a representation of $G$ on a finite dimensional vector space. Then the character of $\pi$ is the function, $\chi_\pi : G \to \mathbb{F}$ defined by, $\chi_\pi(g) = Tr(\pi(g))$.*

**Proposition 2.22.** *[51, Proposition 4.11] Let $G$ be a finite group and let $\pi : G \to GL(V)$ be a representation where $dim(V) = n$ is finite. Then:*

*(i)* $\chi_\pi(e) = n$;

*(ii)* $\chi_\pi(g^{-1}) = \overline{\chi_\pi(g)}$;

*(iii)* *When* $\pi \sim \rho$, *then* $\chi_\pi = \chi_\rho$.

Given a set $X$, we can form a vector space of dimension $card(X)$ with basis, $\{e_x : x \in X\}$. A vector in this space is just a finite linear combination of the form, $\sum_i \lambda_i e_{x_i}$, where two such sums are equal if and only if the set of $x$'s (with non zero coefficients) appearing in the sums are the same and the coefficients of the corresponding $e_x$ are the same. This is often called the **free vector space over** $X$ and is denoted by $\mathbb{F}(X)$. Another, concrete way, to present this space, is to regard it as the set of all functions, $f : X \to \mathbb{C}$ which are finitely supported, i.e., such that the set of $x \in X$, with $f(x) \neq 0$ is finite. Clearly, the usual sum of two finitely supported functions is finitely supported and a scalar multiple of a finitely supported function will be finitely supported.

These two different representations of $\mathbb{F}(X)$ are identified in the following way. If we let $\delta_x$ be the function that is 1 at $x$ and 0 elsewhere, then if $f$ is any finitely supported function, say $f$ is non-zero at $\{x_1, \ldots, x_k\}$, then as functions, $f = \sum_{i=1}^{n} \lambda_i \delta_{x_i}$ where $\lambda_i = f(x_i)$. Clearly, the functions, $\delta_x$ are linearly independent. Thus, $\{\delta_x : x \in X\}$ is a basis for the space of finitely supported functions. Then the map $\delta_x \to e_x$ defines a vector space isomorphism between the space of finitely supported functions on $X$ and the free vector space over $X$.

Let $\mathbb{C}(G)$ be the vector space of finitely supported functions over the group $G$. Given two functions $f_1$, $f_2$ in $\mathbb{C}(G)$, we define

$$(f_1, f_2) = \frac{1}{|G|} \sum_{g \in G} f_1(g) \overline{f_2(g)}$$

Then $(\cdot, \cdot)$ is an *inner product.*

**Theorem 2.23.** *[51, Theorem 4.16] Let* $\pi_i : G \to GL(V_i)$, $i = 1, 2$ *be irreducible* *representations on finite dimensional vector spaces and let $\chi_i$, $i = 1, 2$ be the corresponding characters. Then*

$$(\chi_1, \chi_2) = \begin{cases} 1 & \pi_1 \sim \pi_2 \\ \\ 0 & \pi_1 \not\sim \pi_2 \end{cases}$$

**Corollary 2.24.** *[51, Corollary 4.17] Let $G$ be a finite group and let $\pi : G \to GL(V)$ be a finite dimensional representation. Then $\pi$ is irreducible if and only if $(\chi_\pi, \chi_\pi) = 1$.*

**Corollary 2.25.** *[51, Corollary 4.19] Let $G$ be a finite group and let $\pi_i : G \to GL(V_i)$, $i = 1, 2$ be finite dimensional representations of $G$. Then $\pi_1 \sim \pi_2$ if and only if $\chi_{\pi_1} = \chi_{\pi_2}$.*

### 2.3.3 The left regular representation

**Definition 2.26.** *[51]A (left) action of a group, $G$, on a set, $X$, is a map $\alpha : G \times X \to X$ satisfying $\alpha(e, x) = x$ and $\alpha(g, \alpha(h, x)) = \alpha(gh, x)$ for every $x \in X$ and every $g, h \in G$. Usually, we will write $\alpha(g, x) = g \cdot x$, so that the first property is that $e \cdot x = x$ and the second property is $g \cdot (h \cdot x) = (gh) \cdot x$ which can be seen to be an associativity property.*

Let $G$ be a group, acting on a set $X$. We have seen that each element of $G$ induces a permutation of the elements of $x$, via $x \to g \cdot x$. This permutation extends to a linear map, $\pi(g) : \mathbb{F}(X) \to \mathbb{F}(X)$ by setting $\pi(g)(\sum_i \lambda_i e_{x_i}) = \sum_i \lambda_i e_{g \cdot x_i}$. It is easy to see that $\pi(e)$ is the identity map on $\mathbb{F}(X)$ and that $\pi(g)\pi(h) = \pi(gh)$. Thus, each $\pi(g)$ is invertible and the map, $\pi : G \to GL(\mathbb{F}(X))$ is a homomorphism.

**Definition 2.27.** *[51, Definition 3.7] Let $G$ act on a set $X$. Then the representation of $G$ on $\mathbb{F}(X)$ as above is called the permutation representation induced by the action.*

Let $G$ be a group, define $\alpha_l : G \times G \to G$ by $\alpha_l(g,h) = g \cdot h$. Then $\alpha_l$ is an action of $G$ on itself given by left multiplication.

**Definition 2.28.** *[51, Definition 3.10] Let $G$ act on itself via left multiplication $\alpha_l$ and consider the induced permutation representation on $\mathbb{F}(G)$. This representation is denoted $\lambda : G \to GL(\mathbb{F}(G))$ and is called the (left) regular representation.*

Thus, we have that $\lambda(g)e_h = e_{gh}$. Note that this representation is faithful, since $\lambda(g_1)e_h = \lambda(g_2)e_h$ if and only if $g_1 = g_2$. Also, every vector in the canonical basis is cyclic since $\lambda(G)e_h$ spans $\mathbb{F}(G)$. Algebraists sometimes refer to the left regular representation as the Cayley representation.

**Theorem 2.29.** *[51, Theorem 4.20] Let $G$ be a finite group and let $\pi : G \to GL(n, \mathbb{C})$ be an irreducible representation of $G$. Then $\pi$ is a subrepresentation of $\lambda$ with multiplicity $n$.*

**Theorem 2.30.** *[51, Theorem 4.21] Let $G$ be a finite group. Then there exists a finite number of finite dimensional irreducible representations $\pi_{n_i}$, $i = 1, \ldots, r$ on spaces of dimensions, $n_1, \ldots, n_r$ respectively such that*

$$\lambda \sim \underbrace{\pi_{n_1} \oplus \pi_{n_1} \oplus \ldots \oplus \pi_{n_1}}_{n_1-times} \oplus \ldots\ldots \oplus \underbrace{\pi_{n_r} \oplus \pi_{n_r} \oplus \ldots \oplus \pi_{n_r}}_{n_r-times}$$

*where $|G| = n_1^2 + n_2^2 + \ldots + n_r^2$.*

## 2.4 Number theory and group theory

*The following facts of number theory are taken from [53].*

**Definition 2.31.** *Euler's phi function is defined for positive integer $n$ as the number of elements of $\mathbb{Z}_n^*$. $\phi(n)$ is equal to the number of integers between $0$ and $n-1$ that are relatively prime to $n$.*

**Theorem 2.32.** *[53, Theorem 2.11] For any positive integer $n$ we have*

$$\sum_{d|n} \phi(d) = n,$$

*where the sum is over all positive divisors $d$ of $n$.*

**Definition 2.33.** *The multiplicative order of $a$ modulo $n$ is the smallest positive integer $k$ such that*

$$a^k \equiv 1 \pmod{n}.$$

**Theorem 2.34.** *[53, Theorem 2.16][Fermat's Little Theorem] For any prime $p$, and any integer $a \not\equiv 0 \pmod{p}$, we have $a^{p-1} \equiv 1 \pmod{p}$. Moreover for any integer $a$, we have $a^p \equiv a \pmod{p}$.*

**Definition 2.35.** *For a positive integer $n$, we say that $a \in \mathbb{Z}$ with $gcd(a, n) = 1$ is a primitive root modulo $n$ if the multiplicative order of $a$ modulo $n$ is equal to $\phi(n)$.*

**Definition 2.36.** *For an odd prime $p$ and an integer $a$ with $gcd(a, p) = 1$, the Legendre symbol $(a|p)$ is defined as*

$$(a|p) = \begin{cases} 1 & \textit{if } x^2 = a \pmod{p} \textit{ for some integer } x; \\ -1 & \textit{otherwise.} \end{cases}$$

**Theorem 2.37.** *[53, Theorem 12.4, part(i)] Let $p$ be an odd prime, and let $a, b \in \mathbb{Z}$. Then*

$$(a|p) \equiv a^{\frac{p-1}{2}} \pmod{p}.$$

**Theorem 2.38.** *[53, Theorem 12.8, part(v)] Let $n$ be an odd positive integer. Then*

$$(2|n) = (-1)^{\frac{n^2-1}{8}}$$

*where $(-1)^{\frac{n^2-1}{8}} = 1$ if and only if $n \equiv \pm 1 \pmod{8}$.*

*The following results on group theory are taken from [42].*

**Theorem 2.39.** *[42, Theorem 2.1, II] Every finitely generated abelian group $G$ is (isomorphic to) a finite direct sum of cyclic groups in which the finite cyclic commands (if any) are of orders $m_1, \ldots, m_t$ where $m_1 > 1$ and $m_1|m_2|\ldots|m_t$.*

**Lemma 2.40.** *[42, Theorem 2.3, II] If $m$ is a positive integer and $m = p_1^{n_1} p_2^{n_2} \ldots p_t^{n_t}$, $(p_1, p_2, \ldots, p_t)$ are distinct primes and each $n_i > 0$, then*

$$\mathbb{Z}_m = \mathbb{Z}_{p_1^{n_1}} \oplus Z_{p_2^{n_2}} \oplus \cdots \oplus Z_{p_t^{n_t}}.$$

**Proposition 2.41.** *[42, Exercise 13, Section 5, II] Every group of order $p^2$, p-prime is abelian.*

**Proposition 2.42.** *[42, Proposition 6.1, II] Let $p$, $q$ be primes such that $p > q$. If $q \nmid p - 1$, then every group of order $pq$ is isomorphic to the cyclic group $\mathbb{Z}_{pq}$.*

# Chapter 3

# Unitary Representations and Parseval Frames

In this chapter, we explore an operator theoretic approach to finite frame theory. Motivated by the examples of the type $\{M^x C^y v : x, y \in \mathbb{Z}_k\}$ referred as chirps in [17, 60] and $\{T^x M^y u : x, y \in \mathbb{Z}_k\}$ known as finite Gabor frames, see [31, 34, 36] where

- $M^x$ is the modulation operator defined as

$$M^x : l^2(\mathbb{Z}_k) \to l^2(\mathbb{Z}_k) \quad \text{such that} \quad (M^x f)(t) = e^{\frac{2\pi i x t}{a}} f(t),$$

- $C^x$ is the chirp modulation operator defined as

$$C^x : l^2(\mathbb{Z}_k) \to l^2(\mathbb{Z}_k) \quad \text{such that} \quad C^x f(t) = e^{\pi i x t(t-k)/k} f(t),$$

- $T^x$ is the translation operator defined as

$$T^x : l^2(\mathbb{Z}_k) \to l^2(\mathbb{Z}_k) \quad \text{such that} \quad (T^x f)(t) = f(t-x),$$

and $u, v \in \mathbb{C}^k$, we investigate the conditions on a unitary representation $\pi : G \to GL(\mathbb{C}^k)$ and vectors $v \in \mathbb{C}^k$ such that the collection $\{\pi(g)v\}_{g\in G}$ is a uniform tight frame for $\mathbb{C}^k$.

**Definition 3.1.** *[37] A representation $\pi : G \to GL(\mathbb{C}^k)$ is called a frame representation if there exists a vector $v \in \mathbb{C}^k$ such that $\{\pi(g)v\}_{g\in G}$ is a uniform tight frame for $\mathbb{C}^k$.*

In [37], it has been shown that every frame representation of a group $G$ is unitarily equivalent to a subrepresentation of the left regular representation. Applications of frame theory to group representations, and of the theory of abstract *unitary systems* [33] to wavelet and Gabor analysis are also shown in [33] and [37].

We accomplish the task of constructing frame representations using frames for the Hilbert space $M_k$ of $k \times k$ matrices with the Hilbert Schmidt norm. This alternate approach has provided different proofs of the results shown in [33] and [37]. Because of the use of groups as unifying principle, the procedures suggested in this chapter are simpler and establish easy to verify conditions for a vector $v \in \mathbb{C}^k$ such that the collection $\{\pi(g)v\}_{g\in G}$ is a uniform tight frame for $\mathbb{C}^k$.

We shall call a vector $v \in \mathbb{C}^k$ a **Parseval frame vector** for $\pi$ if $\{\pi(g)v\}_{g\in G}$ is a Parseval frame for $\mathbb{C}^k$.

## 3.1 Unitary representations and frames for $M_k$

The space of $k \times k$ matrices, with the Hilbert Schmidt norm, is a Hilbert space. The inner product on this space is given by

$$\langle A, B\rangle_{HS} = Tr(B^*A), \quad \text{where } A, B \in M_k, \quad B^* = \bar{B}^t. \tag{3.1}$$

Moreover, if $V$ is a vector space over a field $\mathbb{F}$, then by choosing a basis for $V$ we may identify $\mathcal{L}(V)$ with $M_k(\mathbb{F})$. Given a representation $\pi : G \to GL(V)$, we study when is the collection $\{\pi(g)\}_{g\in G}$ a tight frame for $M_k$. We first look at the following facts that leads to the establishment of the conditions on a representation $\pi$ such that the collection $\{\pi(g)\}_{g\in G}$ a tight frame for $M_k$.

**Definition 3.2.** *[51, Definition 3.40] Let $\mathcal{S} \subseteq \mathcal{L}(V)$ be any set. Then the commutant of $\mathcal{S}$ is the set $\mathcal{S}' = \{T \in \mathcal{L}(V) : TS = ST, \quad \forall\ S \in \mathcal{S}\}$.*

Note that since the scalar multiples of the identity commute with every linear transformation, these always belong to the commutant of $S$. If these are the only linear transformations in the commutant of $S$, then we say that $S$ has trivial commutant.

**Lemma 3.3.** *If $\mathcal{A} = \{A_l\}_{l=1}^n$ is a Parseval frame for the space of $k \times k$ matrices, then $\mathcal{A}' = \{\lambda I_k : \ \lambda \in \mathbb{C}\}$.*

*Proof.* Let $B \in \mathcal{A}'$. For $A \in M_k$, by Parseval's identity

$$B^*A = \sum_{l=1}^n \langle B^*A, A_l\rangle A_l = \sum_{l=1}^n \langle A, BA_l\rangle A_l = \sum_{l=1}^n \langle A, A_lB\rangle A_l = \sum_{l=1}^n \langle AB^*, A_l\rangle A_l = AB^*.$$

This is true for all $A \in M_k$. Hence $B^* = \lambda I_k$ for some $\lambda \in \mathbb{C}$. Thus $B = \bar{\lambda} I_k$. $\square$

**Proposition 3.4.** *[32, Proposition 3.4] Let $\mathcal{M}$ be a subspace of $V$ and $P$ be the orthogonal projection onto $\mathcal{M}$. Then $\mathcal{M}$ is invariant under $\pi$ if and only if $P \in \pi(G)'$.*

*Proof.* If $P \in \pi(G)'$ and $v \in \mathcal{M}$, then $\pi(g)v = \pi(g)Pv = P\pi(g)v \in \mathcal{M}$. So $\mathcal{M}$ is invariant under $\pi$. Conversely, if $\mathcal{M}$ is invariant under $\pi$, for all $v \in \mathcal{M}$, we have $\pi(g)Pv = \pi(g)v = P\pi(g)v$ and for all $v \in \mathcal{M}^{\perp}$, $\pi(g)Pv = 0 = P\pi(g)v$. Hence $\pi(g)P = P\pi(g)$. □

The following result establishes a relation between an irreducible representation and its commutant.

**Proposition 3.5.** *[51, Theorem 3.41] Let $G$ be a finite group and $\pi : G \to GL(V)$ be a finite dimensional representation of $G$.*

*(i) If $\pi(G)' = \{\lambda I_V : \lambda \in \mathbb{F}\}$, i.e. if $\pi(G)$ has a trivial commutant, then $\pi$ is irreducible.*

*(ii) When $\mathbb{F} = \mathbb{C}$, then $\pi$ is irreducible if and only if $\pi(G)$ has a trivial commutant.*

*Proof.* (i) If $\pi$ is reducible, then by Definition 2.12 there is a subspace $0 \neq W \neq V$ that is $\pi(G)$-invariant. Let $P$ be the orthogonal projection onto $W$. Then by Proposition 3.4, $P \in \pi(G)' = \{\lambda I_V : \lambda \in \mathbb{F}\}$. Hence for any field, if the commutant is non-trivial, then the representation is irreducible.

(ii) Next, assume that the field is $\mathbb{C}$ and that the commutant is non-trivial. Let $T \in \pi(G)'$ be an operator that is not a scalar multiple of the identity. In this case there exists an eigenvalue, $\lambda$ of $T$, and necessarily, $T - \lambda I_V \neq 0$. Let $W = ker(T - \lambda I_V)$, then $0 \neq W \neq V$. But, if $w \in W$, then $(T - \lambda I_V)(\pi(g)w) = \pi(g)(T - \lambda I_V)w = 0$ and so $W$ is $\pi(G)$-invariant. Hence, $\pi$ is not irreducible. Therefore, when $\mathbb{F} = \mathbb{C}$, if $\pi$ is irreducible, then the commutant of $\pi(G)$ is trivial.

□

Consider the free vector space $\mathbb{F}(X)$ of dimension $card(X)$ with a basis $\{e_x : x \in X\}$. A vector in this space is just a finite linear combination of the form, $\sum_i \lambda_i e_{x_i}$. Another concrete way to present this space is to regard it as the set of all functions, $f : X \to \mathbb{F}$ which are finitely supported, that is, the set of $x \in X$ with $f(x) \neq 0$ is finite.

**Definition 3.6.** *[51]Let $G$ be a finite group, consider $\mathbb{F}(G)$, the free vector space over $G$. We want to give this vector space a product to be such that $e_g e_h = e_{gh}$. Thus,*

$$(\sum_{g \in G} \lambda(g) e_g)(\sum_{h \in G} \mu(h) e_h) = \sum_{k \in g} [\sum_{gh = k} \lambda(g)\mu(h)] e_k$$

*The product defined as above makes $\mathbb{F}(G)$ into an algebra called the* **group algebra***.*

The group algebra has the property that every representation of G, $\pi : G \to GL(V)$ extends uniquely to a unital $*$-homomorphism $\tilde{\pi} : \mathbb{C}(G) \to \mathcal{L}(V)$ by setting

$$\tilde{\pi}(\sum_{g \in G} \lambda_g e_g) = \sum_{g \in G} \lambda_g \pi(g).$$

**Definition 3.7.** *[51, Definition 4.25]A function $f \in \mathbb{C}(G)$ is called a class function if for every $g, h \in G$,*

$$f(h^{-1}gh) = f(g).$$

*The set of class functions is denoted by* $\mathbb{H}(G)$.

**Example 3.8.** *The character* $\chi_\pi : G \to \mathbb{F}$ *is a class function as for all* $g, h \in G$,

$$\begin{aligned}\chi_\pi(h^{-1}gh) &= Tr(\pi(h^{-1}gh))\\ &= Tr(\pi(h)^*\pi(g)\pi(h))\\ &= Tr(\pi(g))\\ &= \chi_\pi(g).\end{aligned}$$

**Proposition 3.9.** *[51, Proposition 4.27] Let* $\pi : G \to GL(V)$ *be an irreducible representation of a group* $G$ *and let* $\tilde{\pi} : \mathbb{C}(G) \to \mathcal{L}(V)$ *be the extension of* $\pi$ *to the group algebra. If* $f \in \mathbb{H}(G)$, *then*

$$\tilde{\pi}(f) = \frac{|G|}{n}(\chi_\pi, \bar{f})I_V$$

*where* $dim(V) = n$.

*Proof.* Note that

$$\begin{aligned}\pi(g^{-1})\tilde{\pi}(f)\pi(g) &= \pi(g^{-1})(\sum_{h\in G} f(h)\pi(h))\pi(g)\\ &= \sum_{h\in G} \pi(g^{-1})f(h)\pi(h)\pi(g)\\ &= \sum_{h\in G} \pi(g^{-1})f(g^{-1}hg)\pi(h)\pi(g)\\ &= \sum_{h\in G} f(g^{-1}hg)\pi(g^{-1}hg)\\ &= \sum_{k\in G} f(k)\pi(k)\\ &= \tilde{\pi}(f).\end{aligned}$$

Thus, $\tilde{\pi}(f) \in \mathcal{C}(\pi)$. Since $\pi$ is irreducible, there exists a $\lambda$ such that $\tilde{\pi}(f) = \lambda I_V$. Thus,

$$\lambda I_V = \sum_{h \in G} f(h)\pi(h).$$

To compute $\lambda$, we note that

$$\begin{aligned} n\lambda &= \sum_{h \in G} f(h)\chi_\pi(h) \\ &= |G|(\chi_\pi, \bar{f}). \end{aligned}$$

Thus we have,

$$\lambda = \frac{|G|}{n}(\chi_\pi, \bar{f})$$

and

$$\tilde{\pi}(f) = \frac{|G|}{n}(\chi_\pi, \bar{f}) I_V.$$

□

The following theorem characterizes group representations that form tight frames for the space of $k \times k$ matrices $M_k$ with the Hilbert-Schmidt norm.

**Theorem 3.10.** *Let $G$ be a group of order $n$ and $\pi : G \to GL(\mathbb{C}^k)$ be a unitary representation. Then $\{\sqrt{\frac{k}{n}}\pi(g) : g \in G\}$ is a Parseval frame for $M_k$ if and only if $\pi$ is an irreducible representation of $G$.*

*Proof.* Assume first that $\{\sqrt{\frac{k}{n}}\pi(g) : g \in G\}$ is a Parseval frame for $M_k$. Then by Proposition 3.3, $\pi(G)' = \{\lambda I_V : \lambda \in \mathbb{F}\}$. Thus by Proposition 3.5, $\pi$ is an irreducible representation.

Conversely, let $\pi$ be an irreducible representation of $G$ on $\mathbb{C}^k$. Then $\pi$ extends to a $*$-homomorphism

$$\tilde{\pi} : \mathbb{C}(G) \to M_k \quad \text{with} \quad \tilde{\pi}(\sum_{g\in G} \lambda_g e_g) = \sum_{g\in G} \lambda_g \pi(g).$$

$\tilde{\pi}(\mathbb{C}(G))$ is a closed $*$-subalgebra of $M_k$ and hence a $C^*$ subalgebra of $M_k$. Since $\tilde{\pi}(\mathbb{C}(G))' \cong \pi(G)' = \lambda I_V$, it follows that $\tilde{\pi}(\mathbb{C}(G)) = M_k$ and we have $\text{span}\{\pi(g) : g \in G\} = M_k$. Thus $\{\pi(g) : g \in G\}$ is a frame for $M_k$. The analysis operator as defined in 2.4 is given by

$$V : M_k \to l^2(G) \quad \text{such that} \quad V(A) = \{\langle A, \pi(g)\rangle_{HS}\}_{g\in G}.$$

Since $\text{span}\{\pi(g) : g \in G\} = M_k$, we can write $A = \sum_{h\in G} \lambda(h)\pi(h)$. Thus we have,

$$\begin{aligned} V(A) &= \{\langle \sum_{h\in G} \lambda(h)\pi(h), \pi(g)\rangle_{HS}\}_{g\in G} \\ &= \{\sum_{h\in G} \lambda(h)\langle \pi(h), \pi(g)\rangle_{HS}\}_{g\in G} \\ &= \{\sum_{h\in G} \lambda(h)\chi_\pi(g^{-1}h)\}_{g\in G}. \end{aligned}$$

The synthesis operator is given by

$$V^* : l^2(G) \to M_k \quad \text{such that} \quad V^*(\{\lambda(g)\}_{g\in G}) = \sum_{g\in G} \lambda(g)\pi(g).$$

For $A = \sum_{h\in G} \lambda(h)\pi(h)$, we have

$$\begin{aligned} V^*V(A) &= V^*(\{\sum_{h\in G} \lambda(h)\chi_\pi(g^{-1}h)\}_{g\in G}) \\ &= \sum_{g\in G}\sum_{h\in G} \lambda(h)\chi_\pi(g^{-1}h)\pi(g) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{\tilde{g}\in G}\sum_{h\in G} \lambda(h)\chi_\pi(\tilde{g})\pi(h\tilde{g}^{-1}) \qquad\qquad (g^{-1}h = \tilde{g}) \\
&= (\sum_{h\in G} \lambda(h)\pi(h))(\sum_{g\in G} \chi_\pi(g^{-1})\pi(g)) \\
&= A(\sum_{g\in G} \overline{\chi_\pi(g)}\pi(g)).
\end{aligned}
$$

Since $\pi$ is an irreducible representation, using Proposition 3.9, we have

$$\sum_{g\in G} \overline{\chi_\pi(g)}\pi(g) = \frac{n}{k}(\chi_\pi, \chi_\pi)I = \frac{n}{k}I.$$

Thus we get,

$$V^*V(A) = A(\sum_{g\in G} \overline{\chi_\pi(g)}\pi(g)) = \frac{n}{k}A.$$

Hence for all $A \in M_k$, we have

$$\frac{n}{k}A = \sum_{g\in G} \langle A, \pi(g)\rangle_{HS}\pi(g),$$

and thus $\{\sqrt{\frac{k}{n}}\pi(g) : g \in G\}$ is a Parseval frame for $M_k$. $\square$

Using Theorem 3.10, next we state a familiar result from the representation theory.

**Corollary 3.11.** *Let $G$ be a group of order $n$. If $\pi : G \to \mathbb{C}^k$ is an irreducible representation, then $k^2 \leq n$.*

*Proof.* Since $\pi : G \to \mathbb{C}^k$ is an irreducible representation, using Theorem 3.10, we have that the collection $\{\sqrt{\frac{k}{n}}\pi(g) : g \in G\}$ is a Parseval frame for $M_k$. Thus we have,

$$k^2 = dim(M_k) \leq card(G) = n.$$

$\square$

**Example 3.12.** *For n-even, consider the Dihedral group $D_n$ of 2n-elements*

$$\{e, R, R^2, \ldots, R^{n-1}, F, FR, FR^2, \ldots, FR^{n-1}\}$$

*where*

- *$R$ is counterclockwise rotation through an angle of $\frac{2\pi}{n}$.*
- *$F$ is reflection about the line $\frac{\pi}{n}$.*

*Then, $R^n = 1$, $F^2 = 1$, $FRF = R^{-1}$. Let $\pi$ be the representation of $D_n$ on $\mathbb{C}^2$ given by*

$$\pi(R^k) = \begin{bmatrix} \omega^k & 0 \\ 0 & \omega^{-k} \end{bmatrix} \quad \text{and} \quad \pi(FR^k) = \begin{bmatrix} 0 & \omega^{-k} \\ \omega^k & 0 \end{bmatrix}$$

*where $\omega = e^{\frac{2\pi i}{n}}$. The corresponding characters $\chi_\pi$ are given by:*

$$\chi_\pi(R^k) = \omega^k + \omega^{-k} \quad \text{and} \quad \chi_\pi(FR^k) = 0.$$

*We have,*

$$(\chi_\pi, \chi_\pi) = \frac{1}{2n} \sum_{g \in D_n} |\chi_\pi(g)|^2 = \frac{1}{2n} \sum_{g \in D_n} |tr(\pi(g))|^2 = \frac{1}{2n} 2n = 1.$$

*Using Corollary 2.24, the representation $\pi$ is irreducible. Thus by Theorem 3.10, $\{\frac{1}{\sqrt{n}}\pi(g) : g \in D_n\}$ is a Parseval frame for $M_2$.*

**Example 3.13.** *For $x \in \mathbb{Z}_k$, the corresponding translation and modulation operators are $T^x, M^x : l^2(\mathbb{Z}_k) \to l^2(\mathbb{Z}_k)$, respectively, where*

$$(T^x f)(t) = f(t - x) \quad \text{and} \quad (M^x f)(t) = e^{\frac{2\pi i x t}{a}} f(t).$$

*The family of operators $H_k = \{\omega^i T^x M^y : i, x, y \in \mathbb{Z}_k\}$ is a group under the operation of composition called the Heisenberg group over $\mathbb{Z}_k$. Let us define*

$$\pi : H_k \to GL(\mathbb{C}^k) \quad \text{such that} \quad \pi(\omega^i T^x M^y) = \omega^i T^x M^y.$$

*Then $\pi$ is a unitary representation of $H_k$ on $\mathbb{C}^k$.*

*Consider,*

$$\begin{aligned}
(\chi_\pi, \chi_\pi) &= \frac{1}{|H_k|} \sum_{i,x,y=1}^{k} \chi_\pi(\omega^i T^x M^y)\overline{\chi_\pi(\omega^i T^x M^y)} \\
&= \frac{1}{k^3} \sum_{i,x,y=1}^{k} tr(\pi(\omega^i T^x M^y))\overline{tr(\pi(\omega^i T^x M^y))} \\
&= \frac{1}{k^3} \sum_{i,x,y=1}^{k} tr(\omega^i T^x M^y)\overline{tr(\omega^i T^x M^y)} \\
&= \frac{1}{k^3} k^3 \\
&= 1.
\end{aligned}$$

*Using Corollary 2.24, $\pi$ is an irreducible representation of $H_k$. Thus by Theorem 3.10, $\{\frac{1}{k}\pi(g) : g \in H_k\}$ is a Parseval frame for $M_k$; that is, the collection $\{\frac{1}{k}\omega^i T^x M^y\}_{i,x,y=1}^{k}$ is a Parseval frame for $M_k$.*

## 3.2 Parseval frames for $M_k$ and Parseval frames for $\mathbb{C}^k$

In this section, we construct frames for $\mathbb{C}^k$ using frames for the space of $k \times k$ matrices $M_k$. We have the following proposition that establishes a relationship between the frames for $M_k$ and frames for $\mathbb{C}^k$.

**Proposition 3.14.** *Let $\{A_l\}_{l=1}^M$ be a Parseval frame for $M_k$. Then for any unit vector $v \in \mathbb{C}^k$, $\{A_l v\}_{l=1}^M$ is a Parseval frame for $\mathbb{C}^k$.*

*Proof.* For $l = 1, \ldots, M$, let $A_l = (a^l_{i,j})_{i,j=1}^k \in M_k$ and $v = (v_j)_{j=1}^k \in \mathbb{C}^k$ be a unit vector. Then, $A_l v = (\sum_{j=1}^k a_{ij} v_j)_{i=1}^k$. For $u \in \mathbb{C}^k$, we have $\langle u, A_l v\rangle = \sum_{i,j=1}^k \bar{a}_{ij}\bar{v}_j u_i$. Using the Hilbert-Schmidt norm on $M_k$, we have

$$\langle uv^*, A_l\rangle_{\mathrm{HS}} = Tr(A_l^*(uv^*)) = \sum_{i,j=1}^k \bar{a}_{ij}\bar{v}_j u_i.$$

Thus we have,

$$\langle u, A_l v\rangle = \langle uv^*, A_l\rangle_{\mathrm{HS}},$$

and hence,

$$\sum_{l=1}^M \langle u, A_l v\rangle A_l v = (\sum_{l=1}^M \langle uv^*, A_l\rangle_{\mathrm{HS}} A_l)v.$$

Since $\{A_l\}_{l=1}^M$ is a Parseval frame for $M_k$, we have $\sum_{l=1}^M \langle uv^*, A_l\rangle_{\mathrm{HS}} A_l = uv^*$. Thus we have,

$$\sum_{l=1}^M \langle u, A_l v\rangle A_l v = uv^* v = u$$

and hence $\{A_l v\}_{l=1}^M$ is a Parseval frame for $\mathbb{C}^k$. □

The following corollary characterizes a vector $v$ as a Parseval frame vector for an irreducible representation $\pi$. A more general result was shown in [37] in the case of unitary systems. Here we employ a technique using groups and tight frames for the Hilbert space of $k \times k$ matrices; that is, Theorem 3.10.

**Corollary 3.15.** *Let $G$ be a group of order $n$ and $\pi : G \to GL(\mathbb{C}^k)$ be an irreducible representation. Then for any unit vector $v \in \mathbb{C}^k$, $\{\sqrt{\frac{k}{n}}\pi(g)v : g \in G\}$ is a Parseval frame for for $\mathbb{C}^k$.*

*Proof.* If $\pi : G \to GL(\mathbb{C}^k)$ is an irreducible representation, then using Theorem 3.10, $\{\sqrt{\frac{k}{n}}\pi(g) : g \in G\}$ is a Parseval frame for for $M_k$. Using Proposition 3.14, $\{\sqrt{\frac{k}{n}}\pi(g)v : g \in G\}$ is a Parseval frame for for $\mathbb{C}^k$. $\square$

**Example 3.16.** *We know from Example 3.12 that if $G$ is the Dihedral group $D_n$ of $2n$-elements, then $\pi : D_n \to \mathbb{C}^2$ given by*

$$\pi(R^k) = \begin{bmatrix} \omega^k & 0 \\ 0 & \omega^{-k} \end{bmatrix} \quad \text{and} \quad \pi(FR^k) = \begin{bmatrix} 0 & \omega^{-k} \\ \omega^k & 0 \end{bmatrix}$$

*where $\omega = e^{\frac{2\pi i}{n}}$, is an irreducible representation. Thus from Corollary 3.15, for any unit vector $v \in \mathbb{C}^2$, $\{\frac{1}{\sqrt{n}}\pi(g)v : g \in D_n\}$ is a Parseval frame for $\mathbb{C}^2$.*

**Example 3.17.** *From Example 3.13, we know that for $G = H_k$, where $H_k$ is the Heisenberg group, $\pi : H_k \to GL(\mathbb{C}^k)$ defined as $\pi(\omega^i T^x M^y) = \omega^i T^x M^y$ is an irreducible representation. From Corollary 3.15, for any unit vector $v \in \mathbb{C}^k$, $\{\frac{1}{k}\pi(g)v : g \in H_k\}$ is a Parseval frame for $\mathbb{C}^k$. That is, for any unit vector $v$, the collection $\{\frac{1}{k}\omega^i T^x M^y v\}_{i,x,y=1}^{k}$ is a Parseval frame for $\mathbb{C}^k$. Since for all $i = 1, \ldots, k$, $|\omega^i| = 1$, the collection $\{\frac{1}{\sqrt{k}}T^x M^y v\}_{x,y \in \mathbb{Z}_k}$ is a Parseval frame for $\mathbb{C}^k$ for any unit* *vector $v \in \mathbb{C}^k$. This collection is known as finite Gabor frames [31, 34, 36]. These frames are known under various names: oversampled DFT filter banks, complex modulated filter banks, short-time Fourier filter banks and Gabor filter banks, and have been studied in [14, 13, 15]. In a recent work [48], the authors study finite dimensional Gabor systems with $n^2$ vectors, in $\mathbb{C}^n$ where $n$-prime and are optimal for $n^2-n$ erasures.*

## 3.3 Unitary representations and Parseval frames for $\mathbb{C}^k$

So far we have seen a construction of tight frames for $\mathbb{C}^k$ using tight frames for $M_k$. We have characterized the representations of a group $G$ on $\mathbb{C}^k$ that form a tight frame for $M_k$. $M_k$ is isometrically isomorphic to $\mathbb{C}^{k^2}$ via the map

$$\Psi : M_k \to \mathbb{C}^{k^2} \quad \text{that takes } A = (a_{ij}) \to \begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix}$$

where $\{c_1, \dots, c_k\}$ are the columns of $A$.

Moreover, note that

$$\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \underbrace{(A \oplus \cdots \oplus A)}_{k-times} \tilde{e} \quad \text{where} \quad \tilde{e} = \begin{pmatrix} e_1 \\ \vdots \\ e_k \end{pmatrix},$$

$\{e_i\}_{i=1}^k$ is the canonical basis for $\mathbb{C}^k$.

Using this isometric isomorphism between $M_k$ and $\mathbb{C}^{k^2}$, we will extend our results in terms of representation theory to construct tight frames for $\mathbb{C}^{k^2}$.

**Proposition 3.18.** *$\{A_l\}_{l=1}^n$ is a tight frame for $M_k$ if and only if $\{\Psi(A_l)\}_{l=1}^n$ is a tight frame for $\mathbb{C}^{k^2}$.*

*Proof.* Let $\{A_l\}_{l=1}^n$ be a tight frame for $M_k$. Then for $u = (u_i)_{i=1}^{k^2} \in \mathbb{C}^{k^2}$, there exist $A \in M_k$ such that $u = \Psi(A)$. Then we have,

$$\sum_{l=1}^{n} |\langle u, \Psi(A_l)\rangle|^2 = \sum_{l=1}^{n} |\langle \Psi(A), \Psi(A_l)\rangle|^2 = \sum_{l=1}^{n} |\langle A, A_l\rangle_{HS}|^2 = \|A\|^2 = \|u\|^2.$$

Conversely, if $A \in M_k$, then

$$\sum_{l=1}^{n} |\langle A, A_l\rangle_{HS}|^2 = \sum_{l=1}^{n} |\langle \Psi(A), \Psi(A_l)\rangle|^2 = \|\Psi(A)\|^2 = \|A\|^2.$$

□

If $\pi : G \to GL(\mathbb{C}^k)$ is a representation of a group $G$, then

$$\Psi(\pi(g)) = \underbrace{(\pi(g) \oplus \cdots \oplus \pi(g))}_{k-times}\,\tilde{e}.$$

Let us denote

$$\tilde{\pi}(g) = \underbrace{\pi(g) \oplus \cdots \oplus \pi(g)}_{k-times}. \tag{3.2}$$

Then $\tilde{\pi}(g)$ is a unitary representation of $G$ on $\mathbb{C}^{k^2}$. We would like to examine the conditions such that $\tilde{\pi}$ is a frame representation. We have the following result that characterizes frame representations for $\mathbb{C}^{k^2}$.

**Theorem 3.19.** *Let $\pi : G \to GL(\mathbb{C}^k)$ be a unitary representation of a group $G$. Then the following are equivalent.*

*(a) $\pi$ is an irreducible representation of $G$.*

*(b)* $\{\sqrt{\frac{k}{n}}\pi(g) : g \in G\}$ *is a Parseval frame for* $M_k$.

*(c)* $\{\sqrt{\frac{k}{n}}\tilde{\pi}(g)\tilde{e} : g \in G\}$ *is a Parseval frame for* $\mathbb{C}^{k^2}$ *where* $\tilde{\pi}(g)$ *is as in* (3.2) *and* $\tilde{e} = \begin{pmatrix} e_1 \\ \vdots \\ e_k \end{pmatrix}$.

*Proof.* The equivalence of (a) and (b) is shown in Theorem 3.10. The equivalence of (b) and (c) follows from Proposition 3.18. □

Let $\pi : G \to GL(\mathbb{C}^k)$ be an irreducible representation of a group $G$. For $r \in \mathbb{N}$, let $m = rk$ and $\tilde{\pi}(g) = \underbrace{\pi(g) \oplus \cdots \oplus \pi(g)}_{r-times}$. Then $\tilde{\pi}(g) : G \to GL(\mathbb{C}^m)$ is a unitary representation of $G$ on $\mathbb{C}^m$. If $v_i \in \mathbb{C}^k$, $\|v_i\| = 1$ for all $i = 1, 2, \ldots, r$ and $\tilde{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_r \end{pmatrix} \in \mathbb{C}^m$, then as before, we would like to investigate that when is $\{\tilde{\pi}(g)v\}_{g \in G}$ a tight frame for $\mathbb{C}^m$.

In the process we have the following lemma.

**Lemma 3.20.** *Let* $\pi : G \to GL(V)$ *be an irreducible representation of a group* $G$ *on a vector space* $V$. *For* $v, w \in V$, *let*

$$Tu = \sum_{g \in G} \langle u, \pi(g)v \rangle \pi(g)w \quad \text{for all } u \in V.$$

*Then* $T = \frac{n}{k}\langle v, w \rangle I_V$.

*Proof.* For $h \in G$ consider,

$$T(\pi(h)u) = \sum_{g \in G} \langle \pi(h)u, \pi(g)v \rangle \pi(g)w$$

$$
\begin{aligned}
&= \sum_{g\in G} \langle u, \pi(h^{-1}g)v\rangle \pi(g)w \\
&= \sum_{g^{'}\in G} \langle u, \pi(g^{'})v\rangle \pi(hg^{'})w \\
&= \sum_{g^{'}\in G} \langle u, \pi(g^{'})v\rangle \pi(h)\pi(g^{'})w \\
&= \pi(h) \sum_{g^{'}\in G} \langle u, \pi(g^{'})v\rangle \pi(g^{'})w \\
&= \pi(h)Tu.
\end{aligned}
$$

Thus, $T \in \pi(G)^{'}$. Since $\pi$ is irreducible, $T = \lambda I$ for some $\lambda \in \mathbb{C}$. Hence we have for all $u \in V$, $\lambda u = \sum_{g\in G}\langle u, \pi(g)v\rangle \pi(g)w$. In particular, if $u_1 \in V$ is a unit vector, then from Corollary 3.15, $\{\sqrt{\frac{k}{n}}\pi(g)u_1 : g \in G\}$ is a Parseval frame for $V$. Thus $\lambda u_1 = \sum_{g\in G}\langle u_1, \pi(g)v\rangle \pi(g)w$ and we have

$$
\begin{aligned}
\lambda = \lambda\langle u_1, u_1\rangle &= \sum_{g\in G} \langle u_1, \pi(g)v\rangle \langle \pi(g)w, u_1\rangle \\
&= \sum_{g\in G} \langle \pi(g^{-1})u_1, v\rangle \langle w, \pi(g^{-1})u_1\rangle \\
&= \sum_{h\in G} \langle \pi(h)u_1, v\rangle \langle w, \pi(h)u_1\rangle \\
&= \frac{n}{k}\langle w, v\rangle.
\end{aligned}
$$

Hence $T = \frac{n}{k}\langle v, w\rangle I_V$. □

**Theorem 3.21.** *Let $\pi : G \to GL(\mathbb{C}^k)$ be an irreducible representation of a group $G$. For $r \in \mathbb{N}$, let $m = rk$ such that $\tilde{\pi}(g) = \underbrace{\pi(g) \oplus \cdots \oplus \pi(g)}_{r-times}$ and $\tilde{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_r \end{pmatrix} \in \mathbb{C}^m$ where each $v_i \in \mathbb{C}^k$, $\|v_i\| = 1$. Then, $\{\sqrt{\frac{k}{n}}\tilde{\pi}(g)\tilde{v} : g \in G\}$ is a Parseval frame for $\mathbb{C}^m$ if and only if $v_i \perp v_j$ for all $i \neq j$.*

*Proof.* For $u \in \mathbb{C}^m$, consider

$$\sum_{g\in G}\langle u, \tilde{\pi}(g)\tilde{v}\rangle\tilde{\pi}(g)\tilde{v} = \sum_{g\in G}\sum_{i=1}^{r}\langle u_i, \pi(g)v_i\rangle\tilde{\pi}(g)\tilde{v} = \sum_{g\in G}\sum_{i,j=1}^{r}\langle u_i, \pi(g)v_i\rangle\pi(g)v_j.$$

Using Lemma 3.20, we have

$$\sum_{g\in G}\sum_{i,j=1}^{r}\langle u_i, \pi(g)v_i\rangle\pi(g)v_j = \frac{n}{k}\sum_{i,j=1}^{r}\langle v_i, v_j\rangle u_i$$

Thus,

$$\begin{aligned}\sum_{g\in G}\langle u, \tilde{\pi}(g)\tilde{v}\rangle\tilde{\pi}(g)\tilde{v} &= \frac{n}{k}\sum_{i,j=1}^{r}\langle v_i, v_j\rangle u_i \\ &= \frac{n}{k}\sum_{i=1}^{r}u_i + \frac{n}{k}\sum_{\substack{i,j=1\\ i\neq j}}^{r}\langle v_i, v_j\rangle u_i \\ &= \frac{n}{k}u + \frac{n}{k}\sum_{\substack{i,j=1\\ i\neq j}}^{r}\langle v_i, v_j\rangle u_i.\end{aligned}$$

Thus, $\sum_{g\in G}\langle u, \tilde{\pi}(g)\tilde{v}\rangle\tilde{\pi}(g)\tilde{v} = \frac{n}{k}u$ if and only if $\langle v_i, v_j\rangle = 0$ for all $i \neq j$ that is $v_i \perp v_j$ for all $i \neq j$. Hence, $\{\sqrt{\frac{k}{n}}\tilde{\pi}(g)\tilde{v} : g \in G\}$ is a Parseval frame for $\mathbb{C}^m$ if and only if $v_i \perp v_j$ for all $i \neq j$. $\square$

Let $G$ be a finite group and $\lambda : G \to \mathcal{B}(l^2(G))$ be the (left) regular representation as defined in 2.28 . Then we know from Theorem 2.30 that there exists a finite number of finite dimensional nonequivalent irreducible representations $\pi_{n_i}$ of dimension $n_i$ such that

$$\lambda \sim \underbrace{\pi_{n_1} \oplus \pi_{n_1} \oplus \ldots \oplus \pi_{n_1}}_{n_1-times} \oplus \ldots\ldots \oplus \underbrace{\pi_{n_r} \oplus \pi_{n_r} \oplus \ldots \oplus \pi_{n_r}}_{n_r-times}$$

where $n = n_1^2 + n_2^2 + \ldots + n_r^2$.

**Definition 3.22.** *[43] Two representations $\pi : G \to GL(\mathbb{C}^k)$ and $\rho : G \to GL(\mathbb{C}^k)$ are said to be (unitarily) equivalent if there exists a unitary $U$ such that*

$$\pi(g) = U^*\rho(g)U \quad \textit{for all} \quad g \in G. \tag{3.3}$$

*We denote $\pi \sim_U \rho$ for the unitary equivalence of $\pi$ and $\rho$.*

If $\pi : G \to GL(\mathbb{C}^k)$ is a unitary representation unitarily equivalent to a subrepresentation of the left regular representation, then there exists a unitary $U$ such that

$$\pi \sim_U \underbrace{\pi_{n_1} \oplus \ldots \oplus \pi_{n_1}}_{m_1-times} \oplus \ldots\ldots \oplus \underbrace{\pi_{n_r} \oplus \ldots \oplus \pi_{n_r}}_{m_r-times} \qquad (*)$$

and $k = n_1m_1 + \ldots + n_rm_r$ where each $m_i \le n_i$. As per our notation used before, we can also write

$$\pi \sim_U \tilde{\pi}_{n_1}^{m_1} \oplus \tilde{\pi}_{n_2}^{m_2} \ldots \oplus \tilde{\pi}_{n_r}^{m_r} \quad \text{where} \quad \tilde{\pi}_{n_i}^{m_i} = \underbrace{\pi_{n_i} \oplus \ldots \oplus \pi_{n_i}}_{m_i-times} \qquad (**)$$

for all $i = 1, \ldots, r$.

Next we consider a representation $\pi : G \to GL(\mathbb{C}^k)$ unitarily equivalent to a subrepresentation of the left regular representation. In [37, Proposition 6.2], it was shown that every frame representation is unitarily equivalent to a subrepresentation of the left regular. We aim to construct Parseval frame vectors by extending our techniques using groups and tight frames for the Hilbert space of $k \times k$ matrices, to a subrepresentation of the left regular representation.

**Lemma 3.23.** *Let $\pi : G \to GL(\mathcal{H})$ and $\rho : G \to GL(\mathcal{H})$ be two irreducible nonequivalent representations of a group $G$. Fix vectors $v$, $w \in \mathcal{H}$, and define the operator $T$ on $\mathcal{H}$ by $Tu = \sum_{g\in G}\langle u, \pi(g)v\rangle\rho(g)w$. Then $T = 0$.*

*Proof.* For $u \in \mathcal{H}$, consider

$$\begin{aligned}
T(\pi(h)u) &= \sum_{g\in G}\langle \pi(h)u, \pi(g)v\rangle \rho(g)w \\
&= \sum_{g\in G}\langle u, \pi(h^{-1}g)v\rangle \rho(g)w \\
&= \sum_{g^{'}\in G}\langle u, \pi(g^{'})v\rangle \rho(hg^{'})w \\
&= \sum_{g^{'}\in G}\langle u, \pi(g^{'})v\rangle \rho(h)\rho(g^{'})w \\
&= \rho(h)(\sum_{g^{'}\in G}\langle u, \pi(g^{'})v\rangle \rho(g^{'})w \\
&= \rho(h)Tu.
\end{aligned}$$

Hence $T \in C(\pi, \rho)$. By Schur's lemma 2.16, we have $C(\pi, \rho) = 0$. Hence $T = 0$. □

**Lemma 3.24.** *Let $\pi : G \to GL(\mathbb{C}^m)$ and $\rho : G \to GL(\mathbb{C}^m)$ be two irreducible and inequivalent representations of a group $G$ and let $\tilde{\pi}(g) = \underbrace{\pi(g) \oplus \cdots \oplus \pi(g)}_{r-times}$ and $\tilde{\rho}(g) = \underbrace{\rho(g) \oplus \cdots \oplus \rho(g)}_{s-times}$. Fix vectors $\tilde{v} \in \mathbb{C}^{rm}$, $\tilde{w} \in \mathbb{C}^{sm}$ such that $Tu = \sum_{g\in G}\langle u, \tilde{\pi}(g)\tilde{v}\rangle \tilde{\rho}(g)\tilde{w}$. Then $T = 0$.*

*Proof.* We have

$$Tu = \sum_{g\in G}\langle u, \tilde{\pi}(g)\tilde{v}\rangle \tilde{\rho}(g)w = \sum_{j=1}^{s}\sum_{i=1}^{r}(\sum_{g\in G}\langle u_i, \pi(g)v_i\rangle \rho(g)w_j).$$

By Lemma 3.23, $T = 0$. □

**Theorem 3.25.** *Let $\pi : G \to GL(\mathbb{C}^k)$ be a representation unitarily equivalent to a subrepresentation of the left regular representation that is by $(**)$ we have*

$$\pi \sim_U \tilde{\pi}_{n_1}^{m_1} \oplus \tilde{\pi}_{n_2}^{m_2} \ldots \oplus \tilde{\pi}_{n_r}^{m_r}.$$

*For* $v_j^{n_i} \in \mathbb{C}^{n_i}$, $\|v_j^{n_i}\| = 1$ *for all* $j = 1, 2, \ldots, m_i$, *let* $\tilde{v}_i = \begin{pmatrix} v_1^{n_i} \\ \vdots \\ v_{m_i}^{n_i} \end{pmatrix} \in \mathbb{C}^{n_i m_i}$ *for all* $i = 1, 2, \ldots, r$. *Let* $v = U^* \begin{pmatrix} \sqrt{n_1}\tilde{v}_1 \\ \vdots \\ \sqrt{n_r}\tilde{v}_r \end{pmatrix} \in \mathbb{C}^k$. *Then the collection* $\{\sqrt{\frac{1}{n}}\pi(g)v : g \in G\}$ *is a Parseval frame for* $\mathbb{C}^k$ *if and only if* $v_j^{n_i} \perp v_l^{n_i}$ *for all* $j \neq l$, $j, l = 1, 2, \ldots, m_i$ *and for all* $i = 1, 2, \ldots, r$.

*Proof.* We have

$$\pi \sim_U \tilde{\pi}_{n_1}^{m_1} \oplus \tilde{\pi}_{n_2}^{m_2} \ldots \oplus \tilde{\pi}_{n_r}^{m_r};$$

that is,

$$\pi = U^*(\tilde{\pi}_{n_1}^{m_1}(g) \oplus \ldots \oplus \tilde{\pi}_{n_r}^{m_r}(g))U.$$

For $u = \begin{pmatrix} u_1 \\ \vdots \\ u_r \end{pmatrix} \in \mathbb{C}^k$, let $Uu = \begin{pmatrix} \tilde{u}_1 \\ \vdots \\ \tilde{u}_r \end{pmatrix}$.
Thus, $\sum_{g \in G} \langle u, \pi(g)v \rangle \pi(g)v$ is equal to

$$\begin{aligned}
&\sum_{g \in G} \langle u, U^*(\tilde{\pi}_{n_1}^{m_1}(g) \oplus \ldots \oplus \tilde{\pi}_{n_r}^{m_r}(g))Uv \rangle U^*(\tilde{\pi}_{n_1}^{m_1}(g) \oplus \ldots \oplus \tilde{\pi}_{n_r}^{m_r}(g))Uv \\
&= U^* \sum_{g \in G} \langle Uu, \tilde{\pi}_{n_1}^{m_1}(g) \oplus \ldots \oplus \tilde{\pi}_{n_r}^{m_r}(g)(Uv) \rangle \tilde{\pi}_{n_1}^{m_1}(g) \oplus \ldots \oplus \tilde{\pi}_{n_r}^{m_r}(g)(Uv) \\
&= U^*(\sum_{g \in G} \sum_{i=1}^{r} n_i \langle \tilde{u}_i, \tilde{\pi}_{n_i}^{m_i}(g)\tilde{v}_i \rangle \tilde{\pi}_{n_i}^{m_i}(g)\tilde{v}_i + \sum_{g \in G} \sum_{\substack{i,j=1 \\ i \neq j}}^{r} n_i \langle \tilde{u}_i, \tilde{\pi}_{n_i}^{m_i}(g)\tilde{v}_i \rangle \tilde{\pi}_{n_j}^{m_j}(g)\tilde{v}_j).
\end{aligned}$$

Since $\pi_{n_i}$ is not equivalent to $\pi_{n_k}$ for all $i \neq k$, we have by Lemma 3.24,

$$\sum_{\substack{i,j=1 \\ i \neq j}}^{r} \sum_{g \in G} n_i \langle \tilde{u}_i, \tilde{\pi}_{n_i}^{m_i}(g)\tilde{v}_i \rangle \tilde{\pi}_{n_j}^{m_j}(g)\tilde{v}_j = 0.$$

Thus we have,

$$\sum_{g \in G} \langle u, \pi(g)v \rangle \pi(g)v = U^*(\sum_{i=1}^{r} \sum_{g \in G} n_i \langle \tilde{u}_i, \tilde{\pi}_{n_i}^{m_i}(g)\tilde{v}_i \rangle \tilde{\pi}_{n_i}^{m_i}(g)\tilde{v}_i).$$

We have,

$$\frac{n_i}{n}\sum_{g\in G}\langle \tilde{u}_i, \tilde{\pi}_{n_i}^{m_i}(g)\tilde{v}_i\rangle \tilde{\pi}_{n_i}^{m_i}(g)\tilde{v}_i = \tilde{u}_i$$

if and only if $\{\sqrt{\frac{n_i}{n}}\tilde{\pi}_{n_i}^{m_i}(g)\tilde{v}_i : g \in G\}$ is a Parseval frame for $\mathbb{C}^{n_i m_i}$ for all $i = 1, \ldots, k$. From Theorem 3.21, this is equivalent to saying $v_j^{n_i} \perp v_l^{n_i}$ for all $j \neq l$, $j, l = 1, 2, \ldots, m_i$.

Hence,

$$\sum_{g\in G}\langle u, \pi(g)v\rangle \pi(g)v = U^*(\sum_{i=1}^{r} n\tilde{u}_i) = nu$$

if and only if $\{\sqrt{\frac{n_i}{n}}\tilde{\pi}_{n_i}^{m_i}(g)\tilde{v}_i : g \in G\}$ is a Parseval frame for $\mathbb{C}^{n_i m_i}$ for all $i = 1, \ldots, r$; that is, $v_j^{n_i} \perp v_l^{n_i}$ for all $j \neq l$, $j, l = 1, 2, \ldots, m_i$. Thus, for $v_j^{n_i} \in \mathbb{C}^{n_i}$, $\|v_j^{n_i}\| = 1$ for all $j = 1, 2, \ldots, m_i$, $\tilde{v}_i = \begin{pmatrix} v_1^{n_i} \\ \vdots \\ v_{m_i}^{n_i} \end{pmatrix} \in \mathbb{C}^{n_i m_i}$ for all $i = 1, 2, \ldots, r$ and $v = U^* \begin{pmatrix} \sqrt{n_1}\tilde{v}_1 \\ \vdots \\ \sqrt{n_r}\tilde{v}_r \end{pmatrix} \in \mathbb{C}^k$, $\{\sqrt{\frac{1}{n}}\pi(g)v : g \in G\}$ is a Parseval frame for $\mathbb{C}^k$ if and only if $v_j^{n_i} \perp v_l^{n_i}$ for all $j \neq l$, $j, l = 1, 2, \ldots, m_i$. $\square$

**Proposition 3.26.** *Let $\pi : G \to GL(\mathbb{C}^k)$ be a unitary representation and $v \in \mathbb{C}^k$ be such that the collection $\{\pi(g)v : g \in G\}$ is a Parseval frame for $\mathbb{C}^k$. If $\rho : H \to GL(\mathbb{C}^k)$ is any unitary representation, then the collection $\{\sqrt{\frac{1}{|H|}}\rho(h)\pi(g)v : g \in G, h \in H\}$ is also a Parseval frame for $\mathbb{C}^k$.*

*Proof.* Consider

$$\sum_{g\in G, h\in H} |\langle u, \rho(h)\pi(g)v\rangle|^2 = \sum_{g\in G, h\in H} |\langle \rho(h^{-1})u, \pi(g)v\rangle|^2$$

$$= \sum_{h \in H} \|\rho(h^{-1})u\|^2$$

$$= |H|\|u\|^2.$$

□

**Example 3.27.** *Let $\pi : \mathbb{Z}_k \to GL(\mathbb{C}^k)$ be a representation such that*

$$\pi(x) = M^x \quad \textit{where} \quad M^x f(t) = e^{2\pi i x t/k} f(t).$$

*Since $\mathbb{Z}_k$ is an abelian group, $\pi = \pi_1 \oplus \ldots \oplus \pi_k$ where for all $i \neq j$, $\pi_i \neq \pi_j$. For $v = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$, $\{\sqrt{\frac{1}{k}}\pi(x)v : x \in \mathbb{Z}_k\}$ is an orthonormal basis for $\mathbb{C}^k$. Let $\rho : \mathbb{Z}_k \to GL(\mathbb{C}^k)$ be such that*

$$\rho(y) = C^y \quad \textit{where} \quad C^y f(t) = e^{\pi i y t(t-k)/k} f(t).$$

*Then $\rho$ is a unitary representation. By Proposition 3.26, $\{\frac{1}{k}\rho(y)\pi(x)v : x, y \in \mathbb{Z}_k\}$ is a Parseval frame for $\mathbb{C}^k$. Thus, the collection $\{\frac{1}{k}C^y M^x \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} : x, y \in \mathbb{Z}_k\}$ is a Parseval frame for $\mathbb{C}^k$. This collection of frames has been earlier shown in [17] where the operator $C^y$ is called the chirp modulation operator.*

# Chapter 4

# Estimating Frame Correlation

In signal processing, one of the primary goals is to obtain a digital representation of the signal of interest that is suitable for storage, transmission, and recovery. The basic problem we are interested in is the transmission of information in the form of a vector $x \in \mathbb{C}^n$ over a channel in such a way that recovery of the information at the receiver is robust to errors introduced by the channel.

In the particular model of interest, we first transform the signal $x$ by forming $y = Fx \in \mathbb{C}^m$. This vector is then quantized in some fashion yielding $\hat{y} = Q(y)$. In other words, we transmit not $x$ but the quantized frame coefficients of $x$. Each such quantized coefficient is considered a packet of data sent over the channel. It is assumed that the channel distorts the transmitted vector by erasing packets at random. Robustness to this sort of distortion means maximizing the number of packets that can be erased while still allowing blind reconstruction of the signal as accurately as possible from the remaining packets.

In [41], it has been shown that tight frames are optimal for signal reconstruction when there is one erasure, and equiangular tight frames are optimal when there are up to two erasures.

**Definition 4.1.** *For a given Parseval frame $\{f_j\}_{j=1}^n$ in $\mathbb{C}^k$, we define the maximal frame correlation $\mathcal{M}(\{f_j\}_{j=1}^n)$ by*

$$\mathcal{M}(\{f_j\}_{j=1}^n) = \max_{j \neq i}\{|\langle f_i, f_j \rangle|\}.$$

All tight frames do not behave the same in applications. Among all tight frames which have the same redundancy, the ones that minimize the maximum correlation tend to perform better in applications. It has been shown in [41] and [55] that equiangular frames minimize the maximum correlation among all tight frames which have the same redundancy.

In this chapter, we focus on frames of the type $\{\pi(g)v\}_{g\in G}$ for $\mathbb{C}^k$. We want to establish conditions such that the maximal correlation $|\langle \pi(g)v, \pi(h)v\rangle|$ for all $g, h \in G$ with $g \neq h$ is as small as possible.

## 4.1 Estimating frame correlation for frames for $\mathbb{C}^{k^2}$

From Theorem 3.19, we know that $\{\sqrt{\frac{k}{n}}\tilde{\pi}(g)\tilde{e} : g \in G\}$ is a Parseval frame for $\mathbb{C}^{k^2}$ if and only if $\pi : G \to GL(\mathbb{C}^k)$ is an irreducible representation of $G$ where $\tilde{\pi}(g)$ is as given in (3.2) and $\tilde{e} = \begin{pmatrix} e_1 \\ \vdots \\ e_k \end{pmatrix}$.

Let us denote

$$\tilde{\pi}(G)\tilde{e} = \{\frac{k}{n}\tilde{\pi}(g)\tilde{e} : g \in G\}.$$

By Definition 4.1 we have

$$\mathcal{M}(\tilde{\pi}(G)\tilde{e}) = \frac{k}{n}\max_{g\neq h}\{|\langle\tilde{\pi}(g)\tilde{e}, \tilde{\pi}(h)\tilde{e}\rangle|\}.$$

Because of the isometric isomorphism between $M_k$ and $\mathbb{C}^{k^2}$, we have

$$\mathcal{M}(\tilde{\pi}(G)\tilde{e}) = \frac{k}{n}\max_{g\neq h}\{|\langle\pi(g), \pi(h)\rangle_{\mathrm{HS}}|\}. \tag{4.1}$$

**Proposition 4.2.** *Let $\pi : G \to GL(\mathbb{C}^k)$ be a unitary representation, then*

$$\mathcal{M}(\tilde{\pi}(G)\tilde{e}) = \frac{k}{n}\max_{g\neq e}\{|\chi_\pi(g)|\}.$$

*Proof.* We have,

$$\begin{aligned}
\mathcal{M}(\tilde{\pi}(G)\tilde{e}) &= \frac{k}{n}\max_{g\neq h}|\langle\pi(g), \pi(h)\rangle_{\mathrm{HS}}| && \text{(by 4.1)}\\
&= \frac{k}{n}\max_{g\neq h}|\langle\pi(h^{-1}g), I\rangle|\\
&= \frac{k}{n}\max_{\tilde{g}\neq e}|\langle\pi(\tilde{g}), I\rangle|\\
&= \frac{k}{n}\max_{g\neq e}\{|\chi_\pi(g)|\}
\end{aligned}$$

□

Since all tight frames do not behave the same in applications and we endeavor to find frames of the type $\{\pi(g)v\}_{g\in G}$ for which $\mathcal{M}(\pi(G)v)$ is as small as possible. We have the following result that compares two such frames in terms of their behavior in applications.

**Proposition 4.3.** *Let $\pi : G \to GL(\mathbb{C}^k)$ and $\rho : G \to GL(\mathbb{C}^k)$ be two inequivalent and irreducible representations with $|G| = n$. Then*

$$\mathcal{M}(\tilde{\pi}(G)\tilde{e}) \leq \mathcal{M}(\tilde{\rho}(G)\tilde{e})$$

*if and only if*

$$\max_{\substack{g \in G \\ g \neq e}} \{|\chi_\pi(g)|\} \leq \max_{\substack{h \in G \\ h \neq e}} \{|\chi_\rho(h)|\}.$$

*Proof.* Follows directly from Proposition 4.2. □

To see how frames of the type $\{\tilde{\pi}(g)\tilde{e}\}_{g \in G}$ perform in applications, in the next result we calculate the bounds on the maximum correlation between the frame elements.

**Proposition 4.4.** *Let $G$ be a group of order $n$ and $\pi : G \to GL(\mathbb{C}^k)$ be an irreducible representation such that $\{\sqrt{\frac{k}{n}}\tilde{\pi}(g)\tilde{e}\}$ is a Parseval frame for $\mathbb{C}^{k^2}$. Then*

$$c_{n,k^2} \leq \mathcal{M}(\tilde{\pi}(G)\tilde{e}) \leq \frac{k^2}{n}$$

*where $c_{n,k^2} = \frac{k}{n}\sqrt{\frac{n-k^2}{n-1}}$.*

*Proof.* From Proposition 4.2, we have $\mathcal{M}(\tilde{\pi}(G)\tilde{e}) = \frac{k}{n} \max_{g \neq e}\{|\chi_\pi(g)|\}$. Let $\{\lambda_i^g\}_{i=1}^k$ be the eigen-values of $\pi(g)$. Since $\pi$ is a unitary representation, $|\lambda_i^g| = 1$ for all $g \in G$. Thus,

$$\mathcal{M}(\tilde{\pi}(G)\tilde{e}) = \frac{k}{n} \max_{g \neq e}\{|\chi_\pi(g)|\} \leq \frac{k}{n} \sum_{i=1}^{k} |\lambda_i^g| = \frac{k^2}{n}$$

Since $\pi$ is an irreducible representation, from Theorem 2.23, we have

$$\begin{aligned}
1 &= \frac{1}{n}\sum_{g\in G}|\chi_\pi(g)|^2 \\
&= \frac{1}{n}[k^2 + \sum_{\substack{g\in G\\ g\neq e}}|\chi_\pi(g)|^2] \\
&\leq \frac{1}{n}[k^2 + (n-1)\max_{g\neq e}|\chi_\pi(g)|^2].
\end{aligned}$$

Again using Proposition 4.2, we have

$$\sqrt{\frac{n-k^2}{n-1}} \leq \max_{g\neq e}|\chi_\pi(g)| = \frac{n}{k}\mathcal{M}(\tilde{\pi}(G)\tilde{e}).$$

Thus, $c_{n,k^2} \leq \mathcal{M}(\tilde{\pi}(G)\tilde{e}) \leq \frac{k^2}{n}$. □

**Corollary 4.5.** *Let $G$ be a group of order $n$ and $\pi : G \to \mathbb{C}^k$ be an irreducible representation. Then,*

$$\max_{g\neq e}\{|\chi_\pi(g)|\} \geq \sqrt{\frac{n-k^2}{n-1}}.$$

*Proof.* Using Proposition 4.2, we have

$$\mathcal{M}(\tilde{\pi}(G)\tilde{e}) = \frac{k}{n}\max_{g\neq e}\{|\chi_\pi(g)|\}.$$

By Proposition 4.4, we have

$$c_{n,k^2} \leq \mathcal{M}(\tilde{\pi}(G)\tilde{e}) \leq \frac{k^2}{n}$$

where $c_{n,k^2} = \frac{k}{n}\sqrt{\frac{n-k^2}{n-1}}$. Thus,

$$\max_{g\neq e}\{|\chi_\pi(g)|\} \geq \sqrt{\frac{n-k^2}{n-1}}.$$

□

**Remark 4.6.** *If $\mathcal{F}$ is the collection of all $(n, k^2)$ tight frames of the type $\{\tilde{\pi}(g)\tilde{e}\}_{g\in G}$, by Proposition 4.4, we infer that*

$$\min_{\mathcal{F}}\{\mathcal{M}(\tilde{\pi}(G)\tilde{e})\} \geq c_{n,k^2} = \sqrt{\frac{k^2(n-k^2)}{n^2(n-1)}}. \tag{4.2}$$

*Grassmanian frames [55] minimize the maximum correlation between among all frames which have the same redundancy. The equality in Equation* (4.2) *is achieved in the case of an equiangular tight frame. We conclude that a frame of the type $\{\tilde{\pi}(g)\tilde{e}\}_{g\in G}$ for $\mathbb{C}^{k^2}$ is equiangular if and only if*

$$\frac{k}{n}|\langle\tilde{\pi}(g)\tilde{e}, \tilde{\pi}(h)\tilde{e}\rangle| = c_{n,k^2}$$

*for all $g, h \in G$ and $g \neq h$ where*

$$c_{n,k^2} = \sqrt{\frac{k^2(n-k^2)}{n^2(n-1)}}.$$

*The quantity $\sqrt{\frac{n-k^2}{k^2(n-1)}}$ is one of the Welch's lower bound [50, 58, 59]. Using character theory of groups, we have re-derived Welch's lower bound in the special case of the frames arising from group representations.*

**Proposition 4.7.** *Let $\pi : G \to GL(\mathbb{C}^k)$ be an irreducible representation of $G$ with $|G| = n$. Then $\{\sqrt{\frac{k}{n}}\tilde{\pi}(g)\tilde{e}\}_{g\in G}$ is an equiangular frame for $\mathbb{C}^{k^2}$ if and only if for all $g \neq e$,*

$$|\chi_\pi(g)| = \sqrt{\frac{n-k^2}{n-1}}.$$

*Proof.* For all $g \neq h$, we have

$$|\langle\tilde{\pi}(g)\tilde{e}, \tilde{\pi}(h)\tilde{e}\rangle| = |\langle\pi(g), \pi(h)\rangle_{\mathrm{HS}}| = |\langle\pi(h^{-1}g), I\rangle| = |\chi_\pi(h^{-1}g)|$$

From Remark 4.6, we have $\{\sqrt{\frac{k}{n}}\tilde{\pi}(g)\tilde{e}\}_{g\in G}$ is an equiangular frame for $\mathbb{C}^{k^2}$ if and only if $\frac{k}{n}|\chi_\pi(h^{-1}g)| = c_{n,k^2}$ for all $g,h\in G$ and $g\neq h$ where $c_{n,k^2} = \sqrt{\frac{k^2(n-k^2)}{n^2(n-1)}}$. Letting $\tilde{g} = h^{-1}g$,

$$\frac{k}{n}|\chi_\pi(\tilde{g})| = c_{n,k^2} \quad \text{for all } e\neq \tilde{g}\in G$$

where $c_{n,k^2} = \sqrt{\frac{k^2(n-k^2)}{n^2(n-1)}}$. Thus $\{\sqrt{\frac{k}{n}}\tilde{\pi}(g)\tilde{e}\}_{g\in G}$ is an equiangular frame for $\mathbb{C}^{k^2}$ if and only if for all $g\neq e$,

$$|\chi_\pi(g)| = \sqrt{\frac{n-k^2}{n-1}}.$$

$\square$

**Example 4.8.** *For $n$-even, consider the Dihedral group $D_n$ of $2n$-elements*

$$\{e, R, R^2, \ldots, R^{n-1}, F, FR, FR^2, \ldots, FR^{n-1}\}$$

*where*

- *$R$ is counterclockwise rotation through an angle of $\frac{2\pi}{n}$.*
- *$F$ is any reflection about the line $\frac{\pi}{n}$.*

*Then, $R^n = 1$, $F^2 = 1$, $FRF = R^{-1}$. Let $\pi$ be the irreducible representation of $D_n$ on $\mathbb{C}^2$ as in Example 3.12 as*

$$\pi(R^k) = \begin{pmatrix} \omega^k & 0 \\ 0 & \omega^{-k} \end{pmatrix}, \quad \pi(FR^k) = \begin{pmatrix} 0 & \omega^k \\ \omega^{-k} & 0 \end{pmatrix}$$

*where $\omega = e^{\frac{2\pi i}{n}}$. The corresponding characters $\chi_\pi$ are given as $\chi_\pi(R^k) = \omega^k + \omega^{-k}$ and $\chi_\pi(FR^k) = 0$ for all $k\in\{1,\ldots,n-1\}$.*

*Thus we have,*

$$\mathcal{M}(\tilde{\pi}(G)\tilde{e}) = \frac{2}{2n} \max_{0<k\leq n-1} \{|\chi_\pi(R^k)|\}.$$

*Since $n$-even, there exists an $m \in \{1, \ldots, n-1\}$ such that $\omega^m = \omega^{-m}$. Hence we have*

$$\max_{0<k\leq n-1} \{|\chi_\pi(R^m)|\} = 2 \quad \textit{and} \quad \mathcal{M}(\tilde{\pi}(G)\tilde{e}) = \frac{2}{n}.$$

*Thus, the Parseval frame obtained in Example 3.12 that is $\{\frac{1}{\sqrt{n}}\pi(g) : g \in D_n\}$ for $\mathbb{C}^2$ is not equiangular.*

From Theorem 3.21 in Chapter 3, we know that if $\pi : G \to GL(\mathbb{C}^k)$ is an irreducible representation of a group $G$ and $\tilde{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_k \end{pmatrix} \in \mathbb{C}^{k^2}$ where each $v_i \in \mathbb{C}^k$, $\|v_i\| = 1$. Then $\{\sqrt{\frac{k}{n}}\tilde{\pi}(g)\tilde{v} : g \in G\}$ is a Parseval frame for $\mathbb{C}^{k^2}$ if and only if $v_i \perp v_j$ for all $i \neq j$.

Next we aim to find frame correlation for the frames of the type $\{\tilde{\pi}(g)\tilde{v}\}_{g\in G}$ for $\mathbb{C}^{k^2}$. We know that from Chapter 2, Definition 2.3 that two $(n, k)$ frames $\{f_i\}_{i=1}^n$ and $\{g_i\}_{i=1}^n$ for $\mathbb{C}^k$ are type-I equivalent if there exists a unitary $U$ such that $Uf_i = g_i$ for all $i$. Thus we have the following result that establishes the equivalence of frames using our representation theoretic construction of frames.

**Proposition 4.9.** *Let $\pi : G \to GL(\mathbb{C}^k)$ and $\rho : G \to GL(\mathbb{C}^k)$ be two equivalent and irreducible representations of $G$ that is $U\pi(g) = \rho(g)U$ for all $g \in G$. Then the following hold.*

*(a) For any unit vector $v \in \mathbb{C}^k$, the frames $\{\pi(g)v\}_{g\in G}$ and $\{\rho(g)Uv\}_{g\in G}$ are type-I equivalent.*

*(b) If $\tilde{U} = \underbrace{U \oplus U \oplus \ldots \oplus U}_{k-times}$, then $\{\tilde{\rho}(g)\tilde{e}\}_{g\in G}$ and $\{\tilde{\pi}(g)(\tilde{U}\tilde{e})\}_{g\in G}$ are type-I equivalent.*

*Proof.* For any unit vector $v \in \mathbb{C}^k$, $\{\pi(g)v\}_{g\in G}$ is a uniform tight frame for $\mathbb{C}^k$. Since $U\pi(g) = \rho(g)U$, we have $U\pi(g)v = \rho(g)Uv$ for all $g \in G$. Thus, the frames $\{\pi(g)v\}_{g\in G}$ and $\{\rho(g)Uv\}_{g\in G}$ are type-I equivalent.
For part (b), if $\tilde{U} = \underbrace{U \oplus U \oplus \ldots \oplus U}_{k-times}$, then $\tilde{\rho}(g) = \tilde{U}^*\tilde{\pi}(g)\tilde{U}$. Thus, $\tilde{\rho}$ is unitarily equivalent to $\tilde{\pi}$. From Theorem 3.19, we know that $\{\tilde{\rho}(g)\tilde{e}\}$ is a tight frame for $\mathbb{C}^{k^2}$. But we have,

$$\tilde{\rho}(g)\tilde{e} = \tilde{U}^*\tilde{\pi}(g)\tilde{U}\tilde{e} \quad \text{for all } g \in G.$$

Thus as frames, $\{\tilde{\rho}(g)\tilde{e}\}_{g\in G}$ and $\{\tilde{\pi}(g)(\tilde{U}\tilde{e})\}_{g\in G}$ are type-I equivalent. □

**Remark 4.10.** *Let $\pi : G \to GL(\mathbb{C}^k)$ be an irreducible representation of $G$ with $|G| = n$ such that $\{\sqrt{\frac{k}{n}}\pi(g)\}_{g\in G}$ is a Parseval frame for $M_k$. Let $\{v_i\}_{i=1}^k$ be an orthonormal basis in $\mathbb{C}^k$ and $\tilde{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_k \end{pmatrix} \in \mathbb{C}^{k^2}$ such that $\{\sqrt{\frac{k}{n}}\tilde{\pi}(g)\tilde{v}\}_{g\in G}$ is a Parseval frame for $\mathbb{C}^{k^2}$. Let*

$$U = \begin{bmatrix} v_1 & v_2 & \ldots & \ldots & v_k \end{bmatrix} \in M_k.$$

*Letting $\rho(g) = U^*\pi(g)U$, for all $g \in G$, we have $\pi$ and $\rho$ as equivalent irreducible representations. From Proposition 4.9, the frames, $\{\tilde{\rho}(g)\tilde{e}\}_{g\in G}$ and $\{\tilde{\pi}(g)(\tilde{U}\tilde{e})\}_{g\in G}$*

*are type-I equivalent. Thus* $\{\tilde{\rho}(g)\tilde{e}\}_{g\in G}$ *and* $\{\tilde{\pi}(g)(\tilde{v})\}_{g\in G}$ *are type-I equivalent and we have,*

$$\mathcal{M}(\tilde{\pi}(G)\tilde{v}) = \mathcal{M}(\tilde{\rho}(G)\tilde{e}). \tag{4.3}$$

Hence we have the similar results for the frame of the type $\{\tilde{\pi}(g)\tilde{v}\}_{g\in G}$.

**Theorem 4.11.** *Let* $\pi : G \to GL(\mathbb{C}^k)$ *be an irreducible representation of* $G$ *with* $|G| = n$ *and let* $\{v_i\}$ *be an orthonormal basis in* $\mathbb{C}^k$ *and* $\tilde{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_k \end{pmatrix} \in \mathbb{C}^{k^2}$ *such that* $\{\sqrt{\frac{k}{n}}\tilde{\pi}(g)\tilde{v}\}_{g\in G}$ *is a Parseval frame for* $\mathbb{C}^{k^2}$*.Then,*

$$c_{n,k^2} \leq \mathcal{M}(\tilde{\pi}(G)\tilde{v}) \leq \frac{k^2}{n}$$

*where* $c_{n,k^2} = \frac{k}{n}\sqrt{\frac{n-k^2}{n-1}}$.

**Theorem 4.12.** *Let* $\pi : G \to GL(\mathbb{C}^k)$ *and* $\rho : G \to GL(\mathbb{C}^k)$ *be two inequivalent and irreducible representations with* $|G| = n$ *and* $\tilde{v}$ *be as above such that* $\{\tilde{\pi}(g)\tilde{v}\}_{g\in G}$ *and* $\{\tilde{\rho}(g)\tilde{v}\}_{g\in G}$ *are uniform tight frames for* $\mathbb{C}^{k^2}$*. Then*

$$\mathcal{M}(\tilde{\pi}(G)\tilde{v}) \leq \mathcal{M}(\tilde{\rho}(G)\tilde{v})$$

*if and only if*

$$\max_{\substack{g\in G \\ g\neq e}}\{|\chi_\pi(g)|\} \leq \max_{\substack{h\in H \\ g\neq e}}\{|\chi_\rho(g)|\}.$$

## 4.2 Estimating frame correlation for frames for $\mathbb{C}^k$

If $\pi : G \to GL(\mathbb{C}^k)$ is an irreducible representation, then we know from 3.15 that for any unit vector $v \in \mathbb{C}^k$, $\{\sqrt{\frac{k}{n}}\pi(g)v : g \in G\}$ is a Parseval frame for for $\mathbb{C}^k$. In this section, we establish results regarding the frame correlation for tight frames of the type $\{\pi(g)v\}_{g\in G}$.

Once again we denote

$$\pi(G)v = \{\sqrt{\frac{k}{n}}\pi(g)v : g \in G\}.$$

From Definition 4.1 we have,

$$\mathcal{M}(\pi(G)v) = \frac{k}{n}\max_{g\neq h}\{|\langle\pi(g)v, \pi(h)v\rangle|\}.$$

**Proposition 4.13.** *Let $\pi : G \to \mathbb{C}^k$ be an irreducible representation and let the linear transformation corresponding to $\pi(g) = (a^g_{i,j})^n_{i,j=1}$. Then*

$$\mathcal{M}(\pi(G)e_m) = \max_{g\neq e}\{|a^g_{m,m}|\}$$

*where $e_m \in \mathbb{C}^k$ is the the canonical basis vector having $1$ at the mth place and $0$ elsewhere.*

*Proof.*

$$\begin{aligned}\mathcal{M}(\pi(G)e_m) &= \max_{g\neq h}\{|\langle\pi(g)e_m, \pi(h)e_m\rangle|\}\\ &= \max_{g\neq e}\{|\langle\pi(g)e_m, e_m\rangle|\}\\ &= \max_{g\neq e}\{|a^g_{m,m}|\}.\end{aligned}$$

$\square$

The following two corollaries gives a comparison of frame correlation between two tight frames constructed using techniques given in Chapter 3. The proofs of these results follow directly from the Proposition 4.13.

**Corollary 4.14.** *Let $\pi : G \to GL(\mathbb{C}^k)$ be an irreducible representation and let the linear transformation corresponding to $\pi(g) = (a^g_{i,j})^n_{i,j=1}$. Then*

$$\mathcal{M}(\pi(G)e_m) \leq \mathcal{M}(\pi(G)e_k)$$

*if and only if*

$$\max_{g\neq e}\{|a^g_{m,m}|\} \leq \max_{g\neq e}\{|a^g_{k,k}|\}$$

*where $e_i$ is the canonical basis vector having 1 at the ith place and 0 elsewhere.*

**Corollary 4.15.** *Let $\pi : G \to GL(\mathbb{C}^k)$ and $\rho : G \to GL(\mathbb{C}^k)$ be two inequivalent and irreducible representations and let the linear transformation corresponding to $\pi(g) = (a^g_{i,j})^n_{i,j=1}$ and $\rho(g) = (b^g_{i,j})^n_{i,j=1}$. Then*

$$\mathcal{M}(\pi(G)e_m) \leq \mathcal{M}(\rho(G)e_m)$$

*if and only if*

$$\max_{g\neq e}\{|a^g_{m,m}|\} \leq \max_{g\neq e}\{|b^g_{m,m}|\}$$

*where $e_m$ is the canonical basis vector having 1 at the mth place and 0 elsewhere.*

# Chapter 5

# Equiangular Tight Frames and Signature Sets in Groups

Equiangular tight frames play an important role in several areas of mathematics, ranging from signal processing (see, e.g. [3], [16], [45], [46], and references therein) to quantum computing. Due to their rich theoretical properties and their numerous practical applications, equiangular tight frames are arguably the most important class of finite dimensional frames.

The problem of the existence of equiangular tight frames is known to be equivalent to the existence of a certain type of matrix called a Seidel matrix [49] or signature matrix [41] with two eigenvalues. A matrix $Q$ is a Seidel matrix provided that it is self-adjoint, its diagonal entries are 0, and its off-diagonal entries are all of modulus one. In the real case, these off-diagonal entries must all be $\pm 1$; such matrices can then be interpreted as (Seidel) adjacency matrices of graphs.

In this chapter we aim to construct equiangular tight frames using subsets of groups with certain properties. We will establish a relationship between these subsets and signature matrices.

## 5.1 Signature sets in groups

Let $G$ be a finite group of order $n$. Let $\lambda : G \to GL(\mathbb{F}(G))$ be the left regular representation as in Definition 2.3.3 such that

$$\lambda(g)e_h = e_{g\cdot h}$$

where $\mathbb{F}(G)$ is the free vector space over $G$. Then we know that $\sum_{g\in G}\lambda(g) = J$ where $J$ is the $n \times n$ matrix of all 1's.

**Definition 5.1.** *Let $G$ be a group of order $n$ and $S \subset G \setminus \{e\}$, $T = S^c \setminus \{e\}$ such that $G \setminus \{e\} = S \cup T$. Form $Q = \sum_{g\in S}\lambda(g) - \sum_{h\in T}\lambda(h)$. Then $Q$ is an $n \times n$ matrix with $Q_{ii} = 0$ and $|Q_{ij}| = 1$ for all $i \neq j$. We call $S$ a* **signature set** *in $G$ for an $(n, k)$-equiangular tight frame if $Q$ is a signature matrix for an $(n, k)$-equiangular tight frame.*

**Proposition 5.2.** *Let $G$ be a finite group of order $n$ and $S \subset G$. If $S$ is a signature set in $G$ for an $(n, k)$-equiangular tight frame, then $T = S^c \setminus \{e\}$ is a signature set in $G$ for an $(n, n-k)$-equiangular tight frame.*

*Proof.* If $S$ is a signature set in $G$ for an $(n, k)$-equiangular tight frame, then $Q$ as in Definition 5.1 is a signature matrix for $(n, k)$-equiangular tight frame.

Let $\tilde{Q} = \sum_{h\in T} \lambda(h) - \sum_{g\in S} \lambda(g)$. Then $\tilde{Q} = -Q$ and thus $\tilde{Q}$ is self adjoint with $\tilde{Q}_{ii} = 0$ and for all $i \neq j$, $|\tilde{Q}_{ij}| = 1$. Consider,

$$\begin{aligned}
\tilde{Q}^2 &= (-Q)^2 \\
&= Q^2 \\
&= (n-1)I + \mu Q && \text{(by Theorem 2.8)} \\
&= (n-1)I - \mu\tilde{Q}.
\end{aligned}$$

Thus $\tilde{Q}$ is a signature matrix for $(n, \tilde{k})$-equiangular tight frame for some $\tilde{k}$. Using Equation (2.4) for $-\mu$, we get $\tilde{k} = n - k$. Thus, $T$ is a signature set in $G$ for an $(n, n-k)$-equiangular tight frame. □

**Definition 5.3.** *Given any subset $S$ of $G$, a subset $\tilde{S}$ of $G$ is said to be conjugate to $S$ if and only if there exists some $\tilde{g}$ in $G$ such that $\tilde{S} = \tilde{g}S\tilde{g}^{-1}$.*

**Proposition 5.4.** *Let $G$ be a finite group of order $n$ and $S \subset G$. If $S$ is a signature set in $G$ for an $(n, k)$-equiangular tight frame, then for any $\tilde{g} \in G$, the set $\tilde{S} = \tilde{g}S\tilde{g}^{-1} = \{\tilde{g}\cdot g\cdot \tilde{g}^{-1} : g \in S\}$ also is a signature set in $G$ for an $(n, k)$-equiangular tight frame.*

*Proof.* Let $T = S^c \setminus \{e\}$ and $\tilde{T} = \tilde{S}^c \setminus \{e\}$. Form $Q = \sum_{g\in S} \lambda(g) - \sum_{h\in T} \lambda(h)$ and $\tilde{Q} = \sum_{g\in\tilde{S}} \lambda(g) - \sum_{h\in\tilde{T}} \lambda(h)$. Then,

$$\begin{aligned}
\tilde{Q} &= \sum_{g\in S} \lambda(\tilde{g}\cdot g\cdot \tilde{g}^{-1}) - \sum_{h\in T} \lambda(\tilde{g}\cdot h\cdot \tilde{g}^{-1}) \\
&= \sum_{g\in S} \lambda(\tilde{g})\lambda(g)\lambda(\tilde{g}^{-1}) - \sum_{h\in T} \lambda(\tilde{g})\lambda(h)\lambda(\tilde{g}^{-1})
\end{aligned}$$

$$= \lambda(\tilde{g})(\sum_{g \in S} \lambda(g) - \sum_{h \in T} \lambda(h))\lambda(\tilde{g}^{-1})$$
$$= \lambda(\tilde{g})Q\lambda(\tilde{g}^{-1}).$$

$Q$ self adjoint implies that $\tilde{Q}$ is self adjoint. Also since for all $g \in G$, $\lambda(g)$ is a permutation, from Proposition 2.10, $\tilde{Q}$ also is a signature matrix for an $(n,k)$-equiangular tight frame and thus $\tilde{S}$ is a signature set in $G$ for an $(n,k)$-equiangular tight frame. □

**Remark 5.5.** *Note that if one of the sets $S$ or $T$ is an empty set, say $T = \emptyset$, then $S = G \setminus \{e\}$. In this case we have*

$$Q = \sum_{g \in S} \lambda(g) = \sum_{g \in G \setminus \{e\}} \lambda(g) = J - I_n$$

*where $J$ is the $n \times n$ matrix of all 1's. Thus, $\mu = n - 2$ and $S$ is a signature set for the trivial $(n,1)$-equiangular tight frame. By Proposition 5.2, $\emptyset$ is the signature set for $(n, n-1)$-equiangular tight frame.*

*From this point onwards, both $S$ and $T$ are taken as non empty subsets of $G$.*

**Notation:** Given subsets $A, B \subseteq G$ and $g \in G$, we denote

$$N^g_{(A,B)} = \#\{(g_1, g_2) \in A \times B : g_1 \cdot g_2 = g\}.$$

**Lemma 5.6.** *Let $G$ be a finite group and $S, T \subset G \setminus \{e\}$ be disjoint such that $G \setminus \{e\} = S \cup T$. Then for all $g \in G$, $N^g_{(S,T)} = N^g_{(T,S)}$.*

*Proof.* Let $|G| = n$ and $|S| = m$. Then $|T| = n - m - 1$. For $g \in S$, assume that $N^g_{(S,T)} = l$. Then there are $l$ ordered pairs $(g_i, h_i) \in S \times T$ such that for all $i \in \{1, \ldots, l\}$, $g_i \cdot h_i = g$. Let us order the elements of $S$ as $\{g, g_1, \ldots, g_l, g_{l+1}, \ldots, g_{m-1}\}$. Thus for all $i \in \{l+1, \ldots, m-1\}$, we have $g_i \cdot g_j = g$ for some $g_j \in S$, $g_j \neq g$. That is $N^g_{(S,S)} = m-1-l$. Again if we order the elements of $T$ as $\{h_1, \ldots, h_l, h_{l+1}, \ldots, h_{n-m-1}\}$, then for all $i \in \{l+1, \ldots, n-m-1\}$, we have $h_j \cdot h_i = g$ for some $h_j \in T$. Thus $N^g_{(T,T)} = n - m - 1 - l$. Since $N^g_{(T,T)} + N^g_{(T,S)} = |T| = n - m - 1$, we have $N^g_{(T,S)} = l$. Similarly we can prove that for all $h \in T$, $N^h_{(S,T)} = N^h_{(T,S)}$. □

Following is a necessary and sufficient condition for a set $S$ in $G$ to be a signature set in $G$ for an $(n, k)$-equiangular tight frame.

**Theorem 5.7.** *Let $G$ be a finite group of order $n$ and $S, T \subset G \setminus \{e\}$ where $T = S^c \setminus \{e\}$ such that $G \setminus \{e\} = S \cup T$. Then there exists a $k$ such that $S$ is a signature set in $G$ for an $(n, k)$-equiangular tight frame if and only if the following hold:*

*(a)* *$g \in S$ implies $g^{-1} \in S$ and $h \in T$ implies $h^{-1} \in T$;*

*(b)* *there exists a real number $\mu$ such that for all $g \in S$;*

$$N^g_{(S,S)} - 2N^g_{(S,T)} + N^g_{(T,T)} = \mu; \tag{5.1}$$

*and for all $h \in T$,*

$$N^h_{(S,S)} - 2N^h_{(S,T)} + N^h_{(T,T)} = -\mu. \tag{5.2}$$

*In this case $k$ and $\mu$ are related by the Equations in* (2.4).

*Proof.* Form $Q = \sum_{g \in S} \lambda(g) - \sum_{h \in T} \lambda(h)$. Then by Definition 5.1, $S$ is a signature set in $G$ for an $(n,k)$-equiangular tight frame if and only if $Q$ is a signature matrix for an $(n,k)$-equiangular tight frame. From Theorem 2.8 we know that an $n \times n$ matrix $Q$ with $Q_{ii} = 0$ and for all $i \neq j$, $|Q_{ij}| = 1$, is a signature matrix for an $(n,k)$ equiangular tight frame if and only if it satisfies the following two conditions:

(a) $Q$ is self adjoint that is $Q = Q^*$; and

(b) $Q^2 = (n-1)I + \mu Q$ for some real number $\mu$.

The condition $Q = Q^*$ is equivalent to

$$\begin{aligned}\sum_{g \in S} \lambda(g) - \sum_{h \in T} \lambda(h) &= (\sum_{g \in S} \lambda(g) - \sum_{h \in T} \lambda(h))^* \\ &= \sum_{g \in S} \lambda(g^{-1}) - \sum_{h \in T} \lambda(h^{-1}).\end{aligned}$$

Thus $g \in S$ implies $g^{-1} \in S$ and $h \in T$ implies $h^{-1} \in T$. The second condition $Q^2 = (n-1)I + \mu Q$, for some real number $\mu$, is equivalent to

$$\begin{aligned}&\sum_{\substack{g_1, g_2 \in S \\ g_1 \neq g_2}} \lambda(g_1 \cdot g_2) - \sum_{\substack{g_1 \in S \\ h_1 \in T}} \lambda(g_1 \cdot h_1) - \sum_{\substack{g_1 \in S \\ h_1 \in T}} \lambda(h_1 \cdot g_1) + \sum_{\substack{h_1, h_2 \in T \\ h_1 \neq h_2}} \lambda(h_1 \cdot h_2) \\ &= (n-1)I + \mu(\sum_{g \in S} \lambda(g) - \sum_{h \in T} \lambda(h)).\end{aligned}$$

By counting arguments, we have $Q^2 = (n-1)I + \mu Q$, for some real number $\mu$, if and only if for all $g \in S$,

$$N^g_{(S,S)} - N^g_{(S,T)} - N^g_{(T,S)} + N^g_{(T,T)} = \mu$$

and for all $h \in T$, we have

$$N^h_{(S,S)} - N^h_{(S,T)} - N^h_{(T,S)} + N^h_{(T,T)} = -\mu.$$

Using Lemma 5.6, we have $Q^2 = (n-1)I + \mu Q$, for some real number $\mu$, if and only if for all $g \in S$,

$$N^g_{(S,S)} - 2N^g_{(S,T)} + N^g_{(T,T)} = \mu$$

and for all $h \in T$, we have

$$N^h_{(S,S)} - 2N^h_{(S,T)} + N^h_{(T,T)} = -\mu.$$

$\square$

**Remark 5.8.** *From the relations given in the Equation* (2.4)*, since $k$ is a function of $\mu$, we shall often use the parameter $\mu$ to specify our frames and denote them as $(n, k(\mu))$-equiangular tight frames.*

Using some counting arguments, we can further simplify conditions (5.1) and (5.2) of Theorem 5.7 given in the following result.

**Theorem 5.9.** *Let $G$ be a group with $|G| = n$. Let $S, T \subset G \setminus \{e\}$ where $T = S^c \setminus \{e\}$ such that $G \setminus \{e\} = S \cup T$. Also let $S = S^{-1}$ and $T = T^{-1}$. Then there exists a $k$ such that $S$ is a signature set in $G$ for an $(n, k)$-equiangular tight frame if and only if for all $g \in S$*

$$N^g_{(S,T)} = \frac{n-2-\mu}{4} \tag{5.3}$$

*and for all $h \in T$,*

$$N^h_{(S,T)} = \frac{n-2+\mu}{4} \tag{5.4}$$

*where $\mu$ and $k$ are related by* (2.4).

*Proof.* Assume $|S| = l$. Since $|G| = n$, we have $|T| = n - 1 - l$. For $g \in S$, let $N^g_{(S,S)} = m$, then since $|S| = N^g_{(S,S)} + N^g_{(S,T)} + 1$, we have $N^g_{(S,T)} = l - 1 - m$. Also by Lemma 5.6, we have $N^g_{(S,T)} = N^g_{(T,S)}$ and using $|T| = N^g_{(T,T)} + N^g_{(T,S)}$, we have

$$N^g_{(T,T)} = n - 1 - l - (l - 1 - m) = n - 2l + m.$$

By Theorem 5.7, $S$ is a signature set in $G$ for an $(n, k(\mu))$ equiangular tight frame if and only if Equations (5.1) and (5.2) hold. That is for all $g \in S$,

$$N^g_{(S,S)} - 2N^g_{(S,T)} + N^g_{(T,T)} = \mu$$

and for all $h \in T$,

$$N^h_{(S,S)} - 2N^h_{(S,T)} + N^h_{(T,T)} = -\mu.$$

$N^g_{(S,S)} - 2N^g_{(S,T)} + N^g_{(T,T)} = \mu$ is equivalent to

$$\begin{aligned} \mu &= m - 2(l - 1 - m) + n - 2l + m \\ &= 4m - 4l + n + 2. \end{aligned}$$

Thus we have,

$$l - m = \frac{n + 2 - \mu}{4}.$$

Since $N^g_{(S,T)} = l - m - 1$, Equation (5.1) holds if and only if for all $g \in S$,

$$N^g_{(S,T)} = \frac{n - 2 - \mu}{4}.$$

Similarly, for $h \in T$, if $N^h_{(S,S)} = \tilde{m}$, then using $|S| = N^h_{(S,S)} + N^h_{(S,T)}$, we have

$$N^h_{(S,T)} = l - \tilde{m}.$$

Also since $|T| = N^h_{(S,T)} + N^h_{(T,T)} + 1$, we have

$$N^h_{(T,T)} = n - 2 - l - (l - \tilde{m}) = n - 2 - 2l + \tilde{m}.$$

The condition $N^h_{(S,S)} - 2N^h_{(S,T)} + N^h_{(T,T)} = -\mu$ is equivalent to

$$\begin{aligned} -\mu &= \tilde{m} - 2(l - \tilde{m}) + n - 2 - 2l + \tilde{m} \\ &= n - 4l + 4\tilde{m} - 2. \end{aligned}$$

Thus we have,

$$l - \tilde{m} = \frac{n - 2 + \mu}{4}.$$

Since $N^h_{(S,T)} = l - \tilde{m}$, Equation (5.2) holds if and only if for all $h \in T$ we have,

$$N^h_{(S,T)} = \frac{n - 2 + \mu}{4}.$$

□

**Corollary 5.10.** *Let $G$ be a group of order $n$. If there exists a signature set $S$ in $G$ corresponding to an $(n, k(\mu))$-equiangular tight frame, then the following hold:*

*(a)* $n \equiv 0 \pmod 2$*;*

*(b)* $\mu \equiv 0 \pmod 2$*;*

*(c)* $n$*,* $\mu$ *satisfies* $-(n-2) \le \mu \le (n-2)$*.*

*Proof.* From Theorem 5.9, if $S$ is a signature set in $G$ for an $(n, k(\mu))$-equiangular tight frame, then Equations (5.3) and (5.4) hold. If we sum the Equations (5.3) and (5.4), we have $n = 0 \pmod 2$ and subtracting (5.3) from (5.4) gives us $\mu \equiv 0 \pmod 2$.

Since $N^g_{(S,T)}, N^h_{(S,T)} \geq 0$, again using (5.3) and (5.4), we have

$$-(n-2) \leq \mu \leq (n-2).$$

□

## 5.2 Equiangular tight frames and signature sets in groups

Using (2.4) and the relations that we have proved in Corollary 5.10, we now classify some of the $(n, k)$-equiangular tight frames arising from signature sets in groups by looking at specific values of $n$ and $\mu$.

**Proposition 5.11.** *Let $G$ be a group and $S \subset G$ be a signature set in $G$ for an $(n, k(\mu))$-equiangular tight frame, then the following hold.*

*(a) If $\mu = 0$, then $n = 2m$ where $m \in \mathbb{N}$ is an odd number and $S$ is a signature set in $G$ for a $(2m, m)$-equiangular tight frame.*

*(b) If $\mu = 2$, then $n = 4a^2$ where $a \in \mathbb{N}$ and $S$ is a signature set in $G$ for $(4a^2, 2a^2 - a)$-equiangular tight frame.*

*(c) If $\mu = -2$, then $n = 4a^2$ where $a \in \mathbb{N}$ and $S$ is a signature set in $G$ for $(4a^2, 2a^2 + a)$-equiangular tight frame.*

(d) *If $n = 2p$ where $p$ is an odd prime, then either $\mu = 0$ and $S$ is a signature set in $G$ for a $(2p, p)$-equiangular tight frame or $\mu = n-2$ and $S$ is a signature set in $G$ for a $(2p, 1)$-equiangular tight frame.*

(e) *If $n = 4p$ where $p$ is an odd prime, then $\mu = n-2$ and $S$ is a signature set in $G$ for a $(4p, 1)$-equiangular tight frame.*

*Proof.* If $\mu = 0$, then from Theorem 5.9, for all $g \in S$ and for all $h \in T$, we have $N^g_{(S,T)} = N^h_{(S,T)} = \frac{n-2}{4}$. Thus $n \equiv 2 \pmod 4$ that is $n = 4l+2$ for $l \in \mathbb{N}$ or equivalently $n = 2m$ where $m \in \mathbb{N}$ is an odd number. Using Equation (2.4), we have $k = m$ and thus $S$ is a signature set in $G$ for $(2m, m)$-equiangular tight frame.

To prove the remaining parts, using Corollary 5.10, we can assume that $n = 2n_1$ and $\mu = 2\mu_1$ where $n_1 \in \mathbb{N}$ and $\mu_1 \in \mathbb{Z}$. Using Equation (2.4), we get

$$k = \frac{2n_1}{2} - \frac{2\mu_1 \cdot 2n_1}{2\sqrt{4(2n_1-1)+4\mu_1^2}} = n_1 - \frac{n_1\mu_1}{\sqrt{(2n_1-1)+\mu_1^2}}. \tag{5.5}$$

Thus $n_1^2\mu_1^2 \equiv 0 \pmod{(2n_1 - 1 + \mu_1^2)}$. If $\mu_1 = \pm 1$, then we have $\mu = \pm 2$. Using Equation (5.5), $n_1$ must be of the form $n_1 = 2a^2$ where $a \in \mathbb{N}$ that is and $n = 4a^2$ where $a \in \mathbb{N}$. Using Equation (2.4) for $\mu = 2$, we get $k = 2a^2 - a$ and $S$ is a signature set in $G$ for $(4a^2, 2a^2 - a)$-equiangular tight frame. Again using (2.4) for $\mu = -2$, we get $k = 2a^2 + a$ and $S$ is a signature set in $G$ for $(4a^2, 2a^2 + a)$-equiangular tight frame.

If $n = 2p$ that is $n_1 = p$, then $n_1^2\mu_1^2 \equiv 0 \pmod{2n_1 - 1 + \mu_1^2}$ implies that $\mu_1^2 - 1 \equiv 0 \pmod p$. If $\mu_1 = 0$, then $\mu = 0$ and by part (a), $S$ is a signature set in $G$ for a $(2p, p)$-equiangular tight frame. If $\mu_1 \neq 0$, then $\mu_1^2 - 1 \equiv 0 \pmod p$ implies that either $\mu_1 - 1 \equiv 0 \pmod p$ or $\mu_1 + 1 \equiv 0 \pmod p$. But from Corollary 5.10, part (c), we have $-(p-1) \leq \mu_1 \leq (p-1)$. Thus we have $\mu_1 = p-1$ that is $\mu = 2p-2 = n-2$. From Equation (2.4), we have $k = 1$. Thus $S$ is a signature set in $G$ for a $(2p, 1)$-equiangular tight frame.

Similarly if $n = 4p$ that is $n_1 = 2p$, then $n_1^2\mu_1^2 \equiv 0 \pmod{2n_1 - 1 + \mu_1^2}$ implies that $\mu_1^2 - 1 \equiv 0 \pmod{4p}$. By part (a) again, $\mu \neq 0$. Thus $\mu_1^2 - 1 \equiv 0 \pmod{4p}$ implies that $\mu_1 - 1 \equiv 0 \pmod 2$ and $\mu_1 + 1 \equiv 0 \pmod 2$. Let $\mu_1 + 1 = 2a$ for some $a \in \mathbb{N}$. Then $\mu_1 - 1 = 2a - 2$ and $\mu_1^2 - 1 \equiv 0 \pmod{4p}$ implies $a(a-1) \equiv 0 \pmod p$. Thus either $a \equiv 0 \pmod p$ or $a - 1 \equiv 0 \pmod p$. Again from Corollary 5.10, part (c), we have $-(2p-1) \leq \mu_1 \leq (2p-1)$ that is $2 - 2p \leq 2a \leq 2p$ or equivalently we have $1 - p \leq a \leq p$. Thus $a = p$ and $\mu_1 = 2p - 1$ that is $\mu = 2\mu_1 = 4p - 2 = n - 2$. Again we get $k = 1$ and thus $S$ is a signature set in $G$ for a $(4p, 1)$-equiangular tight frame.

$\square$

Our goal now is to look for the signature sets in $G$ in a group $G$. The first subsets we look for in the groups are the subgroups. The following result characterizes the frames we get when we take $S$ to be a subgroup of $G$.

**Theorem 5.12.** *Let $G$ be a group of order $n$ and $H$ a proper subgroup of $G$. Then $H \setminus \{e\}$ is a signature set in $G$ of an $(n, k(\mu))$ equiangular tight frame if and only if $H$ is a subgroup of index $2$.*
*In this case $\mu = n - 2$ and thus $k = 1$.*

*Proof.* Let $H$ be a subgroup of $G$. Let $S = H \setminus \{e\}$ and $T = H^c$. Then for all $g \in S$, we have $N^g_{(S,T)} = 0$ and for all $h \in T$, we have $N^h_{(S,T)} = |S|$. From Equations (5.3) and (5.4), $S$ is a signature set in $G$ for an $(n, k(\mu))$ equiangular tight frame if and only if

$$0 = \frac{n-2-\mu}{4}$$

and

$$|S| = \frac{n-2+\mu}{4}$$

which gives us $\mu = n-2$ and $2|S| = n-2$.

Since $|H| = |S| + 1$, we have

$$|H| = \frac{n-2}{2} + 1 = \frac{n}{2}.$$

Thus, $H$ is a subgroup of index 2.

When $\mu = n-2$, using Equations (2.4), we get $k = 1$. Hence we get $(n, 1)$-equiangular tight frame. $\square$

**Remark 5.13.** *From Remark 5.5, $G \setminus \{e\}$ is a signature set in $G$ for the trivial $(n, 1)$-equiangular tight frame. By Theorem 5.12 we have shown one more way to get the trivial $(n, 1)$-equiangular tight frame by taking subgroup of index 2 in the group $G$ as the signature set in $G$.*

**Remark 5.14.** *By Remark 5.5 and Proposition 5.2, the following subsets $S$ of $G$ are signature sets in $G$ for the $(n, n-1)$-equiangular tight frame:*

*(a) $S = \emptyset$ (by Remark 5.5);*

*(b) $S = aH$ where $H$ is a subgroup of index 2 in $G$ and $a \notin H$ (by Theorem 5.12).*

So far we have seen the case of trivial equiangular tight frames only. Following propositions gives us some of the non-trivial equiangular tight frames arising from signature sets in groups of the form $C_n \times C_n$ where $C_n$ is the cyclic group of order $n$.

**Proposition 5.15.** *Let $G \cong C_n \times C_n = \langle a, b : a^n = e, b^n = e, ab = ba \rangle$ and let $S = \{a, a^2, \ldots, a^{n-1}, b, b^2, \ldots, b^{n-1}\}$. Then $S$ is a signature set in $G$ for an $(n^2, k)$-equiangular tight frame if and only if either $n = 2$ and $k = 3$ or $n = 4$ and $k = 6$.*

*Proof.* $|S| = 2(n-1)$ and $|T| = n^2 - 1 - 2(n-1) = n^2 - 2n + 1$. For all $g \in S$, we have $N^g_{(S,S)} = n - 2$. Thus for all $g \in S$,

$$N^g_{(S,T)} = 2(n-1) - 1 - (n-2) = n - 1.$$

Similarly for all $h \in T$, we have $N^h_{(S,S)} = 2$. Thus for all $h \in T$, we have

$$N^h_{(S,T)} = 2(n-1) - 2 = 2n - 4.$$

Using Equations (5.3) and (5.4), $S$ is a signature set in $G$ for an $(n^2, k)$ equiangular tight frame if and only if

$$n - 1 = \frac{n^2 - 2 - \mu}{4},$$

and

$$2n - 4 = \frac{n^2 - 2 + \mu}{4}.$$

This implies that $\mu = n^2 - 4n + 2$ and $\mu = -n^2 + 8n - 14$. Solving for $n$ we get $n = 4$ or $n = 2$. For $n = 4$, we have $\mu = 2$ and for $n = 2$, we have $\mu = -2$. Thus, the equiangular tight frames that we get are $(16, 6)$ and $(4, 3)$ equiangular tight frames. $\square$

Following array demonstrates the construction of signature matrix corresponding to $(16, 6)$ equiangular tight frame.

| $+$ | $e$ | $a^3$ | $a^2$ | $a$ | $b^3$ | $a^3b^3$ | $a^2b^3$ | $ab^3$ | $b^2$ | $a^3b^2$ | $a^2b^2$ | $ab^2$ | $b$ | $a^3b$ | $a^2b$ | $ab$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $e$ | 0 | 1 | 1 | 1 | 1 | −1 | −1 | −1 | 1 | −1 | −1 | −1 | 1 | −1 | −1 | −1 |
| $a$ | 1 | 0 | 1 | 1 | −1 | 1 | −1 | −1 | −1 | 1 | −1 | −1 | −1 | 1 | −1 | −1 |
| $a^2$ | 1 | 1 | 0 | 1 | −1 | −1 | 1 | −1 | −1 | −1 | 1 | −1 | −1 | −1 | 1 | −1 |
| $a^3$ | 1 | 1 | 1 | 0 | −1 | −1 | −1 | 1 | −1 | −1 | −1 | 1 | −1 | −1 | −1 | 1 |
| $b$ | 1 | −1 | −1 | −1 | 0 | 1 | 1 | 1 | 1 | −1 | −1 | −1 | 1 | −1 | −1 | −1 |
| $ab$ | −1 | 1 | −1 | −1 | 1 | 0 | 1 | 1 | −1 | 1 | −1 | −1 | −1 | 1 | −1 | −1 |
| $a^2b$ | −1 | −1 | 1 | −1 | 1 | 1 | 0 | 1 | −1 | −1 | 1 | −1 | −1 | −1 | 1 | −1 |
| $a^3b$ | −1 | −1 | −1 | 1 | 1 | 1 | 1 | 0 | −1 | −1 | −1 | 1 | −1 | −1 | −1 | 1 |
| $b^2$ | 1 | −1 | −1 | −1 | 1 | −1 | −1 | −1 | 0 | 1 | 1 | 1 | 1 | −1 | −1 | −1 |
| $ab^2$ | −1 | 1 | −1 | −1 | −1 | 1 | −1 | −1 | 1 | 0 | 1 | 1 | −1 | 1 | −1 | −1 |
| $a^2b^2$ | −1 | −1 | 1 | −1 | −1 | −1 | 1 | −1 | 1 | 1 | 0 | 1 | −1 | −1 | 1 | −1 |
| $a^3b^2$ | −1 | −1 | −1 | 1 | −1 | −1 | −1 | 1 | 1 | 1 | 1 | 0 | −1 | −1 | −1 | 1 |
| $b^3$ | 1 | −1 | −1 | −1 | 1 | −1 | −1 | −1 | 1 | −1 | −1 | −1 | 0 | 1 | 1 | 1 |
| $ab^3$ | −1 | 1 | −1 | −1 | −1 | 1 | −1 | −1 | −1 | 1 | −1 | −1 | 1 | 0 | 1 | 1 |
| $a^2b^3$ | −1 | −1 | 1 | −1 | −1 | −1 | 1 | −1 | −1 | −1 | 1 | −1 | 1 | 1 | 0 | 1 |
| $a^3b^3$ | −1 | −1 | −1 | 1 | −1 | −1 | −1 | 1 | −1 | −1 | −1 | 1 | 1 | 1 | 1 | 0 |

**Proposition 5.16.** *Let $G \cong C_n \times C_n = \langle a, b : a^n = e, b^n = e, ab = ba \rangle$ and let $S = \{a, \ldots, a^{n-1}, b, \ldots, b^{n-1}, ab, \ldots, a^{n-1}b^{n-1}\}$. Then $S$ is a signature set in $G$ for an $(n^2, k)$-equiangular tight frame if and only if either $n = 4$ and $k = 10$ or $n = 6$ and $k = 15$.*

*Proof.* $|S| = 3(n-1)$ and $|T| = n^2 - 1 - 3(n-1) = n^2 - 3n + 2$. For all $g \in S$, we have $N^g_{(S,S)} = n$. Thus for all $g \in S$,

$$N^g_{(S,T)} = 3(n-1) - 1 - n = 2n - 4.$$

Similarly for all $h \in T$, we have $N^h_{(S,S)} = 6$. Thus for all $h \in T$,

$$N^h_{(S,T)} = 3(n-1) - 6 = 3n - 9.$$

By Theorem 5.9 and Equations (5.4), $S$ is a signature set in $G$ of an $(n^2, k)$ equiangular tight frame if and only if

$$2n - 4 = \frac{n^2 - 2 - \mu}{4},$$

and

$$3n - 9 = \frac{n^2 - 2 + \mu}{4}.$$

Solving for $n$ we get $n^2 - 8n + 14 = -n^2 + 12n - 34$ which gives us $n^2 - 10n + 24 = 0$. Thus either $n = 4$ or $n = 6$. For $n = 6$, we have $\mu = 2$ and for $n = 4$, we have $\mu = -2$. Thus the frames that we get are $(36, 15)$ and $(16, 10)$ equiangular tight frames. $\square$

**Definition 5.17.** *[23]A real $n \times n$ matrix $H$ is called a* **Hadamard matrix** *provided that $h_{i,j} = \pm 1$ and $H^*H = nI$.*

**Remark 5.18.** *Let us denote the signature matrices obtained from the Propositions 5.15 and 5.16 by $Q_1$ and $Q_2$ respectively. Then from Example 3.8 in [41], we infer that the matrix $I - Q_i$ is a Hadamard matrix for $i = 1, 2$.*

# Chapter 6

# Signature Sets and Difference Sets

In [44], a relation between difference sets and complex equiangular cyclic frames was shown. In this section we will present a relation between the two type of subsets: signature sets and difference sets in groups.

## 6.1 Difference sets

**Definition 6.1.** *Let $G$ be an additively written group of order $n$. A subset $D$ of $G$ with $|D| = k$ is a $(n, k, \lambda)$-difference set of $G$ if for for some fixed number $\lambda$, every non zero element of $G$ can be written as a difference of two elements of $D$ in exactly $\lambda$ ways.*

**Example 6.2.** *The set $\{1, 3, 4, 5, 9\}$ is a $(11, 5, 2)$-difference set in $\mathbb{Z}_{11}$.*

**Example 6.3.** *Consider the signature set in $G$ that we are getting in Proposition 5.15. For $n = 4$ we have $G = \mathbb{Z}_4 \times \mathbb{Z}_4$ and*

$$S = \{(1,0),(0,1),(2,0),(0,2),(3,0),(0,3)\}$$

*Then $S$ is also a $(16,6,2)$ difference set with $\lambda = 2$.*

**Example 6.4.** *Consider the signature set in $G$ that we are getting in Proposition 5.16. For $n = 6$ we have $G = \mathbb{Z}_6 \times \mathbb{Z}_6$ and*

$$S = \{(1,0),(2,0),\ldots,(5,0),(0,1),(0,2),\ldots,(0,5),(1,1),(2,2),\ldots,(5,5)\}$$

*Then $S$ is also a $(36,15,3)$ difference set.*

**Proposition 6.5.** *If $D$ is a $(n,k,\lambda)$ difference set in $G$, then the following hold:*

*(a) [22, Remark 18.7 (2)] $\lambda = \dfrac{k(k-1)}{n-1}$;*

*(b) [22, Remark 18.4 (2)] $D^c$ is a $(n, n-k, \tilde{\lambda})$ difference set where*

$$\tilde{\lambda} = \frac{(n-k)(n-k-1)}{n-1}.$$

*Proof.* Since $|D| = k$, the number of ordered pairs $(x,y) \in D \times D$ such that $x \neq y$ is equal to $k(k-1)$. On the other hand, $D$ has $n-1$ non-zero elements, and for each non-zero element $a \in G$, there are $\lambda$ ordered pairs $(x,y) \in D \times D$ such that $a = x - y$. Hence, $k(k-1) = \lambda(n-1)$.

Since every non-zero element in $G$ can be written as a difference of two elements of $D$ in exactly $\lambda$ ways, it follows that every non-zero element in $G$ can be written as a difference of an element of $D$ and $D^c$ in exactly $k - \lambda$ ways. Thus every non-zero element in $G$ can be written as a difference of elements of $D^c$ in exactly $\tilde{\lambda} = n - k - (k - \lambda)$ ways.

Using $\lambda = \dfrac{k(k-1)}{n-1}$, we get

$$\tilde{\lambda} = n - k - (k - \lambda) = \frac{(n-k)(n-k-1)}{n-1}.$$

Thus, $D^c$ is a $(n, n-k, \tilde{\lambda})$ difference set where $\tilde{\lambda} = \dfrac{(n-k)(n-k-1)}{n-1}$. □

**Definition 6.6.** *[25]A difference set $D$ in a group $G$ is called* **reversible** *if*

$$-D = \{-d : d \in D\} = D.$$

**Remark 6.7.** *Let $D$ be a reversible $(n, k, \lambda)$ difference set in an additive group $G$. Then for any $g \in G$,*

$$N^g_{D,D} = \#\{(g_1, g_2) \in D \times D : g_1 + g_2 = g\} = \#\{(g_1, g_2) \in D \times D : g_1 - g_2 = g\}.$$

*In addition, for all $g \in G$, we have*

*(a)* $N^g_{(D,D)} = \lambda$;

*(b)* $N^g_{(D^c,D^c)} = \tilde{\lambda}$ *where $\tilde{\lambda}$ is as in Proposition 6.5, part (b);*

*(c)* $N^g_{(D,D^c)} = \frac{k(n-k)}{n-1}$, *using the equation*

$$N^g_{(D,D)} + 2N^g_{(D,D^c)} + N^g_{(D^c,D^c)} = n.$$

## 6.2 Difference sets and signature sets

**Lemma 6.8.** *Let $D$ be a $(n,k,\lambda)$ reversible difference set in a group $G$ such that $0 \notin D$. Let $T = D^c \setminus \{0\}$, then for all $g \in D$ and for all $h \in T$, the following hold:*

*(a)* $N^g_{(D,T)} + 1 = N^h_{(D,T)}$;

*(b)* $N^g_{(T,T)} = N^g_{(D^c,D^c)} = \tilde{\lambda}$ *where $\tilde{\lambda}$ is as in Proposition 6.5, part (b);*

*(c)* $N^h_{(T,T)} + 2 = N^h_{(D^c,D^c)}$.

*Proof.* Since $D$ is a $(n,k,\lambda)$ difference set, by Proposition 6.5, part (b), $D^c$ is also a difference set. Since $0 \in D^c$, for every $g \in D$, we have $(g,0) \in \{(g_1,g_2) \in D \times D^c : g_1 - g_2 = g\}$. Thus, $N^g_{(D,T)} = N^g_{(D,D^c)} - 1$. But for any $h \in T$ we have, $N^h_{(D,T)} = N^h_{(D,D^c)}$. Using Remark 6.7, we have $N^g_{(D,T)} + 1 = N^h_{(D,T)}$.
For $g \in D$, if $g = h_1 + h_2$, $h_1, h_2 \in D^c$, then $h_1 \neq 0$ and $h_2 \neq 0$. Thus by Remark 6.7, $N^g_{(T,T)} = N^g_{(D^c,D^c)} = \tilde{\lambda}$.
If $h \in T$, then $(h,0),(0,h) \in N^h_{(D^c,D^c)} = \#\{(h_1,h_2) \in D^c \times D^c : h_1 - h_2 = h\}$. Since $0 \notin T$, we have $N^h_{(T,T)} + 2 = N^h_{(D^c,D^c)}$. □

The following result gives us a relation between the difference sets and the signature sets in $G$.

**Theorem 6.9.** *Let $G$ be a group of order $n$ and $D$ be a $(n,k,\lambda)$ difference set in $G$.*

*(a) If $0 \notin D$, then $D$ is a signature set in $G$ for an $(n,k)$-equiangular tight frame if and only if $D$ is reversible and* $k = \dfrac{n-\sqrt{n}}{2}$.

*(b) If $0 \in D$, then $D \setminus \{0\}$ is a signature set in $G$ for an $(n,k)$-equiangular tight frame if and only if $D$ is reversible and $k = \dfrac{n+\sqrt{n}}{2}$.*

*Proof.* Since $D$ is a $(n,k,\lambda)$ difference set in $G$, from Remark 6.7 and Proposition 6.5, we have for all $g \in G$, $N^g_{(D,D)} = \lambda$ and $N^g_{(D^c,D^c)} = \tilde{\lambda}$. For part (a), let us first assume that $D$ is a signature set in $G$ for an $(n,k)$-equiangular tight frame. Let $T = D^c \setminus \{0\}$. From Theorem 5.9, Equations (5.3) and (5.4) hold. That is we have for all $g \in D$

$$N^g_{(S,T)} = \frac{n-2-\mu}{4},$$

and for all $h \in T$,

$$N^h_{(S,T)} = \frac{n-2+\mu}{4}.$$

Also, $D$ signature set in $G$ for an $(n,k)$-equiangular tight frame implies that $D$ is reversible. Using Lemma 6.8, part (a), we have for all $g \in D$ and for all $h \in T$, $N^g_{(D,T)} + 1 = N^h_{(D,T)}$. Thus we have,

$$\frac{n-2-\mu}{4} + 1 = \frac{n-2+\mu}{4}.$$

Solving for $\mu$ we get $\mu = 2$ and using (2.4), we get $k = \dfrac{n-\sqrt{n}}{2}$.
Conversely, assume that $D$ is reversible and $k = \dfrac{n-\sqrt{n}}{2}$. We claim that $D$ is a signature set in $G$ for an $(n,k)$-equiangular tight frame. For $g \in S$, we have

$$\begin{aligned} |G| - 2 &= N^g_{(D,D)} + 2N^g_{(D,T)} + N^g_{(T,T)} \\ &= \lambda + 2N^g_{(D,T)} + \tilde{\lambda} \qquad\qquad \text{(by Lemma 6.8, part (b))}. \end{aligned}$$

Thus we get, $n - 2 = \lambda + 2N^g_{(D,T)} + \tilde{\lambda}$. Using Proposition 6.5 and $k = \dfrac{n-\sqrt{n}}{2}$, we get

$$2N^g_{(D,T)} = (n-2) - \frac{n-2\sqrt{n}}{4} - \frac{n+2\sqrt{n}}{4}$$

$$= \frac{n}{2} - 2.$$

Thus for all $g \in D$, $N^g_{(D,T)} = \frac{n}{4} - 1$ and from Lemma 6.8, we have $N^g_{(D,T)} + 1 = N^h_{(D,T)}$. Thus for all $h \in T$, we have $N^h_{(D,T)} = \frac{n}{4}$. By Theorem 5.9, $D$ is a signature set in $G$ for an $(n,k)$-equiangular tight frame.

For part (b), since $D^c$ is a $(n, n-k, \tilde{\lambda})$ difference set with $0 \notin D^c$, using the same argument as in part (a) for $D^c$, we have that $D^c$ is a signature set in $G$ for an $(n, n-k)$-equiangular tight frame if and only if $D^c$ is reversible and $n - k = \frac{n - \sqrt{n}}{2}$ that is $k = \frac{n + \sqrt{n}}{2}$. Using Proposition 5.2, $D \setminus \{0\}$ is a signature set in $G$ for $(n,k)$-equiangular tight frame if and only if $D$ is reversible and $k = \frac{n + \sqrt{n}}{2}$. $\square$

**Remark 6.10.** *Note that in Theorem 6.9 part (a), we are getting* $k = \frac{n - \sqrt{n}}{2}$. *Using* (2.4)*, the corresponding value of* $\mu$ *is* 2*. From Proposition 5.11, we know that when* $\mu = 2$*, then* $n = 4m^2$*,* $m$*-positive integer. Thus we have,* $k = \frac{n - \sqrt{n}}{2} = 2m^2 - m$ *and* $\lambda = \frac{k(k-1)}{n-1} = m^2 - m$.

**Definition 6.11.** *[22] A difference set* $D$ *with parameters* $(4m^2; 2m^2 - m; m^2 - m)$ *(*$m$ *a positive integer) is called a* **Hadamard difference set***.*

**Corollary 6.12.** *Let* $G$ *be a group of order* $n$ *and* $D$ *be a* $(n,k,\lambda)$ *difference set such that* $0 \notin D$*. Then* $D$ *is a signature set in* $G$ *for an* $(n,k)$*-equiangular tight frame if and only if* $D$ *is a reversible Hadamard difference set.*

From Corollary 6.12, we infer that the problem of finding signature sets in $G$ for $(n, \frac{n - \sqrt{n}}{2})$-equiangular tight frames is equivalent to finding reversible Hadamard difference sets. Dillon [27], gave an explicit construction of a reversible Hadamard difference set in $\mathbb{Z}_{2^{a+1}}^2$ for all $a \in \mathbb{N}$. Thus by Corollary 6.12, there exists $(n, \frac{n-\sqrt{n}}{2})$-equiangular tight frames for all $n$ of the type $2^{2b}$ where $b \in \mathbb{N}$ and $b \geq 2$. Following is an example of such a difference set in [22].

**Example 6.13.** *Let $G = \mathbb{Z}_8 \times \mathbb{Z}_8$. Then the set $D = A \cup A^{-1}$ where*

$$A = \{ab^4, ab^5, ab^6, ab^7, a^2b^2, a^2b^3, a^2b^6, a^2b^7, a^3b^2, a^3b^4, a^3b^5, a^3b^7, a^4b, a^4b^3\}$$

*is a $(64, 28, 12)$ reversible Hadamard difference set. Since $e \notin D$, from Corollary 6.12, $D$ is a signature set in $G$ for a $(64, 28)$-equiangular tight frame.*

**Remark 6.14.** *The sets studied in Proposition 5.15 and Proposition 5.16 are reversible Hadamard difference sets not containing the identity. This is another way to see that these sets are signature sets in $G$ for $(16, 6)$ and $(36, 15)$-equiangular tight frames respectively.*

The following proposition gives us a relation between reversible Hadamard difference sets and Hadamard matrices.

**Proposition 6.15.** *Let $D$ be a $(n, k, \lambda)$ reversible Hadamard difference set with $0 \notin D$ and let $Q = \sum_{g \in D} \lambda(g) - \sum_{g \in T} \lambda(g)$, where $T = D^c \setminus \{0\}$. Then the matrix $H = I - Q$ is a Hadamard matrix.*

*Proof.* Since $D$ is a $(n, k, \lambda)$ reversible Hadamard difference set with $0 \notin D$, by Corollary 6.12, $D$ is a signature set in $G$ for an $(n, k)$-equiangular tight frame. Thus

$Q = \sum_{g \in D} \lambda(g) - \sum_{g \in T} \lambda(g)$ is a signature matrix for the $(n, k)$-equiangular tight frame. Let $H = I - Q$. Then for all $i, j$, $h_{i,j} = \pm 1$. Consider

$$\begin{aligned} H^2 &= (I - Q)^2 \\ &= I - 2Q + Q^2 \\ &= I - 2Q + (n-1)I + 2Q && \text{(by Theorem 2.8)} \\ &= nI. \end{aligned}$$

Also $Q = Q^*$ implies $H = H^*$. Thus we have $HH^* = nI$ and hence $H = I - Q$ is a Hadamard matrix. $\square$

# Chapter 7

# Equiangular Tight Frames and Quasi-signature Sets in Groups

If $Q$ is a Seidel matrix, we say that $Q$ is in a standard form if its first row and column contains only 1's except on the diagonal, as shown below:

$$Q = \begin{bmatrix} 0 & 1 & \dots & \dots & 1 \\ 1 & 0 & * & \dots & * \\ \vdots & * & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 1 & * & \dots & \dots & 0 \end{bmatrix}$$

We say that it is trivial if it has a standard form which has all of its off-diagonal entries equal to 1 and nontrivial if at least one off-diagonal entry is not equal to 1.

**Definition 7.1.** *[52] Two Seidel matrices $Q$ and $Q^{'}$ are switching equivalent if they* *can be obtained from each other by conjugating with a diagonal unitary and a permutation matrix.*

One can verify by conjugation with an appropriate diagonal unitary that the equivalence class of any Seidel matrix contains a matrix of standard form. So we only need to examine when matrices of this form satisfies either of the conditions (b) or (c) of Theorem 2.8.

In the real case, the off-diagonal entries of $Q$ are in the set $\{-1, 1\}$ and in the complex case the off-diagonal entries of $Q$ are roots of unity as shown in [12] where off-diagonal entries are cube roots of unity.

## 7.1 Quasi-signature sets in groups

Let $G$ be a group of order $m$. Let $\lambda : G \longrightarrow GL(\mathbb{F}(G))$ be the left regular representation such that $\lambda(g)e_h = e_{gh}$. Then we know that $\sum_{g \in G} \lambda(g) = J$ where $J$ is the $m \times m$ matrix of all 1's. As in Chapter 5, we form signature matrices using the left regular representation of a group, but in the standard form.
Lets observe the following definition in analogy with the Definition 5.1 for constructing signature matrices in the standard form.

**Definition 7.2.** *Let $S \subset G \setminus \{e\}$ and $T = S^c \setminus \{e\}$ such that $G \setminus \{e\} = S \cup T$. Let*

$$Q = \sum_{g \in S} \lambda(g) - \sum_{h \in T} \lambda(h).$$

*Form*

$$\tilde{Q} = \left[\begin{array}{c|c} 0 & C^t \\ \hline C & Q \end{array}\right] \quad \textit{where} \quad C = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \in \mathbb{C}^m.$$

*Then $\tilde{Q}$ is an $(m+1) \times (m+1)$ matrix with $\tilde{Q}_{ii} = 0$ and $|\tilde{Q}_{ij}| = 1$ for all $i \neq j$. Let $n = m+1$, then we call $S$ a quasi-signature set in $G$ for an $(n,k)$ equiangular tight frame if $\tilde{Q}$ is a signature matrix for an $(n,k)$ equiangular tight frame.*

**Remark 7.3.** *If any of the two subsets $S$ or $T$ of the group $G$ is an empty set, say $T = \emptyset$, then $S = G \setminus \{e\}$ and*

$$Q = \sum_{g \in S} \lambda(g) = \sum_{g \in G \setminus \{e\}} \lambda(g) = J - I_m$$

*where $J$ is the $m \times m$ matrix of all 1's. Hence $\tilde{Q}_{ii} = 0$ and $\tilde{Q}_{ij} = 1$ for all $i, j = 1, \ldots, n$. Thus, $\tilde{Q}$ is a signature matrix for the trivial $(n,1)$-equiangular tight frame. Hence $S = G \setminus \{e\}$ is a quasi-signature set for the trivial $(n,1)$-equiangular tight frame.*

*From this point onwards, both $S$ and $T$ are taken as non empty subsets of $G$.*

Following is a necessary and sufficient condition for a set in a group $G$ to be a quasi-signature set in $G$ for an $(n,k)$-equiangular tight frame.

**Theorem 7.4.** *Let $G$ be a group of order $m$ and $S \subset G \setminus \{e\}$, $T = S^c \setminus \{e\}$ such that $G \setminus \{e\} = S \cup T$. Then there exists a $k$ such that $S$ is a quasi-signature set in $G$ for an $(n,k)$-equiangular tight frame, where $n = m+1$, if and only if the following hold:*

*(a)* $g \in S$ *implies* $g^{-1} \in S$ *and* $h \in T$ *implies* $h^{-1} \in T$*;*

*(b) for all* $g \in S$*,*

$$N^g_{(S,S)} - 2N^g_{(S,T)} + N^g_{(T,T)} = \mu - 1,$$

*and for all* $h \in T$*,*

$$N^h_{(S,S)} - 2N^h_{(S,T)} + N^h_{(T,T)} = -\mu - 1,$$

*where* $\mu = |S| - |T|$*.*

$k$ *is related to* $\mu$ *by the equations given in* (2.4)*.*

*Proof.* Form $Q = \sum_{g \in S} \lambda(g) - \sum_{h \in T} \lambda(h)$, and

$$\tilde{Q} = \begin{bmatrix} 0 & C^t \\ C & Q \end{bmatrix} \quad \text{where} \quad C = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \in \mathbb{C}^m.$$

Then by Definition 7.2, $S$ is a quasi-signature set in $G$ for an $(n,k)$-equiangular tight frame if and only if $\tilde{Q}$ forms a signature matrix for an $(n,k)$-equiangular tight frame. From Theorem 2.8, $\tilde{Q}$ will form a signature matrix for an $(n,k)$ equiangular tight frame if and only if it satisfies the following two conditions:

(a) $\tilde{Q}$ is self adjoint; that is, $\tilde{Q} = \tilde{Q}^*$, and

(b) $\tilde{Q}^2 = (n-1)I + \mu\tilde{Q}$ for some real number $\mu$.

The condition $\tilde{Q} = \tilde{Q}^*$ is equivalent to $Q = Q^*$ which is further equivalent to

$$\begin{aligned}\sum_{g\in S}\lambda(g) - \sum_{h\in T}\lambda(h) &= (\sum_{g\in S}\lambda(g) - \sum_{h\in T}\lambda(h))^* \\ &= \sum_{g\in S}\lambda(g^{-1}) - \sum_{h\in T}\lambda(h^{-1}).\end{aligned}$$

Thus, $g \in S$ implies $g^{-1} \in S$ and $h \in T$ implies $h^{-1} \in T$.

For the second condition, we need $\tilde{Q}^2 = (n-1)I + \mu\tilde{Q}$. We have

$$\tilde{Q}^2 = \begin{bmatrix} n-1 & \tilde{C}^t \\ \tilde{C} & J + Q^2 \end{bmatrix},$$

where $\tilde{C} = (|S| - |T|)C$, and $J$ is the $m \times m$ matrix of all 1's. Thus, $\tilde{Q}^2 = (n-1)I + \mu\tilde{Q}$ if and only if

(a) $|S| - |T| = \mu$, and

(b) $J + Q^2 = (n-1)I + \mu Q$ that is $Q^2 = (n-1)I + \mu Q - J$.

Since $J = \sum_{g\in G}\lambda(g)$, we have

$$\begin{aligned}Q^2 &= (n-1)I + \mu Q - J \\ &= (n-2)I + \mu(\sum_{g\in S}\lambda(g) - \sum_{h\in T}\lambda(h)) - \sum_{g\in G\setminus\{e\}}\lambda(g) \\ &= (n-2)I + (\mu-1)\sum_{g\in S}\lambda(g) - (\mu+1)\sum_{h\in T}\lambda(h).\end{aligned}$$

By the same counting arguments as used before in Theorem 5.7, we get

$$Q^2 = (n-2)I + (\mu-1)\sum_{g\in S}\lambda(g) - (\mu+1)\sum_{h\in T}\lambda(h)$$

if and only if for all $g \in S$,

$$N^g_{(S,S)} - 2N^g_{(S,T)} + N^g_{(T,T)} = \mu - 1,$$

and for all $h \in T$,

$$N^h_{(S,S)} - 2N^h_{(S,T)} + N^h_{(T,T)} = -\mu - 1,$$

where $\mu = |S| - |T|$. □

**Remark 7.5.** *From Theorem 7.4, note that if $S \subset G$ is a quasi-signature set in $G$ for an $(n, k(\mu))$-equiangular tight frame, then $|S| - |T| = \mu$.*

**Proposition 7.6.** *Let $G$ be a group of order $m$ and $S \subset G \setminus \{e\}$, $T = S^c \setminus \{e\}$. Then the condition $|S| - |T| = \mu$ for some integer $\mu$ is equivalent to*

$$|S| = \frac{n-2+\mu}{2} \quad \textit{and} \quad |T| = \frac{n-2-\mu}{2}$$

*where $n = m + 1$.*

*Proof.* Let $\mu \in \mathbb{Z}$. If $|S| = \frac{n-2+\mu}{2}$ and $|T| = \frac{n-2-\mu}{2}$, then $|S| - |T| = \mu$.
If $|S| - |T| = \mu$, then since $G \setminus \{e\} = S \cup T$, we have $|S| + |T| = m - 1 = n - 2$. Thus solving for $|S|$ and $|T|$, we get that

$$|S| = \frac{n-2+\mu}{2} \quad \text{and} \quad |T| = \frac{n-2-\mu}{2}$$

where $n = m + 1$. □

**Example 7.7.** *We know that there exists a $(6,3)$-equiangular tight frame. In this example we will show that this frame comes from a quasi-signature set in $(\mathbb{Z}_5,+)$. Let $G=(\mathbb{Z}_5,+)$. Since $\mu=0$, we must have $|S|=|T|=2$. If we take $S=\{1,4\}$ and $T=\{2,3\}$ and form $Q=\sum_{g\in S}\lambda(g)-\sum_{h\in T}\lambda(h)$, then $Q$ can be obtained from the following multiplication table:*

| + | *0* | *1* | *2* | *3* | *4* |
|---|---|---|---|---|---|
| *0* | *0* | *1* | *-1* | *-1* | *1* |
| *4* | *1* | *0* | *1* | *-1* | *-1* |
| *3* | *-1* | *1* | *0* | *1* | *-1* |
| *2* | *-1* | *-1* | *1* | *0* | *1* |
| *1* | *1* | *-1* | *-1* | *1* | *0* |

*That is,*

$$Q=\begin{bmatrix} 0 & 1 & -1 & -1 & 1 \\ 1 & 0 & 1 & -1 & -1 \\ -1 & 1 & 0 & 1 & -1 \\ -1 & -1 & 1 & 0 & 1 \\ 1 & -1 & -1 & 1 & 0 \end{bmatrix}.$$

*Thus we can form $\tilde{Q}$ as follows:*

$$\tilde{Q} = \begin{bmatrix} 0 & C^t \\ C & Q \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & -1 & -1 & 1 \\ 1 & 1 & 0 & 1 & -1 & -1 \\ 1 & -1 & 1 & 0 & 1 & -1 \\ 1 & -1 & -1 & 1 & 0 & 1 \\ 1 & 1 & -1 & -1 & 1 & 0 \end{bmatrix}$$

*It can be easily verified that $\tilde{Q} = \tilde{Q}^*$ and $\tilde{Q}^2 = 5I$. Thus, $S$ is a quasi-signature set in $G$ for $(6,3)$-equiangular tight frame.*

We simplify the conditions given in Theorem 7.4 as follows:

**Theorem 7.8.** *Let $G$ be a group with $|G| = m$ and $S, T \subset G \setminus \{e\}$ be disjoint such that $G \setminus \{e\} = S \cup T$. Also let $S = S^{-1}$ and $T = T^{-1}$. Then there exists a $k$ such that $S$ is a quasi-signature set in $G$ for an $(n,k)$-equiangular tight frame, where $n = m+1$, if and only if for all $g \in S$,*

$$N^g_{(S,S)} = \frac{n + 3\mu - 6}{4}, \tag{7.1}$$

*and for all $h \in T$,*

$$N^h_{(T,T)} = \frac{n - 3\mu - 6}{4}, \tag{7.2}$$

*where $\mu = |S| - |T|$. Here $k$ and $\mu$ are related by Equation (2.4).*

*Proof.* Let us assume that $|S| - |T| = \mu$, for some $\mu \in \mathbb{Z}$. Using Proposition 7.6, we have $|S| = \frac{n-2+\mu}{2}$. For $g \in S$, let $N^g_{(S,S)} = m_1$. Since $|S| - 1 = N^g_{(S,S)} + N^g_{(S,T)}$, we have $N^g_{(S,T)} = |S| - 1 - m_1$. Also, we have $N^g_{(S,T)} + N^g_{(T,T)} = |T| = |G| - 1 - |S|$. Thus,

$$\begin{aligned}
N^g_{(T,T)} &= |G| - 1 - |S| - N^g_{(S,T)} \\
&= n - 2 - |S| - (|S| - 1 - m_1) \\
&= n - 2|S| + m_1 - 1.
\end{aligned}$$

From Theorem 7.4, $S$ is a quasi-signature set in $G$ for an $(n,k)$-equiangular tight frame if and only if condition (7.4) holds.

Thus, $N^g_{(S,S)} - 2N^g_{(S,T)} + N^g_{(T,T)} = \mu - 1$ is equivalent to

$$\begin{aligned}
\mu - 1 &= m_1 - 2(|S| - 1 - m_1) + n - 2|S| + m_1 - 1 \\
&= 4m_1 - 4|S| + 1 + n
\end{aligned}$$

or equivalently, we have $n + 2 - \mu = 4(|S| - m_1)$.

Thus,

$$\begin{aligned}
m_1 &= |S| - \frac{n + 2 - \mu}{4} \\
&= \frac{n + 3\mu - 6}{4}.
\end{aligned}$$

Now if we let $h \in T$ and $N^h_{(T,T)} = m_2$, then as above $N^h_{(S,T)} = |T| - 1 - m_2$. Also, $N^h_{(S,T)} + N^h_{(S,S)} = |S| = |G| - 1 - |T|$. Thus

$$\begin{aligned}
N^h_{(S,S)} &= n - 2 - |T| - N^h_{(S,T)} \\
&= n - 2 - |T| - (|T| - 1 - m_2) \\
&= n - 2|T| + m_2 - 1.
\end{aligned}$$

Again using Theorem 7.4, condition (7.4), we have $-N^h_{(S,S)}+2N^h_{(S,T)}-N^h_{(T,T)}=\mu+1$ which is equivalent to

$$\begin{aligned}\mu+1 &= -(n-2|T|+m_2-1)+2(|T|-1-m_2)-m_2\\ &= -4m_2+4|T|-n-1.\end{aligned}$$

Hence, we have

$$\begin{aligned}m_2 &= |T|-\frac{\mu+n+2}{4}\\ &= \frac{n-3\mu-6}{4}.\end{aligned}$$

Thus, $S$ is a quasi-signature set in $G$ for an $(n,k)$-equiangular tight frame if and only if for all $g\in S$, we have

$$N^g_{(S,S)}=\frac{n+3\mu-6}{4},$$

and for all $h\in T$, we have

$$N^h_{(T,T)}=\frac{n-3\mu-6}{4}.$$

□

**Remark 7.9.** *Since* $|S|-1=N^g_{(S,S)}+N^g_{(S,T)}$*, we can find* $N^g_{(S,T)}$ *in terms of* $n$ *and* $\mu$*. Thus,*

$$\begin{aligned}N^g_{(S,T)} &= |S|-1-N^g_{(S,S)}\\ &= \frac{n-2-\mu}{4}.\end{aligned}$$

*Similarly for all* $h\in T$*, we have*

$$\begin{aligned}N^h_{(S,T)} &= |T|-1-N^h_{(T,T)}\\ &= \frac{n-2+\mu}{4}.\end{aligned}$$

**Corollary 7.10.** *Let $G$ be a group of order $m$. If there exists a quasi-signature set $S$ in $G$ for an $(n, k(\mu))$-equiangular tight frame, where $n = m + 1$, then*

*(a)* $\mu \equiv 0 \pmod 2$*;*

*(b)* $n \equiv 0 \pmod 2$*;*

*(c)* $n$, $\mu$ *satisfies*

$$2 - \frac{n}{3} \leq \mu \leq \frac{n}{3} - 2. \tag{7.3}$$

*Proof.* If there exists a quasi-signature set $S$ in $G$ for an $(n, k(\mu))$-equiangular tight frame, then we know from Theorem 7.8, conditions 7.1 and 7.2 hold. Adding 7.1 and 7.2, we get $n \equiv 0 \pmod 2$. Subtracting 7.1 from 7.2, we get $\mu \equiv 0 \pmod 2$.
Also $0 \leq N^g_{(S,S)} = \frac{3\mu+n-6}{4}$ implies that $2 - \frac{n}{3} \leq \mu$, and $0 \leq N^h_{(T,T)} = \frac{-3\mu+n-6}{4}$ implies that $\mu \leq \frac{n}{3} - 2$. Thus we have,

$$2 - \frac{n}{3} \leq \mu \leq \frac{n}{3} - 2.$$

□

## 7.2 Equiangular tight frames and quasi-signature sets in groups

Note that in the case of quasi-signature sets in $G$, we have a better bound on the value of $\mu$ as compared to the case of signature sets in $G$. Thus we have the following proposition that eliminates some of the cases in which $S \subset G$ can be a quasi-signature set in $G$ for an $(n, k(\mu))$-equiangular tight frame.

**Proposition 7.11.** *Let $G$ be a group of order $m$, and $n = m + 1$, then the following hold.*

*(a) If there exists a quasi-signature set in $G$ for an $(n, \frac{n}{2})$-equiangular tight frame, then $n = 2a$ where $a \in \mathbb{N}$ is an odd number.*

*(b) For an odd prime $p$, if there exists a quasi-signature set in $G$ for a $(2p, k)$-equiangular tight frame, then $k = p$.*

*(c) For an odd prime $p$, there does not exist a quasi-signature set in $G$ for a $(4p, k)$-equiangular tight frame for any value of $k$.*

*Proof.* If there exists a quasi-signature set $S$ in $G$ for an $(n, \frac{n}{2})$-equiangular tight frame, then using Equation (2.4), $\mu = 0$. From Theorem 7.8, we have for all $g \in S$ and for all $h \in T$,

$$N^g_{(S,S)} = N^h_{(T,T)} = \frac{n-6}{4}.$$

Thus, $n \equiv 6 \pmod 4$ or equivalently $n \equiv 2 \pmod 4$. Hence, $n$ is of the form $n = 2a$ where $a \in \mathbb{N}$ is odd.

For the second part assume that $S$ is a quasi-signature set in $G$ for a $(2p, k)$-equiangular tight frame where $p$ is an odd prime. Using Corollary 7.10, let $\mu = 2\mu_1$ for some $\mu_1 \in \mathbb{Z}$. Then from Equation (2.4), we have

$$k = \frac{2p}{2} - \frac{2\mu_1 \cdot 2p}{2\sqrt{4(2p-1) + 4\mu_1^2}} = p - \frac{p\mu_1}{\sqrt{(2p-1) + \mu_1^2}}. \tag{7.4}$$

Thus $p^2\mu_1^2 \equiv 0 \pmod{(2p-1+\mu_1^2)}$. If $\mu_1 = 0$, then from part (a), we have that $S$ forms a signature set in $G$ for a $(2p,p)$-equiangular tight frame. Clearly $\mu_1 \neq \pm 1$ because in that case $p \equiv 0 \pmod 2$ which contradicts that $p$ is a prime. If $\mu_1 \neq 0$, then $p^2\mu_1^2 \equiv 0 \pmod{(2p{-}1{+}\mu_1^2)}$ implies $\mu_1^2{-}1 \equiv 0 \pmod p$. Thus, either $\mu_1{-}1 \equiv 0 \pmod p$ or $\mu_1 + 1 \equiv 0 \pmod p$. But, this contradicts (7.3) in Corollary 7.10 as

$$2 - \frac{2p}{3} \leq \mu \leq \frac{2p}{3} - 2 \quad \text{implies} \quad 1 - \frac{p}{3} \leq \mu_1 \leq \frac{p}{3} - 1.$$

Thus, $S$ is a quasi-signature set in $G$ for a $(2p,p)$-equiangular tight frame.

Similarly, if $S$ is a quasi-signature set in $G$ for a $(4p,k)$-equiangular tight frame, where $p$ is a prime, then once again using Equation (2.4), we get

$$k = \frac{4p}{2} - \frac{2\mu_1 \cdot 4p}{2\sqrt{4(4p-1)+4\mu_1^2}} = 2p - \frac{2p\mu_1}{\sqrt{(4p-1)+\mu_1^2}}. \tag{7.5}$$

From part (a), $\mu \neq 0$. Using the same argument as discussed above, $\mu \neq \pm 1$. Thus, $4p^2\mu_1^2 \equiv 0 \pmod{(4p-1+\mu_1^2)}$ implies that $\mu_1^2 - 1 \equiv 0 \pmod{4p}$. Thus, both $\mu_1 - 1$ and $\mu_1 + 1$ are even integers. Let $\mu_1 + 1 = 2a$ for some $a \in \mathbb{Z}$, $a \neq 0$. Thus, $\mu_1^2 - 1 \equiv 0 \pmod{4p}$ implies that $a(a-1) \equiv 0 \pmod p$. But, this contradicts (7.3) in Corollary 7.10 as

$$2 - \frac{4p}{3} \leq 2(2a-1) \leq \frac{4p}{3} - 2 \quad \text{is equivalent to} \quad 1 - \frac{p}{3} \leq a \leq \frac{p}{3} - 2.$$

Hence, in the case of $n = 4p$, there does not exist a quasi-signature set $S$ in $G$ for an $(n,k)$-equiangular tight frame for any value of $k$. □

As in Chapter 5, we consider group $G$ of the form $G = C_N \times C_N$, the direct product of groups of order $N$. We will be constructing equiangular tight frames by taking subsets in $G$ as before. But, these subsets will act as quasi-signature sets in $G$. The following two propositions illustrate the type of equiangular tight frames we get when $G = C_N \times C_N$.

**Proposition 7.12.** *Let $G = C_N \times C_N = \langle a, b : a^N = e, b^N = e, ab = ba \rangle$ and let $S = \{a, a^2, \ldots, a^{N-1}, b, b^2, \ldots, b^{N-1}\}$. Then, $S$ is a quasi-signature set in $G$ for an $(n,k)$-equiangular tight frame where $n = N^2 + 1$ if and only if $N = 3$ and $k = 5$.*

*Proof.* Since $|S| = 2(N-1)$ and $|T| = (N-1)^2$ we have $\mu = |S| - |T| = -N^2 + 4N - 3$. For all $g \in S$, we have $N^g_{(S,S)} = N - 2$ and for all $h \in T$, we have $N^h_{(S,S)} = 2$. Thus,

$$N^h_{(S,T)} = 2N - 2 - 2 = 2N - 4,$$

and hence

$$\begin{aligned} N^h_{(T,T)} &= |T| - 1 - N^h_{(S,T)} \\ &= (N-1)^2 - 1 - 2N - 4 \\ &= (N-2)^2 - 1 - 2N + 4 \\ &= (N-2)^2. \end{aligned}$$

Using Theorem 7.8, we have that $S$ is a quasi-signature set in $G$ for an $(n, k(\mu))$-equiangular tight frame if and only if Equations 7.1 and 7.2 hold. Thus, we have

$$N - 2 = \frac{N^2 - 5 + 3\mu}{4}$$

which gives us

$$3\mu = 4N - N^2 - 3,$$

and

$$(N-2)^2 = \frac{N^2 - 5 - 3\mu}{4}$$

which gives us

$$3\mu = -3N^2 + 16N - 21.$$

Solving for $N$, we get $N = 3$ and hence we get $\mu = 0$. Thus we get a $(10, 5)$ equiangular tight frame. $\square$

**Proposition 7.13.** *Let $G = C_N \times C_N = \langle a, b : a^N = e, b^N = e, ab = ba \rangle$ and let $S = \{a, \ldots, a^{N-1}, b, \ldots, b^{N-1}, ab, \ldots, a^{n-1}b^{n-1}\}$, then $S$ is a quasi-signature set in $G$ for an $(n, k(\mu))$-equiangular tight frame where $n = N^2 + 1$ if and only if $N = 5$ and $k = 13$.*

*Proof.* Since $|S| = 3(N-1)$ and $|T| = (N-1)(N-2)$ we have $\mu = |S| - |T| = -N^2 + 6N - 5$. For all $g \in S$, we have $N^g_{(S,S)} = N$, and for all $h \in S$ we have $N^h_{(S,S)} = 6$. Thus,

$$N^h_{(S,T)} = 3(N-1) - 6 = 3N - 9,$$

and

$$\begin{aligned} N^h_{(T,T)} &= |T| - 1 - N^h_{(S,T)} \\ &= (N-1)^2 - 1 - 2N - 4 \\ &= (N-2)^2 - 1 - 2N + 4 \\ &= (N-2)^2. \end{aligned}$$

By Theorem 7.8, we have $S$ is a quasi-signature set in $G$ for an $(n, k(\mu))$-equiangular tight frame if and only if Equations 7.1 and 7.2 hold. Thus we have,

$$N = \frac{N^2 - 5 + 3\mu}{4}$$

which gives us

$$3\mu = -N^2 + 4N + 5,$$

and

$$N^2 - 6N + 10 = \frac{N^2 - 5 - 3\mu}{4}$$

which gives us

$$3\mu = -3N^2 + 24N - 45.$$

Solving for $N$, we get $-3N^2 + 24N - 45 = -N^2 + 4N + 5$ which implies $2N^2 - 20N + 50 = 0$; that is, $N = 5$ and $\mu = 0$. Thus we get a $(26, 13)$ equiangular tight frame. $\square$

**Definition 7.14.** *[23] A real $n \times n$ matrix $C$ with $c_{i,i} = 0$ and $c_{i,j} = \pm 1$ for $i \neq j$ is called a* **conference matrix** *provided $C^*C = (n-1)I$.*

Note that from the above two propositions, we are getting equiangular tight frames of the type $(n, \frac{n}{2})$. It has been shown in [41] that every symmetric conference matrix is a signature matrix with $\mu = 0$ and $k = \frac{n}{2}$. There are sufficient number of examples in the literature, see [11] and [20], of the equiangular tight frames of type $(p + 1, \frac{p+1}{2})$ where $p$ is a prime. In the next two results, we will characterize some of the equiangular tight frames of the type $(p + 1, \frac{p+1}{2})$, where $p$ is a prime that arise from a quasi-signature set in $G$ in the group $(\mathbb{Z}_p, +)$.

**Theorem 7.15.** *Let $G = (\mathbb{Z}_p, +)$ where $p$-prime. If $(\mathbb{Z}_p, \cdot) = \langle 2 \rangle$, then the subgroup $\langle 2^2 \rangle$ of $(\mathbb{Z}_p, \cdot)$ is a quasi-signature set in $G$ for a $(p+1, k)$-equiangular tight frame if and only if $p \equiv 5 \pmod 8$. In this case $k = \frac{p+1}{2}$.*

*Proof.* Let us denote $S = \langle 2^2 \rangle$. Using Fermat's Little theorem 2.34, we know that $2^{p-1} \equiv 1 \pmod p$. Thus, $S = \langle 2^2 \rangle = \{2^{2k} : k = 1, \dots, \frac{p-1}{2}\}$. Also, $G = \langle 2 \rangle$ implies that $2 \notin S$. Since $S$ is a subgroup of $(\mathbb{Z}, \cdot)$ of index 2 and $2 \notin S$, we have $(\mathbb{Z}, \cdot) = S \cup 2 \cdot S$. Thus $T = 2 \cdot S$ and we have $|S| = |T|$. Hence $\mu = |S| - |T| = 0$. We will prove this theorem in two steps. First we will show that $S = S^{-1}$ in $(\mathbb{Z}_p, +)$. Secondly we will verify the conditions of Theorem 7.8.

Let $\tilde{g} \in S$, then $\tilde{g}$ is of the form $2^{2m}$ for some $m \in \{1, \dots, \frac{p-1}{2}\}$. Since $2^{\frac{p-1}{2}} \equiv (p-1) \pmod p$, we have

$$\begin{aligned} 2^{2m} + 2^{\frac{p-1}{2}+2m} &= 2^{2m}(1 + 2^{\frac{p-1}{2}}) \\ &\equiv 0 \pmod p. \end{aligned}$$

Thus, $2^{\frac{p-1}{2}+2m} \pmod p$ is the inverse of $\tilde{g}$ in $(\mathbb{Z}_p, +)$. But $2^{\frac{p-1}{2}+2m} \in S$ if and only if $p-1 \equiv 0 \pmod 4$. Thus, $S$ is closed under inverses with respect to $(\mathbb{Z}_p, +)$ if and only if $(p-1) \equiv 0 \pmod 4$. Also note that if $\frac{p-1}{4} + m > \frac{p-1}{2}$ that is $2(\frac{p-1}{4} + m) > p-1$, then $2(\frac{p-1}{4} + m) = p - 1 + 2s$ where $s \in \{1, \dots, \frac{p-1}{2}\}$. By using Fermat's Little theorem 2.34 again, we have $2^{2(\frac{p-1}{4}+m)} \equiv 2^{2s} (\mathrm{mod}\ p)$.

For the second part assume that $N^{\tilde{g}}_{(S,S)} = N$ for some $N \in \mathbb{N}$. Then for any $g \in S$, $g$ is of the form $2^{2l}$ for some $l \in \{1, \dots, \frac{p-1}{2}\}$.

Let us denote

$$S \times_g S = \{(g_1, g_2) \in S \times S : g_1 + g_2 = g\}.$$

For $i, j \in \{1, \ldots, \frac{p-1}{2}\}$, since $2^{2i} + 2^{2j} \equiv 2^{2m} \pmod{p}$ if and only if $2^{2(l-m+i)} + 2^{2(l-m+j)} \equiv 2^{2l} (\mathrm{mod}\ p)$, the map $\phi : S \times_{2^{2m}} S \longrightarrow S \times_{2^{2l}} S$ such that

$$\phi((2^{2i}, 2^{2j})) = (2^{2(l-m+i)} (\mathrm{mod}\ p), 2^{2(l-m+j)} (\mathrm{mod}\ p))$$

is one to one and onto. Thus $|S \times_{2^{2m}} S| = |S \times_{2^{2l}} S|$. But $|S \times_{2^{2m}} S| = N^{2^{2m}}_{(S,S)}$. Thus for all $g \in S$, we have $N^g_{(S,S)} = N$.

Now if $2^{2m} \in S$ then $2^{2m} + 2^{2m} = 2 \cdot 2^{2m} = 2^{2m+1} \in T$. Thus for $g \in S$, if $(g_i, g_j) \in S \times_g S$, then $(g_j, g_i) \in S \times_g S$. Also for all $i$, $(g_i, g_i) \notin S \times_g S$. Hence $N^g_{(S,S)} = |S \times_g S| = N$ must be an even number and let $N = 2r$ for some $r \in \mathbb{N}$.

Similarly if $h \in T$, then $h = 2^{2\tilde{m}+1}$ for some $\tilde{m} \in \{1, \ldots, \frac{p-1}{2}\}$. For $i, j \in \{1, \ldots, \frac{p-1}{2}\}$, $2^{2i+1} + 2^{2j+1} \equiv 2^{2\tilde{m}+1} \pmod{p}$ if and only if $2^{2i} + 2^{2j} \equiv 2^{2\tilde{m}} \pmod{p}$. Thus the map that takes $S \times_{2^{2\tilde{m}}} S \to T \times_{2^{2\tilde{m}+1}} T$ such that

$$(2^{2i}, 2^{2j}) \longrightarrow (2^{2i+1} (\mathrm{mod}\ p), 2^{2j+1} (\mathrm{mod}\ p))$$

is one to one and onto. Thus, $N^{2^{2\tilde{m}+1}}_{(T,T)} = N^{2^{2\tilde{m}}}_{(S,S)}$. But we have $N^g_{(S,S)} = 2r$ for all $g \in S$. Thus for all $h \in T$, we have $N^h_{(T,T)} = 2r$. By Theorem 7.8, Conditions 7.1 and 7.2 hold if and only if we have $\frac{p+1-6}{4} = 2r$ that is $p = 8r + 5$ or equivalently $p \equiv 5 \pmod{8}$. But we know from the first part of the proof that $S = S^{-1}$ if and only if $p \equiv 1 \pmod{4}$. Since $p \equiv 5 (\mathrm{mod}\ 8)$ implies $p \equiv 1 (\mathrm{mod}\ 4)$, we have that $S$ is a quasi-signature set in $G$ for a $(p + 1, k)$-equiangular tight frame if and only if $p \equiv 5 (\mathrm{mod}\ 8)$. Also $\mu = 0$ implies that $k = \frac{p+1}{2}$. □

**Remark 7.16.** *Note that if $G = (\mathbb{Z}_p, +)$, then $|G| = p$. From Chapter 5, Corollary 5.10, we know that there cannot be any signature set in $G$. But if we look at*

*$G = (\mathbb{Z}_p, \cdot)$, then $|G| = p - 1$. Since the set $S$ in Theorem 7.15 is a subgroup of $(\mathbb{Z}_p, \cdot)$ of index 2, from Theorem 5.12 in Chapter 5, $S$ is a signature set in $G$ for the trivial $(p-1, 1)$-equiangular tight frame.*

**Example 7.17.** *Let $G = (\mathbb{Z}_{13}, +)$ where $(\mathbb{Z}_{13}, \cdot) = \langle 2 \rangle$. Then using Theorem 7.8, $S = \{2^2, 2^4, 2^6, 2^8, 2^{10}, 2^{12}\}$ is a quasi-signature set in $G$ for $(14, 7)$-equiangular tight frame. Thus we have, $S = \{4, 3, 12, 9, 10, 1\}$, and $T = \{2, 5, 6, 7, 8, 11\}$.*

*We have the following signature matrix $\tilde{Q}$ for $(14, 7)$ equiangular tight frame:*

$$\tilde{Q} = \begin{bmatrix}
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & 1 \\
1 & 1 & 0 & 1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 \\
1 & -1 & 1 & 0 & 1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\
1 & 1 & -1 & 1 & 0 & 1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 \\
1 & 1 & 1 & -1 & 1 & 0 & 1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 \\
1 & -1 & 1 & 1 & -1 & 1 & 0 & 1 & -1 & 1 & 1 & -1 & -1 & -1 \\
1 & -1 & -1 & 1 & 1 & -1 & 1 & 0 & 1 & -1 & 1 & 1 & -1 & -1 \\
1 & -1 & -1 & -1 & 1 & 1 & -1 & 1 & 0 & 1 & -1 & 1 & 1 & -1 \\
1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & 1 & 0 & 1 & -1 & 1 & 1 \\
1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & 1 & 0 & 1 & -1 & 1 \\
1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & 1 & 0 & 1 & -1 \\
1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & 1 & 0 & 1 \\
1 & 1 & -1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & -1 & 1 & 0
\end{bmatrix} \tag{7.6}$$

Next we will state an algorithm to generate equiangular tight frames of the type $(p+1, \frac{p+1}{2})$ using Theorem 7.15. We will consider primes of the type $p \equiv 5 \pmod 8$

and then examine which groups of the type $(\mathbb{Z}_p, \cdot)$ are generated by 2. Note that if $p = 109$, then $p \equiv 5 \pmod 8$ but $(\mathbb{Z}_p, \cdot)$ is not generated by 2. Also not every group $(\mathbb{Z}_p, \cdot)$ that is generated by 2 has $p \equiv 5 \pmod 8$ for example $(\mathbb{Z}_{19}, \cdot) = \langle 2 \rangle$ but $19 \not\equiv 5 \pmod 8$.

**Algorithm 7.1 Generating $(p+1, \frac{p+1}{2})$ Equiangular Frames**

1: Begin by taking a positive integer $m$.
2: Check whether $8m + 5$ is a prime, call it $p$.
3: For $p$ obtained in Step 1, evaluate $l = 2^i \pmod p$ for each $i \in \{1, \ldots, p-2\}$.
4: If $l \neq 1$ for all $i \in \{1, \ldots, p-2\}$, then the set $\{2^{2r} : 1 \leq r \leq \frac{p-1}{2}\}$ is a quasi-signature set in $G$ for a $(p+1, \frac{p+1}{2})$-equiangular tight frame.

Table 7.1: *Equiangular frames obtained using Algorithm 7.1 for $m < 35$*

| $m$ | 0 | 1 | 3 | 4 | 6 | 7 |
|---|---|---|---|---|---|---|
| $(n, k)$ | $(6, 3)$ | $(14, 7)$ | $(30, 15)$ | $(38, 19)$ | $(54, 27)$ | $(62, 31)$ |
| $m$ | 12 | 18 | 21 | 22 | 24 | 33 |
| $(n, k)$ | $(102, 51)$ | $(150, 75)$ | $(174, 87)$ | $(182, 91)$ | $(198, 99)$ | $(270, 135)$ |

*A comprehensive table for $m < 500$ is given in Chapter 10, Table 10.2.*

**Remark 7.18.** *Note that $p \equiv 5 \pmod 8$ is equivalent to saying that $p = 4q + 1$ where $q \in \mathbb{N}$ is odd. To obtain equiangular tight frames using Theorem 7.15, we have considered groups of the form $(\mathbb{Z}_p, \cdot)$, where $p \equiv 5 \pmod 8$, that are generated by 2; that is, for which 2 is a primitive root (mod p) Definition 2.35. This relates to* **Artin's conjecture***[4] which states that "Every integer a, not equal to $-1$ or to a square, is a primitive root (mod p) of infinitely many primes". In the nineteenth century, several mathematicians proved (see chapter VII in [26] for references) that* *whenever $p$ is of the form $4q+1$, $q$-odd prime, 2 is a primitive root (mod $p$). In addition to odd primes, we are also looking for all odd numbers $q$ such that $p = 4q+1$ with 2 as primitive root (mod $p$). For example as we can see from Table 7.1, that we have $(38, 19)$-equiangular tight frame where $37 = 4 \cdot 9 + 1$.*

Note from Table 7.1 that we are not getting all the equiangular tight frames of the type $(p+1, \frac{p+1}{2})$. For example we did not get $(18, 9)$-equiangular tight frame. The next result will enable us to construct some more equiangular tight frames again using the group $(\mathbb{Z}_p, +)$.

**Theorem 7.19.** *Let $p$ be a prime of the form $p \equiv 1(mod\ 4)$ and $G = (\mathbb{Z}_p, +)$. If $\langle 2 \rangle \subset (\mathbb{Z}_p, \cdot)$ is a subgroup of index 2, then $\langle 2 \rangle$ is a quasi-signature set in $G$ for a $(p+1, k)$-equiangular tight frame if and only if $p \equiv 1(mod\ 8)$. In this case $k = \frac{p+1}{2}$.*

*Proof.* Let us denote $S = \langle 2 \rangle = \{2^k : k = 1, 2, \ldots, \frac{p-1}{2}\}$. Since $S$ is a subgroup of index 2 in $(\mathbb{Z}_p, \cdot)$, then for $a \notin S$, we have $T = a \cdot S$. Thus, $|S| = |T|$ and let $\mu = |S| - |T|$. As before, we will first prove that $S = S^{-1}$ in $(\mathbb{Z}_p, +)$ and then we will verify the conditions of Theorem 7.8.
Let $\tilde{g} \in S$, then $\tilde{g}$ is of the form $2^m$ for some $m \in \{1, 2, \ldots, \frac{p-1}{2}\}$. Since $2^{\frac{p-1}{2}} \equiv 1(\text{mod } p)$ and $\langle 2 \rangle$ is a subgroup of index 2 in $(\mathbb{Z}_p, \cdot)$, we have $2^{\frac{p-1}{4}} \equiv (p-1)(\text{mod } p)$. Thus,

$$\begin{aligned} 2^m + 2^{\frac{p-1}{4}+m} &= 2^m(1 + 2^{\frac{p-1}{4}}) \\ &\equiv 0(\text{mod } p). \end{aligned}$$

Thus, $2^{\frac{p-1}{4}+m}$ (mod $p$) is the inverse of $\tilde{g}$ in $(\mathbb{Z}_p, +)$. Also note that if $\frac{p-1}{4} + m > \frac{p-1}{2}$, then $\frac{p-1}{4}+m=\frac{p-1}{2}+s$ where $s\in\{1,\ldots,\frac{p-1}{2}\}$. Thus, $2^{\frac{p-1}{4}}\equiv 2^s(\text{mod } p)\in S$ where $s\in\{1,\ldots,\frac{p-1}{2}\}$. Hence $S$ is closed under inverses in the group $(\mathbb{Z}_p,+)$.

For the second part, assume that $N^{\tilde{g}}_{(S,S)}=N$ for some $N\in\mathbb{N}$. For any $g\in S$, $g$ is of the form $2^l$ for some $l\in\mathbb{N}$.

Let us denote

$$S\times_g S=\{(g_1,g_2)\in S\times S: g_1+g_2=g\}.$$

For $i,j\in\{1,\ldots,\frac{p-1}{2}\}$, since $2^i+2^j\equiv 2^m(\text{mod } p)$ if and only if $2^{l-m+i}+2^{l-m+j}\equiv 2^l$ $(\text{mod } p)$, the map $\phi: S\times_{2^m} S\to S\times_{2^l} S$ such that

$$\phi((2^i,2^j))=(2^{l-m+i}(\text{mod } p),2^{l-m+j}(\text{mod } p))$$

is one to one and onto. Thus, $|S\times_{2^m} S|=|S\times_{2^l} S|$. But $|S\times_{2^m} S|=N^{2^m}_{(S,S)}$. Thus, for all $g\in S$, we have $N^g_{(S,S)}=N$.

Now if $2^m\in S$ then $2^m+2^m=2\cdot 2^m=2^{m+1}\in S$. Thus for $g\in S$, if $(g_i,g_j)\in S\times_g S$, then $(g_j,g_i)\in S\times_g S$. Also for all $i$, $(g_i,g_i)\in S\times_g S$. Hence, $N^g_{(S,S)}=|S\times_g S|=N$ must be an odd number and let $N=2r-1$ for some $r\in\mathbb{N}$.

Similarly if $h\in T$, then $h=a\cdot 2^{\tilde{m}}$ for some $\tilde{m}\in\{1,\ldots,\frac{p-1}{2}\}$ and $a\notin S$. Since $a\cdot 2^i+a\cdot 2^j\equiv a\cdot 2^{\tilde{m}}$ $(\text{mod } p)$ if and only if $2^i+2^j\equiv 2^{\tilde{m}}$ $(\text{mod } p)$, the map that takes

$$S\times_{2^{\tilde{m}}} S\to T\times_{a\cdot 2^{\tilde{m}}} T,\quad (2^i,2^j)\longrightarrow(a\cdot 2^i(\text{mod } p),a\cdot 2^j(\text{mod } p))$$

is one to one and onto. Thus $N^{a\cdot 2^{\tilde{m}}}_{(T,T)}=N^{2^{\tilde{m}}}_{(S,S)}$. But $N^g_{(S,S)}=2r-1$ for all $g\in S$. Thus for all $h\in T$, we have $N^h_{(T,T)}=2r-1$. By Theorem 7.8, Conditions 7.1 and 7.2 hold if and only if we have $\frac{p+1-6}{4}=2r-1$; that is, $p=8r+1$ or equivalently $p\equiv 1(\text{mod } 8)$. Also $\mu=0$ implies that $k=\frac{p+1}{2}$. □

Let us look at the example when $p = 17$.

**Example 7.20.** *Let* $G = (\mathbb{Z}_{17}, +)$*. Then* $\langle 2 \rangle$ *is a subgroup of index* 2 *in* $(\mathbb{Z}_{17}, \cdot)$*. By Theorem 7.19, if we take* $S = \langle 2 \rangle$ *that is* $S = \{2, 4, 8, 16, 15, 13, 9, 1\}$*, then* $S$ *is a quasi-signature set in* $G$ *for* $(18, 9)$*-equiangular tight frame.*

**Remark 7.21.** *Using Theorems 2.37 and 2.38, we know that* $2^{\frac{p-1}{2}} \equiv 1 \pmod{p}$ *if and only if* $p \equiv \pm 1 \pmod{8}$*. Thus, in the algorithm for generating equiangular tight frames of the type* $(p+1, \frac{p+1}{2})$ *from Theorem 7.19, we will take primes of the form* $p = 8m + 1$*,* $m \in \mathbb{N}$ *and then will check for which of the groups of the type* $(\mathbb{Z}_p, \cdot)$*, we have* $\langle 2 \rangle$ *as a subgroup of index* 2*. Note that* $73 = 8 \cdot 9 + 1$ *but* $\langle 2 \rangle$ *is not a subgroup of index* 2 *in* $(\mathbb{Z}_{73}, \cdot)$ *and conversely* $\langle 2 \rangle$ *is a subgroup of index* 2 *in* $(\mathbb{Z}_7, \cdot)$ *but* $7 \not\equiv 1 \pmod{8}$.

**Algorithm 7.2 Generating** $(p+1, \frac{p+1}{2})$ **Equiangular Frames**

1: Begin by taking a positive integer $m$.
2: Check whether $8m + 1$ is a prime, if so, call it $p$.
3: For $p$ obtained in Step 1, evaluate $l = 2^i \pmod{p}$ for each $i \in \{1, \ldots, \frac{p-3}{2}\}$.
4: If $l \neq 1$, for all $i \in \{1, \ldots, \frac{p-3}{2}\}$, then then the set $\{2^r : 1 \leq r \leq \frac{p-1}{2}\}$ is a quasi-signature set in $G$ for a $(p+1, \frac{p+1}{2})$-equiangular tight frame.

Table 7.2: *Equiangular frames obtained using Algorithm 7.2 for* $m < 75$

| $m$ | 0 | 2 | 5 | 12 | 17 | 24 |
|---|---|---|---|---|---|---|
| $(n, k)$ | $(2, 1)$ | $(18, 9)$ | $(42, 21)$ | $(98, 49)$ | $(138, 69)$ | $(194, 97)$ |
| $m$ | 39 | 50 | 51 | 56 | 65 | 71 |
| $(n, k)$ | $(314, 157)$ | $(402, 201)$ | $(410, 205)$ | $(450, 225)$ | $(522, 261)$ | $(570, 285)$ |

*A comprehensive table for* $m < 800$ *is given in Chapter 10, Table 10.3.*

# Chapter 8

# Cube Roots of Unity and Signature Pairs of Sets in Groups

In [12], the authors studied nontrivial signature matrices whose off-diagonal entries are cube roots of unity. Also in [12], a number of necessary and sufficient conditions for such a signature matrix of an $(n, k)$-equiangular tight frame to exist are presented. In this chapter, we extend our techniques used in the case of real equiangular tight frames to the case when entries of $Q$ are cube roots of unity.

**Definition 8.1.** *[12] A matrix $Q$ a cube root Seidel matrix if it is self-adjoint, has vanishing diagonal entries, and off-diagonal entries which are all cube roots of unity. If $Q$ has exactly two eigenvalues, then we say that it is the cube root signature matrix of an equiangular tight frame.*

## 8.1 Signature pairs of sets in groups

Let $\omega = \frac{-1}{2} + i\frac{\sqrt{3}}{2}$. Then the set $\{1, \omega, \omega^2\}$ is the set of cube roots of unity. Note also that $\omega^2 = \bar{\omega}$ and $1 + \omega + \omega^2 = 0$.

**Definition 8.2.** *Let $G$ be a group with $|G| = n$ and $\lambda : G \longrightarrow GL(\mathbb{F}(G))$ be the left regular representation. Let $S, T \subset G \setminus \{e\}$ be disjoint such that $G \setminus \{e\} = S \cup T \cup V$ where $V = (S \cup T)^c \setminus \{e\}$. Form $Q = \sum_{g \in S} \lambda(g) + \omega \sum_{g \in T} \lambda(g) + \omega^2 \sum_{g \in V} \lambda(g)$. We call $(S, T)$ a signature pair of sets in $G$ for a cube root $(n, k)$-equiangular tight frame if $Q$ forms the cube root signature matrix of an $(n, k)$-equiangular tight frame.*

*We shall call frames arising from such signature pairs as $(n, k)$-cube root equiangular tight frames.*

Similar to Theorem 5.7 in Chapter 5, we have the following result in the case of $(n, k)$-cube root equiangular tight frames:

**Theorem 8.3.** *Let $G$ be a group with $|G| = n$. Let $S, T \subset G \setminus \{e\}$-disjoint such that $G \setminus \{e\} = S \cup T \cup V$ where $V$ is as in Definition 8.2. Then there exists a $k$ such that $(S, T)$ is a signature pair of sets in $G$ for an $(n, k)$-cube root equiangular tight frame if and only if the following hold:*

1. *$S = S^{-1}$ and $T^{-1} = V$;*

2. *there exists an integer $\mu$ such that*

(a) *for all $g \in S$, we have*

$$N^g_{(S,S)} + \omega^2 N^g_{(T,T)} + \omega N^g_{(V,V)} + \omega(N^g_{(S,T)} + N^g_{(T,S)}) + \omega^2(N^g_{(S,V)} + N^g_{(V,S)}) + N^g_{(T,V)} + N^g_{(V,T)} = \mu; \tag{8.1}$$

(b) *for all $h \in T$, we have*

$$N^h_{(S,S)} + \omega^2 N^h_{(T,T)} + \omega N^h_{(V,V)} + \omega(N^h_{(S,T)} + N^h_{(T,S)}) + \omega^2(N^g_{(S,V)} + N^g_{(V,S)}) + N^h_{(T,V)} + N^h_{(V,T)} = \omega\mu; \tag{8.2}$$

(c) *for all $\tilde{h} \in V$, we have*

$$N^{\tilde{h}}_{(S,S)} + \omega^2 N^{\tilde{h}}_{(T,T)} + \omega N^{\tilde{h}}_{(V,V)} + \omega(N^{\tilde{h}}_{(S,T)} + N^{\tilde{h}}_{(T,S)}) + \omega^2(N^{\tilde{h}}_{(S,V)} + N^{\tilde{h}}_{(V,S)}) + N^{\tilde{h}}_{(T,V)} + N^{\tilde{h}}_{(V,T)} = \omega^2\mu. \tag{8.3}$$

*In this case $k$ and $\mu$ are related by the equations given in* (2.4)

*Proof.* Form $Q = \sum_{g\in S}\lambda(g) + \omega\sum_{g\in T}\lambda(g) + \omega^2\sum_{g\in V}\lambda(g)$. Then by Definition 8.2, $(S,T)$ is a signature pair of sets in $G$ for an $(n,k)$-cube root equiangular tight frame if and only if $Q$ is a signature matrix for an $(n,k)$-cube root equiangular tight frame. From Theorem 2.8 we know that an $n \times n$ matrix $Q$ is a signature matrix for an $(n,k)$-equiangular tight frame if and only if it satisfies the following two conditions:

(a) $Q$ is self adjoint that is $Q = Q^*$, and

(b) $Q^2 = (n-1)I + \mu Q$ for some real number $\mu$.

The condition $Q = Q^*$ is equivalent to

$$\sum_{g\in S}\lambda(g)+\omega\sum_{h\in T}\lambda(h)+\omega^2\sum_{\tilde{h}\in V}\lambda(\tilde{h}) = (\sum_{g\in S}\lambda(g)+\omega\sum_{h\in T}\lambda(h)+\omega^2\sum_{\tilde{h}\in V}\lambda(\tilde{h}))^*$$
$$= \sum_{g\in S}\lambda(g^{-1})+\omega^2\sum_{h\in T}\lambda(h^{-1})+\omega\sum_{\tilde{h}\in V}\lambda(\tilde{h}^{-1}).$$

Thus $g \in S$ implies $g^{-1} \in S$ and $h \in T$ implies $h^{-1} \in V$. By using counting arguments as before in Theorem 5.7, the second condition $Q^2 = (n-1)I + \mu Q$ for some real number $\mu$, is equivalent to

(a) for all $g \in S$, we have

$$N^g_{(S,S)} + \omega^2 N^g_{(T,T)} + \omega N^g_{(V,V)} + \omega(N^g_{(S,T)} + N^g_{(T,S)}) + \omega^2(N^g_{(S,V)} + N^g_{(V,S)}) +$$
$$N^g_{(T,V)} + N^g_{(V,T)} = \mu;$$

(b) for all $h \in T$, we have

$$N^h_{(S,S)} + \omega^2 N^h_{(T,T)} + \omega N^h_{(V,V)} + \omega(N^h_{(S,T)} + N^h_{(T,S)}) + \omega^2(N^g_{(S,V)} + N^g_{(V,S)}) +$$
$$N^h_{(T,V)} + N^h_{(V,T)} = \omega\mu;$$

(c) for all $\tilde{h} \in V$, we have

$$N^{\tilde{h}}_{(S,S)} + \omega^2 N^{\tilde{h}}_{(T,T)} + \omega N^{\tilde{h}}_{(V,V)} + \omega(N^{\tilde{h}}_{(S,T)} + N^{\tilde{h}}_{(T,S)}) + \omega^2(N^{\tilde{h}}_{(S,V)} + N^{\tilde{h}}_{(V,S)}) +$$
$$N^{\tilde{h}}_{(T,V)} + N^{\tilde{h}}_{(V,T)} = \omega^2\mu.$$

□

**Theorem 8.4.** *Let $G$ be a group with $|G| = n$ and $S, T \subset G \setminus \{e\}$ -disjoint such that $G \setminus \{e\} = S \cup T \cup V$ where $V = (S \cup T)^c \setminus \{e\}$. Also let $S = S^{-1}$ and $T^{-1} = V$. If $(S, T)$ is a signature pair of sets in $G$ for an $(n, k(\mu))$-cube root equiangular tight frame, then the following hold.*

*(a) For all $g \in S$,*

$$N^g_{(S,T)} + N^g_{(T,S)} + N^g_{(T,T)} = \frac{n - 2 - \mu}{3}. \tag{8.4}$$

*(b) For all $h \in T$,*

$$N^h_{(V,V)} + N^h_{(S,T)} + N^h_{(S,V)} = \frac{\mu + n - 1}{3}. \tag{8.5}$$

*(c) For all $\tilde{h} \in V$,*

$$N^{\tilde{h}}_{(T,T)} + N^{\tilde{h}}_{(S,T)} + N^{\tilde{h}}_{(S,V)} = \frac{\mu + n - 1}{3}. \tag{8.6}$$

*Proof.* From Theorem 8.3, since $\mu$ is real, using Equation (8.1) we have,

$$N^g_{(T,T)} + N^g_{(S,V)} + N^g_{(V,S)} = N^g_{(V,V)} + N^g_{(S,T)} + N^g_{(T,S)}. \tag{8.7}$$

Assume, $|S| = l$ and for $g \in S$, $N^g_{(T,T)} = m$. Then, $|T| = |V| = \frac{n-l-1}{2}$, and $N^g_{(T,S)} + N^g_{(T,V)} = \frac{n-1-l}{2} - m$. Thus, by (8.7), we have

$$m + N^g_{(S,V)} + N^g_{(V,S)} = N^g_{(V,V)} + \frac{n-1-l}{2} - m - N^g_{(T,V)} + \frac{n-1-l}{2} - m - N^g_{(V,T)};$$

that is,

$$3m + N^g_{(S,V)} + N^g_{(T,V)} + N^g_{(V,S)} + N^g_{(V,T)} = N^g_{(V,V)} + n - 1 - l.$$

But, we also have

$$N^g_{(S,V)} + N^g_{(T,V)} + N^g_{(V,V)} = \frac{n-1-l}{2},$$

and

$$N^g_{(V,S)} + N^g_{(V,T)} + N^g_{(V,V)} = \frac{n-1-l}{2}.$$

Thus,

$$3m + \frac{n-1-l}{2} - N^g_{(V,V)} + \frac{n-1-l}{2} - N^g_{(V,V)} = N^g_{(V,V)} + n - 1 - l.$$

Hence we get $N^g_{(V,V)} = m$ and for all $g \in S$,

$$N^g_{(T,T)} = N^g_{(V,V)}. \tag{8.8}$$

Again using (8.1), we have $N^g_{(S,V)} + N^g_{(V,S)} = N^g_{(S,T)} + N^g_{(T,S)}$. For all $g \in S$, $N^g_{(S,V)} + N^g_{(S,T)} = N^g_{(V,S)} + N^g_{(T,S)}$. Thus, for all $g \in S$, we have

$$N^g_{(S,V)} = N^g_{(T,S)} \quad \text{and} \quad N^g_{(V,S)} = N^g_{(S,T)}. \tag{8.9}$$

Using Equations (8.8) and (8.9), from (8.1) we get,

$$\begin{aligned}
\mu &= N^g_{(S,S)} - N^g_{(T,T)} - N^g_{(S,T)} - N^g_{(T,S)} + N^g_{(T,V)} + N^g_{(V,T)} \\
&= N^g_{(S,S)} - N^g_{(T,T)} - N^g_{(S,T)} - N^g_{(T,S)} + (|T| - N^g_{(T,T)} - N^g_{(T,S)}) + \\
&\quad (|T| - N^g_{(T,T)} - N^g_{(S,T)}) \\
&= N^g_{(S,S)} - 3N^g_{(T,T)} + 2|T| - 2(N^g_{(S,T)} + N^g_{(T,S)}) \\
&= N^g_{(S,S)} - 3N^g_{(T,T)} + 2|T| - 2(N^g_{(S,T)} + N^g_{(S,V)}) \\
&= N^g_{(S,S)} - 3N^g_{(T,T)} + 2|T| - 2(l - 1 - N^g_{(S,S)})
\end{aligned}$$

$$= 3N^g_{(S,S)} - 3N^g_{(T,T)} + n - 1 - l - 2l + 2$$
$$= 3N^g_{(S,S)} - 3l - 3N^g_{(T,T)} + n + 1.$$

Thus, $n + 1 - \mu = 3(l - N^g_{(S,S)} + N^g_{(T,T)})$ that is $\frac{n+1-\mu}{3} = (l - N^g_{(S,S)} + N^g_{(T,T)})$. But we know that $N^g_{(S,T)} + N^g_{(S,V)} = l - 1 - N^g_{(S,S)}$. Thus,

$$N^g_{(S,T)} + N^g_{(S,V)} + N^g_{(T,T)} = \frac{n-2-\mu}{3}$$

or by (8.9), we have

$$N^g_{(S,T)} + N^g_{(T,S)} + N^g_{(T,T)} = \frac{n-2-\mu}{3}.$$

To simplify Condition (8.2), assume that for $h \in T$, $N^h_{(S,S)} = m$. Then as before, since $\mu$ is a real number, we must have from Condition (8.2),

$$N^h_{(S,S)} + N^h_{(T,V)} + N^h_{(V,T)} = N^h_{(T,T)} + N^h_{(S,V)} + N^h_{(V,S)}. \tag{8.10}$$

Also, we have

$$N^h_{(S,T)} + N^h_{(S,V)} = l - m \quad and \quad N^h_{(T,S)} + N^h_{(V,S)} = l - m \tag{8.11}$$

Thus, (8.10) changes to $m + N^h_{(T,V)} + N^h_{(V,T)} = N^h_{(T,T)} + l - m - N^h_{(S,T)} + l - m - N^h_{(T,S)}$ that is $m + N^h_{(T,V)} + N^h_{(T,S)} + N^h_{(V,T)} + N^h_{(S,T)} = N^h_{(T,T)} + 2l - 2m$. But $N^h_{(T,V)} + N^h_{(T,S)} = |T| - 1 - N^h_{(T,T)}$ and $N^h_{(V,T)} + N^h_{(S,T)} = |T| - 1 - N^h_{(T,T)}$. Thus we have $m + 2(|T| - 1 - N^h_{(T,T)}) = N^h_{(T,T)} + 2l - 2m$ that is $N^h_{(T,T)} = m - l - 1 + \frac{n}{3}$. Again using (8.10) we get

$$m + N^h_{(T,V)} + N^h_{(V,T)} = m - l - 1 + \frac{n}{3} + N^h_{(S,V)} + N^h_{(V,S)}.$$

Thus,

$$N^h_{(T,V)} - N^h_{(S,V)} + N^h_{(V,T)} - N^h_{(V,S)} = \frac{n}{3} - l - 1. \tag{8.12}$$

But $N^h_{(T,V)} + N^h_{(S,V)} = |V| - N^h_{(V,V)}$ and $N^h_{(V,T)} + N^h_{(V,S)} = |V| - N^h_{(V,V)}$. Thus we have, $|V| - N^h_{(V,V)} - 2N^h_{(S,V)} + |V| - N^h_{(V,V)} - 2N^h_{(V,S)} = \frac{n}{3} - l - 1$ that is $2|V| - 2N^h_{(V,V)} - 2N^h_{(S,V)} - 2N^h_{(V,S)} = \frac{n}{3} - l - 1$. If $N^h_{(V,V)} = \tilde{m}$, then we have

$$2(N^h_{(S,V)} + N^h_{(V,S)}) = n - l - 1 - 2\tilde{m} - \frac{n}{3} + l + 1.$$

That is,

$$N^h_{(S,V)} + N^h_{(V,S)} = \frac{n - 2\tilde{m} - \frac{n}{3}}{2} = \frac{n - 3\tilde{m}}{3}.$$

Thus using (8.12) we have,

$$N^h_{(T,V)} + N^h_{(V,T)} = \frac{n}{3} - l - 1 + \frac{n - 2\tilde{m} - \frac{n}{3}}{2} = \frac{2n - 3l - 3 - 3\tilde{m}}{3}$$

and using (8.11), we get

$$N^h_{(T,S)} + N^h_{(S,T)} = 2(l - m) - (\frac{n - 3\tilde{m}}{3}) = \frac{6l - 6m - n + 3\tilde{m}}{3}.$$

Substituting the values obtained above in (8.2) we have,

$$\begin{aligned}
\omega\mu &= m + \omega^2(m - l - 1 + \frac{n}{3}) + \omega\tilde{m} + \omega(\frac{6l - 6m - n + 3\tilde{m}}{3}) + \omega^2(\frac{n - 3\tilde{m}}{3}) + \\
&\quad \frac{2n - 3l - 3 - 3\tilde{m}}{3} \\
&= \frac{3m + 2n - 3l - 3\tilde{m} - 3 + \omega^2(3m - 3l - 3 + 2n - 3\tilde{m}) + \omega(6\tilde{m} + 6l - 6m - n)}{3} \\
&= \frac{\omega(9\tilde{m} + 9l - 9m - 3n + 3)}{3} \\
\mu &= 3\tilde{m} + 3l - 3m - n + 1.
\end{aligned}$$

Thus,

$$\frac{\mu + n - 1}{3} = \tilde{m} + l - m;$$

that is,

$$N^h_{(V,V)} + N^h_{(S,T)} + N^h_{(S,V)} = \frac{\mu + n - 1}{3}.$$

By symmetry, Condition (8.3) reduces to for all $\tilde{h} \in V$, we have

$$N^{\tilde{h}}_{(T,T)} + N^{\tilde{h}}_{(S,T)} + N^{\tilde{h}}_{(S,V)} = \frac{\mu + n - 1}{3}.$$

□

We have the following corollary, a more general case of which was shown in [12] as Proposition 3.3. Here we prove it in case of cube root equiangular tight frames arising from signature pairs of sets in groups.

**Corollary 8.5.** *Let $G$ be a group of order $n$. If there exists a signature pair $(S, T)$ associated with an $(n, k)$-cube root equiangular tight frame, then the following hold.*

*(a)* $n \equiv 0 \pmod 3$.

*(b)* *$\mu$ is an integer and* $\mu \equiv 1 \pmod 3$.

*(c)* *The integer $4(n-1) + \mu^2$ is a perfect square and in addition* $4(n-1) + \mu^2 \equiv 0 \pmod 9$.

*Proof.* Let $G$ be a group of order $n$ and assume that there exists a signature pair $(S, T)$ associated with an $(n, k)$-cube root equiangular tight frame. Then by Theorem 8.4, Equations (8.4) and (8.5) hold. Adding Equations (8.4) and (8.5), we get

$n \equiv 0 \pmod 3$. Since $n$ is an integer and from Equation(8.4), $\frac{n-2-\mu}{3}$ is an integer, thus $\mu$ is also an integer. Let $n = 3m$ for some $m$ in $\mathbb{N}$. From Equation (8.6), we know that $\frac{\mu+n-1}{3}$ is an integer, say $l$. Thus we have, $\mu = 3(l-m)+1$ that is $\mu = 1 \pmod 3$.

For the third part, using the relation given in (2.4) between $k$ and $\mu$, we have

$$k = \frac{3m}{2}(1 - \frac{3\tilde{m}+1}{\sqrt{4(3m-1)+(3\tilde{m}+1)^2}})$$

where $n = 3m$ for $m \in \mathbb{N}$ and $\mu = 3\tilde{m}+1$ for $\tilde{m} \in \mathbb{Z}$. Thus $4(3m-1)+(3\tilde{m}+1)^2$ should be perfect square. But we have,

$$4(3m-1)+(3\tilde{m}+1)^2 = 3(3\tilde{m}^2+2\tilde{m}+4m-1).$$

Thus, $4(3m-1)+(3\tilde{m}+1)^2 \equiv 0 \pmod 3$. But since $4(3m-1)+(3\tilde{m}+1)^2$ is a perfect square, we have $4(3m-1)+(3\tilde{m}+1)^2 \equiv 0 \pmod 9$. □

**Example 8.6.** *Let $G = (\mathbb{Z}_3, +)$. If we take $S = \{1,2\}$ and $T = \emptyset$, then for $g \in S$, $N^g_{(S,S)} = 1$. Since it is the only non zero value, using Equation (8.1), we have $\mu = 1$. The signature matrix is*

$$Q = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

*which gives rise to the trivial $(3,1)$-equiangular tight frame.*

*If we take $S = \emptyset$, $T = \{1\}$ and $V = \{2\}$, then for $h \in T$, $N^h_{(V,V)}$ is the only non-zero* *value and equals* 1*. Using Equation* (8.2)*, we have* $\mu = 1$*. In this case we get the following signature matrix*

$$Q = \begin{bmatrix} 0 & \omega & \omega^2 \\ \omega^2 & 0 & \omega \\ \omega & \omega^2 & 0 \end{bmatrix}$$

*which is again a signature matrix for the trivial* $(3, 1)$*-equiangular tight frame.*

## 8.2 Investigating some values of $n$ and $\mu$

Next we would like to examine the case of non-trivial cube root equiangular tight frames arising from a signature pair of sets in $G$. In the process, we will look at some specific values of $n$ and $\mu$ and use the theory we have so far to investigate the possibility of the existence of non-trivial cube root equiangular tight frames arising from a signature pair of sets in $G$.

**Lemma 8.7.** *Let $G$ be a finite abelian group of order $m$ where $m \in \mathbb{N}$ is odd. Then for every element $e \neq g \in G$, there exists a unique $h \neq e$ such that $g = h^2$.*

*Proof.* Firstly we will show that for $g \in G$ there exists an $h \in G$ such that $g = h^2$. The order of $G$ is odd implies that the order of $g$ is odd. Thus, there exists an $r \in \mathbb{N}$ such that $g^{2r+1} = e$. Thus $g^{2r+1} \cdot g = e \cdot g$ that is $(g^{r+1})^2 = g$. Taking $h = g^{r+1} \in G$, we have $h^2 = g$. Now suppose that there exists $e \neq h, e \neq \tilde{h} \in G$ such that $g = h^2$ and $g = \tilde{h}^2$. Then $h^2\tilde{h}^{-2} = e$. Since $G$ is abelian, we have $(h\tilde{h}^{-1})^2 = e$. This is only possible when $h\tilde{h}^{-1} = e$ that is $h = \tilde{h}$. $\square$

**Proposition 8.8.** *Let $G$ be an abelian group of order $n$ where $n \equiv 3 \pmod 6$ and let $\mu \equiv 4 \pmod 6$. Then there does not exist a signature pair of sets in $G$ associated with an $(n, k(\mu))$-cube root equiangular tight frame.*

*Proof.* Suppose on the contrary that there exists a signature pair $(S, T)$ associated with $(n, k(\mu))$-cube root equiangular tight frame. Then using (8.4) in Theorem 8.4, we have

$$N^g_{(S,T)} + N^g_{(T,S)} + N^g_{(T,T)} = \frac{n-2-\mu}{3}.$$

Since $n \equiv 3 \pmod 6$ and $\mu \equiv 4 \pmod 6$, there exists $k \in \mathbb{N}$, and $k' \in \mathbb{Z}$ such that

$$\frac{n-2-\mu}{3} = \frac{6k+4-2-(6k'+4)}{3} = 2(k-k') - 1.$$

Also, $G$ abelian implies $N^g_{(S,T)} = N^g_{(T,S)}$. Hence for all $g \in S$, we have

$$2N^g_{(S,T)} + N^g_{(T,T)} = 2(k-k') - 1. \tag{8.13}$$

Since the right hand side of (8.13) is odd, $N^g_{(T,T)}$ must be a positive odd integer. For $\tilde{g} \in S$, by Lemma 8.7 we know that there exists a unique $e \neq h \in G$ such that $h^2 = \tilde{g}$. Since $N^{\tilde{g}}_{(T,T)}$ is odd, $h$ must be in $T$. Thus $h^{-1} \in V$. Since $S$ is closed under inverses, we have $\tilde{g}^{-1} \in S$ and $\tilde{g}^{-1} = h^{-2} = (h^{-1})^2$. This contradicts that $N^{\tilde{g}^{-1}}_{(T,T)}$ is odd. Thus there does not exist a signature pair of sets in $G$ associated with the $(n,k)(\mu))$-cube root equiangular tight frame. $\square$

**Remark 8.9.** *It was shown in [12] that there exists a $(9, 6)$-cube root equiangular tight frame with $\mu = -2$. Also we know from Theorem 2.39 and Proposition 2.41 that there are two distinct groups of order 9 that is $\mathbb{Z}_9$ and $\mathbb{Z}_3 \times \mathbb{Z}_3$, both abelian. Since*

$9 \equiv 3 \pmod 6$ *and* $-2 \equiv 4 \pmod 6$*, using Proposition 8.8, there does not exist a signature pair of sets in* $\mathbb{Z}_9$ *or* $\mathbb{Z}_3 \times \mathbb{Z}_3$ *associated with the* $(9, 6)$*-cube root equiangular tight frame. The next possible* $(n, k)$ *value of a cube root equiangular tight frame listed in [12] is* $(33, 11)$ *with* $\mu = 4$*. Once again* $33 \equiv 3 \pmod 6$ *and* $4 \equiv 4 \pmod 6$*. Using Proposition 2.42, we infer that every group of order* $33$ *is isomorphic to the cyclic group* $\mathbb{Z}_{33}$*. Thus using Proposition 8.8, we can conclude that there does not exist a signature pair of sets in* $\mathbb{Z}_{33}$ *associated with a* $(33, 11)$*-cube root equiangular tight frame.*

This motivates us to explore the quasi-signature case. Similar to Section 7, where we had real signature matrices in the standard form, next we will look at the cube root signature matrices in the standard form.

# Chapter 9

# Cube Roots of Unity and Quasi-signature Pairs of Sets

In this chapter we consider the cube root signature matrices in the standard form with entries as cube roots of unity.

**Lemma 9.1.** *[12, Lemma 2.2] If $Q'$ is an $n \times n$ cube root Seidel matrix, then it is switching equivalent to a cube root Seidel matrix of the form*

$$Q = \begin{bmatrix} 0 & 1 & \cdots & \cdots & 1 \\ 1 & 0 & * & \cdots & * \\ \vdots & * & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & * \\ 1 & * & \cdots & * & 0 \end{bmatrix} \tag{9.1}$$

*where the $*$'s are cube roots of unity. Moreover, $Q'$ is the signature matrix of an equiangular $(n, k)$-frame if and only if $Q$ is the signature matrix of an equiangular*

*$(n, k)$-frame.*

*Proof.* Suppose that $Q^{'}$ is an $n \times n$ cube root Seidel matrix. Then $Q'$ is self-adjoint with $|Q_{ij}| = 1$ for $i \neq j$, and by Theorem 2.8 we have that $(Q')^2 = (n-1)I + \mu Q'$ for some real number $\mu$. If we let $U$ be the diagonal matrix

$$U := \begin{bmatrix} 1 & & & & \\ & Q'_{12} & & & \\ & & Q'_{13} & & \\ & & & \ddots & \\ & & & & Q'_{1n} \end{bmatrix}$$

then $U$ is a unitary matrix (since $|Q'_{ij}| = 1$ when $i \neq j$), and we see that $Q := U^*Q'U$ is a self-adjoint $n \times n$ matrix with $Q_{ii} = 0$ and $|Q_{ij}| = 1$ for $i \neq j$. We see that the off-diagonal elements of $Q$ are cube roots of unity and $Q$ has the form shown in (9.1). (To see that the off-diagonal elements in the first row and column are 1's, recall that $Q'_{ij} = \overline{Q'_{ji}}$.) Thus $Q$ is a cube root Seidel matrix that is unitarily equivalent to $Q'$. Since $Q$ and $Q'$ have the same eigenvalues, if one of them is the signature matrix of an equiangular $(n, k)$-frame, then the same holds for the other matrix. $\square$

## 9.1 Quasi-signature pairs of sets in groups

We have the following definition when cube root signature matrix is in the standard form:

**Definition 9.2.** *Let $G$ be a group such that $|G| = m$. Let $S, T \subset G \setminus \{e\}$ be disjoint such that $G \setminus \{e\} = S \cup T \cup V$ where $V = (S \cup T)^c \setminus \{e\}$. For $\omega = \frac{-1}{2} + i\frac{\sqrt{3}}{2}$, form*

$Q = \sum_{g\in S}\lambda(g) + \omega\sum_{h\in T}\lambda(g) + \omega^2\sum_{\tilde{h}\in V}\lambda(\tilde{h})$ *as in Chapter 8. Let*

$$\tilde{Q} = \left[\begin{array}{c|c} 0 & C^t \\ \hline C & Q \end{array}\right] \quad \textit{where} \quad C = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \in \mathbb{C}^m.$$

*Then we call $(S,T)$ a quasi-signature pair of sets in $G$ for an $(n,k)$-cube root equiangular tight frame, where $n = m+1$, if $\tilde{Q}$ is a cube root signature matrix for an $(n,k)$-cube root equiangular tight frame.*

Analogous to Theorem 7.4 that gives us a necessary and sufficient condition for the existence of quasi-signature set, we have the following result about the quasi-signature pair of sets in $G$:

**Theorem 9.3.** *Let $G$ be a group with $|G| = m$. Let $S,T \subset G\setminus\{e\}$ be disjoint such that $G\setminus\{e\} = S\cup T\cup V$ where $V = (S\cup T)^c\setminus\{e\}$. Then there exists a $k$, such that $(S,T)$ is a quasi-signature pair of sets in $G$ for an $(n,k)$-cube root equiangular tight frame if and only if the following hold:*

1. *$S = S^{-1}$ and $T^{-1} = V$;*
2. *(a) for all $g \in S$,*

$$N^g_{(S,S)} + \omega^2 N^g_{(T,T)} + \omega N^g_{(V,V)} + \omega(N^g_{(S,T)} + N^g_{(T,S)}) + \omega^2(N^g_{(S,V)} + N^g_{(V,S)}) + N^g_{(T,V)} + N^g_{(V,T)} = \mu - 1;$$

*(b) for all* $h \in T$,

$$N^h_{(S,S)} + \omega^2 N^h_{(T,T)} + \omega N^h_{(V,V)} + \omega(N^h_{(S,T)} + N^h_{(T,S)}) + \omega^2(N^h_{(S,V)} + N^h_{(V,S)}) + N^h_{(T,V)} + N^h_{(V,T)} = \omega\mu - 1;$$

*(c) for all* $\tilde{h} \in V$,

$$N^{\tilde{h}}_{(S,S)} + \omega^2 N^{\tilde{h}}_{(T,T)} + \omega N^{\tilde{h}}_{(V,V)} + \omega(N^{\tilde{h}}_{(S,T)} + N^{\tilde{h}}_{(T,S)}) + \omega^2(N^{\tilde{h}}_{(S,V)} + N^{\tilde{h}}_{(V,S)}) + N^{\tilde{h}}_{(T,V)} + N^{\tilde{h}}_{(V,T)} = \omega^2\mu - 1,$$

*where* $\mu = |S| - |T|$ *and is related to* $k$ *by equations given in* (2.4).

*Proof.* Form $Q = \sum_{g\in S} \lambda(g) + \omega \sum_{h\in T} \lambda(h) + \omega^2 \sum_{\tilde{h}\in V} \lambda(\tilde{h})$ and

$$\tilde{Q} = \left[\begin{array}{c|c} 0 & C^t \\ \hline C & Q \end{array}\right] \quad \text{where} \quad C = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \in \mathbb{C}^m.$$

By Definition 9.2, $(S, T)$ is a quasi-signature pair of sets in $G$ for an $(n, k)$-cube root equiangular tight frame if and only if $Q$ is a signature matrix for an $(n, k)$-equiangular tight frame. From Theorem 2.8, $\tilde{Q}$ is a signature matrix for an $(n, k)$ equiangular tight frame if and only if it satisfies the following two conditions:

(a) $\tilde{Q}$ is self adjoint that is $\tilde{Q} = \tilde{Q}^*$; and

(b) $\tilde{Q}^2 = (n-1)I + \mu\tilde{Q}$ for some real number $\mu$.

The condition $\tilde{Q} = \tilde{Q}^*$ is equivalent to $Q = Q^*$ which is equivalent to saying that $g \in S$ implies $g^{-1} \in S$ and $h \in T$ implies $h^{-1} \in V$.

For the second condition we need $\tilde{Q}^2 = (n-1)I + \mu\tilde{Q}$. We have

$$\tilde{Q}^2 = \begin{bmatrix} n-1 & \tilde{C}^t \\ \tilde{C} & J + Q^2 \end{bmatrix}$$

where $\tilde{C} = \alpha C$, $\alpha = |S| + \omega|T| + \omega^2|V| = |S| - |T|$. Thus, $\tilde{Q}^2 = (n-1)I + \mu\tilde{Q}$ if and only if

(a) $\alpha = |S| - |T| = \mu$, and

(b) $J + Q^2 = (n-1)I + \mu Q$ that is $Q^2 = (n-1)I + \mu Q - J$. Since $J = \sum_{g \in G} \lambda(g)$, we have

$$\begin{aligned} Q^2 &= (n-1)I + \mu Q - J \\ &= (n-2)I + \mu(\sum_{g \in S} \lambda(g) + \omega \sum_{h \in T} \lambda(h) + \omega^2 \sum_{\tilde{h} \in T} \lambda(\tilde{h})) - \sum_{g \in G \setminus \{e\}} \lambda(g) \\ &= (n-2)I + (\mu - 1) \sum_{g \in S} \lambda(g) + (\omega\mu - 1) \sum_{h \in T} \lambda(h) + (\omega^2\mu - 1) \sum_{\tilde{h} \in T} \lambda(\tilde{h}). \end{aligned}$$

By the same counting arguments as before we have that $\tilde{Q}^2 = (n-1)I + \mu\tilde{Q}$ if and only if for all $g \in S$,

$$N^g_{(S,S)} + \omega^2 N^g_{(T,T)} + \omega N^g_{(V,V)} + \omega(N^g_{(S,T)} + N^g_{(T,S)}) + \omega^2(N^g_{(S,V)} + N^g_{(V,S)}) +$$
$$N^g_{(T,V)} + N^g_{(V,T)} = \mu - 1,$$

for all $h \in T$,

$$N^h_{(S,S)} + \omega^2 N^h_{(T,T)} + \omega N^h_{(V,V)} + \omega(N^h_{(S,T)} + N^h_{(T,S)}) + \omega^2(N^h_{(S,V)} + N^h_{(V,S)}) +$$
$$N^h_{(T,V)} + N^h_{(V,T)} = \omega\mu - 1,$$

for all $\tilde{h} \in V$,

$$N^{\tilde{h}}_{(S,S)} + \omega^2 N^{\tilde{h}}_{(T,T)} + \omega N^{\tilde{h}}_{(V,V)} + \omega(N^{\tilde{h}}_{(S,T)} + N^{\tilde{h}}_{(T,S)}) + \omega^2(N^{\tilde{h}}_{(S,V)} + N^{\tilde{h}}_{(V,S)}) +$$
$$N^{\tilde{h}}_{(T,V)} + N^{\tilde{h}}_{(V,T)} = \omega^2\mu - 1$$

where $\mu = |S| - |T|$. $\square$

**Remark 9.4.** *From Theorem 9.3, note that if $S, T \subset G$ is a quasi-signature pair of sets in $G$ for an $(n, k(\mu))$-cube root equiangular tight frame, then $|S| - |T| = \mu$.*

**Proposition 9.5.** *Let $G$ be a group of order $m$ and $S, T, V \subset G \setminus \{e\}$ be pairwise disjoint such that $G \setminus \{e\} = S \cup T \cup V$ where $S = S^{-1}$ and $V = T^{-1}$. Then the condition $|S| - |T| = \mu$ for some integer $\mu$ is equivalent to*

$$|S| = \frac{n + 2\mu - 2}{3} \quad \textit{and} \quad |T| = \frac{n - 2 - \mu}{3}, \tag{9.2}$$

*where $n = m + 1$.*

*Proof.* Let $\mu$ be an integer. Since $G \setminus \{e\} = S \cup T \cup V$ where $S, T, V$ are pairwise disjoint with $S = S^{-1}$ and $V = T^{-1}$, we have $|S| + 2|T| = n - 2$. If $|S| - |T| = \mu$, then solving these equations for $|S|$ and $|T|$, we get

$$|S| = \frac{n + 2\mu - 2}{3} \quad \text{and} \quad |T| = \frac{n - 2 - \mu}{3}.$$

Conversely if $|S| = \dfrac{n + 2\mu - 2}{3}$ and $|T| = \dfrac{n - 2 - \mu}{3}$, then $|S| - |T| = \mu$. $\square$

## 9.2 The $(9,6)$-cube root equiangular tight frame

We know from [12] that there exists a $(9,6)$-cube root equiangular tight frame. In the following example we show this frame arises from the group of quaternions.

**Example 9.6.** *Let $G = \{1, -1, i, -i, j, -j, k, -k\}$ be the group of quaternions where $i^2 = -1$, $j^2 = -1$, $k^2 = -1$, $i \cdot j = k$, $j \cdot k = i$, $k \cdot i = j$, $j \cdot i = -k$, $k \cdot j = -i$ and $i \cdot k = -j$. Using Proposition 9.5, let us take $S$, $T$, $V$ as follows:*

$$S = \{-1\}, \quad T = \{i, j, k\}, \quad V = \{-i, -j, -k\}.$$

*Then all the conditions of the Theorem 9.3 are satisfied and we get $\mu = -2$. Using 2.4, we get $k = 6$. Hence $(S, T)$ is a quasi-signature pair of sets for the $(9,6)$-cube root equiangular tight frame. The signature matrix for the $(9,6)$-cube root equiangular tight frame is given below in the standard form (here $n = |G| + 1 = 9$):*

$$\tilde{Q} = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & \omega & \omega^2 & \omega & \omega^2 & \omega & \omega^2 \\ 1 & 1 & 0 & \omega^2 & \omega & \omega^2 & \omega & \omega^2 & \omega \\ 1 & \omega^2 & \omega & 0 & 1 & \omega^2 & \omega & \omega & \omega^2 \\ 1 & \omega & \omega^2 & 1 & 0 & \omega & \omega^2 & \omega^2 & \omega \\ 1 & \omega^2 & \omega & \omega & \omega^2 & 0 & 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 & \omega^2 & \omega & 1 & 0 & \omega & \omega^2 \\ 1 & \omega^2 & \omega & \omega^2 & \omega & \omega & \omega^2 & 0 & 1 \\ 1 & \omega & \omega^2 & \omega & \omega^2 & \omega^2 & \omega & 1 & 0 \end{bmatrix}$$

# Chapter 10

# Results

In this chapter we list the examples of equiangular tight frames arising from groups in Chapters 5 to 9. In Table 10.1 we consolidate all the examples of equiangular tight frames obtained via group constructions. Here $(n, k)$ is our usual notation of a frame of $n$ vectors in a $k$ dimensional Hilbert space, $G$ is the group associated with this frame, $S$, $QS$, $SP$, $QSP$ are the corresponding sets: signature, quasi-signature, signature pair and quasi-signature pair. In the fourth column, we explicitely describe these sets. The last column of the table gives us the result associated with that particular frame.

In the next two tables, we give a comprehensive list of the equiangular tight frames obtained using quasi-signature sets in groups in Chapter 7. In Table 10.2, we list equiangular tight frames arising from Algorithm 7.1 for $m < 500$. In Table 10.3 we list equiangular tight frames arising from Algorithm 7.2 for $m < 800$.

## 10.1 Tables

Table 10.1: *Equiangular frames obtained using groups*

| $(n,k)$ | $G$ | Type | Set/s | Result |
|---|---|---|---|---|
| $(n,1)$ | $G$ | $S$ | subgroup of index2 | Thm. 5.12 |
| $(16,6)$ | $C_4 \times C_4$ | $S$ | $\{a,a^2,a^3,b,b^2,b^3\}$ | Prop. 5.15 |
| $(36,15)$ | $C_6 \times C_6$ | $S$ | $\{a,\dots,a^5,b,\dots,b^5,ab,\dots,a^5b^5\}$ | Prop. 5.16 |
| $(n,\frac{n-\sqrt{n}}{2})$ | $\mathbb{Z}^2_{2^{a+1}}$, $a \in \mathbb{N}$ | $S$ | reversible Hadamard difference set | Thm. 6.12 |
| $(10,5)$ | $C_3 \times C_3$ | $QS$ | $\{a,a^2,b,b^2\}$ | Prop. 7.12 |
| $(26,13)$ | $C_5 \times C_5$ | $QS$ | $\{a,\dots,a^4,b,\dots,b^4,ab,\dots,a^4b^4\}$ | Prop. 7.13 |
| $(p+1,\frac{p+1}{2})$ | $\underset{p\equiv 5 \pmod 8}{(\mathbb{Z}_p,\cdot)}$ | $QS$ | $\{2^{2r} : 1 \le r \le \dfrac{p-1}{2}\}$ | Thm. 7.15 |
| $(p+1,\frac{p+1}{2})$ | $\underset{p\equiv 1 \pmod 8}{(\mathbb{Z}_p,\cdot)}$ | $QS$ | $\{2^{r} : 1 \le r \le \dfrac{p-1}{2}\}$ | Thm. 7.19 |
| $(9,6)$ | Quaternions | $QSP$ | $\{-1\}$,$\{i,j,k\}$ | Ex. 9.6 |

Table 10.2: *Equiangular frames obtained using Algorithm 7.1 for* $m < 500$

| $m$ | $(n,k)$ | $m$ | $(n,k)$ | $m$ | $(n,k)$ |
|---|---|---|---|---|---|
| 0 | $(6,3)$ | 138 | $(1110,555)$ | 304 | $(2438,1219)$ |
| 1 | $(14,7)$ | 139 | $(1118,559)$ | 309 | $(2478,1239)$ |
| 3 | $(30,15)$ | 151 | $(1214,607)$ | 318 | $(2550,1275)$ |
| 4 | $(38,19)$ | 153 | $(1230,615)$ | 319 | $(2558,1279)$ |
| 6 | $(54,27)$ | 154 | $(1238,619)$ | 327 | $(2622,1311)$ |
| 7 | $(62,31)$ | 159 | $(1278,639)$ | 334 | $(2678,1339)$ |
| 12 | $(102,51)$ | 162 | $(1302,651)$ | 336 | $(2694,1347)$ |
| 18 | $(150,75)$ | 171 | $(1374,687)$ | 342 | $(2742,1371)$ |
| 21 | $(174,87)$ | 172 | $(1382,691)$ | 348 | $(2790,1395)$ |
| 22 | $(182,91)$ | 181 | $(1454,727)$ | 349 | $(2798,1399)$ |
| 24 | $(198,99)$ | 186 | $(1494,747)$ | 354 | $(2838,1419)$ |
| 33 | $(270,135)$ | 193 | $(1550,775)$ | 357 | $(2862,1431)$ |
| 36 | $(294,147)$ | 202 | $(1622,811)$ | 363 | $(2910,1455)$ |
| 39 | $(318,159)$ | 204 | $(1638,819)$ | 369 | $(2958,1479)$ |
| 43 | $(350,175)$ | 208 | $(1670,835)$ | 379 | $(3038,1519)$ |
| 46 | $(374,187)$ | 211 | $(1694,847)$ | 397 | $(3182,1591)$ |
| 48 | $(390,195)$ | 216 | $(1734,867)$ | 403 | $(3230,1615)$ |
| 52 | $(422,211)$ | 217 | $(1742,871)$ | 406 | $(3254,1627)$ |
| 57 | $(462,231)$ | 232 | $(1862,931)$ | 421 | $(3374,1687)$ |
| 63 | $(510,255)$ | 234 | $(1878,939)$ | 426 | $(3414,1707)$ |
| 67 | $(542,271)$ | 237 | $(1902,951)$ | 432 | $(3462,1731)$ |
| 69 | $(558,279)$ | 243 | $(1950,975)$ | 433 | $(3470,1735)$ |
| 76 | $(614,307)$ | 246 | $(1974,987)$ | 439 | $(3518,1759)$ |
| 81 | $(654,327)$ | 249 | $(1998,999)$ | 441 | $(3534,1767)$ |
| 82 | $(662,331)$ | 253 | $(2030,1015)$ | 444 | $(3558,1779)$ |
| 84 | $(678,339)$ | 256 | $(2054,1027)$ | 447 | $(3582,1791)$ |
| 87 | $(702,351)$ | 258 | $(2070,1035)$ | 451 | $(3614,1807)$ |
| 88 | $(710,355)$ | 267 | $(2142,1071)$ | 454 | $(3638,1819)$ |
| 94 | $(758,379)$ | 276 | $(2214,1107)$ | 459 | $(3678,1839)$ |
| 96 | $(774,387)$ | 277 | $(2222,1111)$ | 462 | $(3702,1851)$ |
| 99 | $(798,399)$ | 279 | $(2238,1119)$ | 463 | $(3710,1855)$ |
| 102 | $(822,411)$ | 283 | $(2270,1135)$ | 466 | $(3734,1867)$ |
| 103 | $(830,415)$ | 286 | $(2294,1147)$ | 474 | $(3798,1899)$ |
| 106 | $(854,427)$ | 288 | $(2310,1155)$ | 481 | $(3854,1927)$ |
| 109 | $(878,439)$ | 291 | $(2334,1167)$ | 484 | $(3878,1939)$ |
| 117 | $(942,471)$ | 294 | $(2358,1179)$ | 489 | $(3918,1959)$ |
| 132 | $(1062,531)$ | 298 | $(2390,1195)$ | 498 | $(3990,1995)$ |

Table 10.3: *Equiangular frames obtained using Algorithm 7.2 on for $m < 800$*

| $m$ | $(n,k)$ | $m$ | $(n,k)$ | $m$ | $(n,k)$ |
|---|---|---|---|---|---|
| 0 | $(2,1)$ | 270 | $(2162,1081)$ | 542 | $(4338,2169)$ |
| 2 | $(18,9)$ | 287 | $(2298,1149)$ | 555 | $(4442,2221)$ |
| 5 | $(42,21)$ | 297 | $(2378,1189)$ | 557 | $(4458,2229)$ |
| 12 | $(98,49)$ | 302 | $(2418,1209)$ | 570 | $(4562,2281)$ |
| 17 | $(138,69)$ | 315 | $(2522,1261)$ | 581 | $(4650,2325)$ |
| 24 | $(194,97)$ | 326 | $(2610,1305)$ | 584 | $(4674,2337)$ |
| 39 | $(314,157)$ | 327 | $(2618,1309)$ | 599 | $(4794,2397)$ |
| 50 | $(402,201)$ | 329 | $(2634,1317)$ | 600 | $(4802,2401)$ |
| 51 | $(410,205)$ | 339 | $(2714,1357)$ | 602 | $(4818,2409)$ |
| 56 | $(450,225)$ | 341 | $(2730,1365)$ | 611 | $(4890,2445)$ |
| 65 | $(522,261)$ | 344 | $(2754,1377)$ | 617 | $(4938,2469)$ |
| 71 | $(570,285)$ | 347 | $(2778,1389)$ | 621 | $(4970,2485)$ |
| 95 | $(762,381)$ | 350 | $(2802,1401)$ | 626 | $(5010,2505)$ |
| 96 | $(770,385)$ | 362 | $(2898,1449)$ | 654 | $(5234,2617)$ |
| 101 | $(810,405)$ | 375 | $(3002,1501)$ | 659 | $(5274,2637)$ |
| 107 | $(858,429)$ | 380 | $(3042,1521)$ | 660 | $(5282,2641)$ |
| 116 | $(930,465)$ | 396 | $(3170,1585)$ | 674 | $(5394,2697)$ |
| 122 | $(978,489)$ | 401 | $(3210,1605)$ | 677 | $(5418,2709)$ |
| 126 | $(1010,505)$ | 416 | $(3330,1665)$ | 690 | $(5522,2761)$ |
| 141 | $(1130,565)$ | 429 | $(3434,1717)$ | 707 | $(5658,2829)$ |
| 162 | $(1298,649)$ | 449 | $(3594,1797)$ | 725 | $(5802,2901)$ |
| 170 | $(1362,681)$ | 452 | $(3618,1809)$ | 731 | $(5850,2925)$ |
| 176 | $(1410,705)$ | 462 | $(3698,1849)$ | 732 | $(5858,2929)$ |
| 186 | $(1490,745)$ | 471 | $(3770,1885)$ | 735 | $(5882,2941)$ |
| 212 | $(1698,849)$ | 474 | $(3794,1897)$ | 737 | $(5898,2949)$ |
| 234 | $(1874,937)$ | 491 | $(3930,1965)$ | 759 | $(6074,3037)$ |
| 249 | $(1994,997)$ | 509 | $(4074,2037)$ | 764 | $(6114,3057)$ |
| 260 | $(2082,1041)$ | 527 | $(4218,2109)$ | 765 | $(6122,3061)$ |
| 267 | $(2138,1069)$ | 530 | $(4242,2121)$ | 777 | $(6218,3109)$ |
| 269 | $(2154,1077)$ | 536 | $(4290,2145)$ | 782 | $(6258,3129)$ |

# Chapter 11

# Conclusions and Future Work

In this thesis we have seen techniques to construct Parseval frames using finite groups. The use of group representations has been one of the key tools to understand frame theory. Using group representations, we have seen a construction of uniform tight frames. We are able to construct *Parseval frame vectors* for representations unitarily equivalent to subrepresentations of the left regular representation. This has provided us an insight into applications using character theory of groups. We are able to compare the behavior of two such frames in applications using characters of group representations. There is scope to characterize equiangular tight frames of the type $\{\pi(g)v\}_{g\in G}$ for $\mathbb{C}^k$ using the results seen in Chapter 4. This can also lead to solve the problem of the existence of finite Gabor equiangular tight frames. Also, in Chapter 4, we compare two Parseval frame vectors in terms of the behavior of the corresponding Parseval frames in applications. To find a vector $v \in \mathbb{C}^k$ that minimizes the maximum correlation between the frame elements can be examined further using characters of group representations.

We have demonstrated a novel way to construct equiangular tight frames by taking subsets of groups having certain properties. We have seen necessary and sufficient conditions for a group to give rise to a real equiangular tight frame. As a result, we are able to show that a lot of real equiangular tight frames are associated with signature sets and quasi-signature sets. We observed a new correspondence between signature sets of real equiangular tight frames of the type $(n, \frac{n-\sqrt{n}}{2})$ and reversible Hadamard difference sets. Finding groups that admit reversible Hadamard difference sets is an active area of research and can be explored further via this correspondence. We also discovered a relationship between Artin's conjecture [4] in the case of $a = 2$ and equiangular tight frames of the type $(p+1, \frac{p+1}{2})$ where $p$ is a prime of the form $p \equiv 5 \pmod 8$.

From the results seen in Chapters 5 to 9, we see that many but not all equiangular tight frames arise from groups. When they do arise, then we know from [11] that there are at most finitely many such frame equivalence classes and hence the problem of determining equiangular tight frames is reduced to the problem of finding representatives for each equivalence class and determining which one of these finitely many equivalence classes is optimal. Finding equivalence classes of frames using groups is another important area of investigation that can be explored in future research.

In the case of cube root equiangular tight frames, we have shown that the $(9, 6)$-cube root equiangular tight frame arises naturally from the group of quaternions. Our methods extend to show that there exists a $(45, 12)$ equiangular tight frame.

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