A FINITE-ELEMENT APPROACH FOR PRICING SWING OPTIONS UNDER STOCHASTIC VOLATILITY

A Dissertation

Presented to

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In Partial Fulfillment

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Doctor of Philosophy

By

Muhu Wang

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Abstract

Option pricing plays an important role in financial,energy, and commodity markets. The Black-Scholes model is an indispensable framework for the option pricing. This thesis studies the pricing of a swing option under stochastic volatility. A swing option is an American-style contract with multiple exercise rights. As such, it is an optimal multiple-stopping time problem. In this dissertation, we reduce the problem to a sequence of optimal single stopping time problems. We propose an algorithm based on the finite element method to value the option. In real-world applications, volatility is typically not a constant. Stochastic volatility models are commonly chosen for modeling dynamic changes of volatility. Here we use the finite element approach to handle this added complication and present numerical results. For benchmark comparisons, we develop Monte Carlo methods to simulate the swing option under stochastic volatility. We compare the results obtained from both approaches and demonstrate that the finite element method is accurate and efficient, whereas the Monte Carlo method is easy to implement.

Contents

Chapter 1

Introduction

1.1 Preface

A large body of literature on option pricing has emerged in the past thirty years. The earliest model was proposed by Louis Bachelier in 1900. In 1973, Fischer Black, Myron Scholes published their milestone paper: *The Pricing of Options and Corporate Liabilities*[10]. In that paper, they gave the famous Black-Scholes model and the associated Black-Scholes equation, which has become an indispensable tool for pricing options in continuous time. The Black-Scholes equation is a second-order parabolic differential equation. Unfortunately, only for some limited cases, such as a European call/put option, this partial differential equation (PDE) has an analytical solution. For most applications, the corresponding Black-Scholes equation has no analytical solution. Numerical methods or simulation methods are needed to calculate the approximate solution. Independent of Black and Scholes, at about the same time, Robert Merton proposed a similar approach to study the valuation of contingent claims. His work is encapsulated in a treatise entitled: *Continuous Time*

Finance[48].

In this dissertation, we consider swing options, used commonly in energy markets - especially in the power sector. Since power prices often change rapidly, the assumption that the volatility of the underlying asset is constant is too simplistic to match the reality as evidenced by the market data. As an alternative, it is common to assume that the volatility changes over time. In this dissertation, we investigate swing options under stochastic volatility. Specifically, we will explore the application of the finite element method (FEM) for the numerical solution of swing options under stochastic volatility. We will compare the results obtained from such endeavors with those found from Monte Carlo simulations.

The dissertation is organized as follows: In the remaining of this chapter, we give a brief account of the option pricing problem. We focus on the Black-Scholes model and the associated partial differential equation (PDE). This line of research has attracted an inordinate amount of attention in computational finance. We also review numerical methods for American options and swing options.

In Chapter 2, we survey the existing methods for pricing swing options. Most of the methods for handling swing options are recursive in nature, i.e., in the spirit of dynamic programming. They include Monte Carlo methods, tree-based methods, or PDE approaches. We will survey some notable work using these approaches. More recently, stochastic programming has been considered for pricing swing options. We will also include a brief exposition about the method. Finally, we introduce the transform-based approach for pricing swing options under constant volatility. The results obtained from using this approach will be compared with those found from applying the FEM given in Chapter 4.

In the third chapter, we propose a Monte Carlo approach for pricing swing options. We first apply the generalized Least-Square Monte Carlo method to American options, then extend this method to a swing option under stochastic volatility. Through simulations, we obtain the approximate solutions for swing options under two different stochastic volatility models, i.e., the Stein-Stein's model and the Heston's Model. We also present convergence analyses for the Monte Carlo simulations. We will compare the simulation results with those based on the FEM developed in Chapter 4.

In the fourth chapter, we analyze the pricing of swing options based on Carmona and Touzi's paradigm[13]. There they showed that the pricing of a swing option can be converted to a sequence of European and American options. In this dissertation, we extend their approach to the case of a swing option under stochastic volatility. After that, we introduce the finite element method (FEM), and develop an algorithm to solve a swing option under stochastic volatility.

In the fifth chapter, we give the numerical results using FEM. We study two special cases as well as the general case. We also examine the convergence behaviors of the algorithm, and compared the results with those from Monte Carlo simulations.

In the last chapter, we give some concluding remarks and describe possible future work.

1.2 Options

In finance, an option is a financial contract between two parties, where the value of the option is derived from an underlying asset. The option does not represent ownership rights in the underlying asset. The simplest option, a European call option, gives the buyer the right, but not the obligation, to buy an agreed quantity of the underlying asset at a specified time (the maturity date) for a prescribed price (the strike price). The seller is obligated to sell the underlying asset if the buyer decides to exercise the right. In return, the buyer has to pay a premium to the seller to obtain the right. A put option is defined analogously.

Why do we need an option? The basic role about the option is to reduce exposure to the risk triggered by economic and political uncertainties, or caused by the volatility of the financial market. For a call option, the holder can decide whether to exercise the right or not at the maturity date. When in the money, he can exercise the right and get some profit. When out of the money, he can choose to give up the right to avoid losses. Since the financial market is volatile, especially in energy market, option holders can reduce the risk and avoid big losses.

The most commonly used options are European options and American options. For European options, holders are allowed to exercise their rights only on the option maturity date. For American options, holders can exercise their rights at any time prior to the maturity date. There is an option between the European option and the American option. Just like Bermuda is positioned between the European continent and the American continent, the Bermudan option is an option between a European option and an American option. It may be exercised only on some specified dates until maturity date. These options, as well as other options which have the similar payoff processes, are referred to as "vanilla options".

Generally speaking, an option which is not a vanilla option is an exotic option. Most of exotic options are more complicated than European options and American options. The payoff function of most exotic options depends on the path of the underlying asset price as well as its value at the maturity date. For example, an Asian option is a fully path-dependent option. The payoff function depends on the average of the underlying asset over a specific time period. Another example of the exotic option is the barrier option. For a barrier option, the right of the exercise is either activated (an in barrier) or forfeited (an out barrier) when the underlying asset price hits a prescribed value at some time before the maturity date. Exotic options have some advantages to reduce the risk of financial market. The disadvantage is that it is relatively complicated to calculate the price or set up a hedge strategy.

1.3 The Option Pricing Problem

The option pricing is an old problem, but it plays a prominent role in the financial market. The modern computational finance begins in the early period of the 20th century. In 1900, French mathematician Louis Bachelier[4] finished his Ph.D. dissertation: " Théorie de la Spéculation". This is the first paper that builds the option model based on the Brownian motion process. In his model, the non-dividend-paying stock price S_t follows the following stochastic differential equation:

$$dS_t = \sigma dW_t \tag{1.1}$$

where the W_t is a standard Brownian Motion process, and σ is the volatility of the stock price S_t.

Based on this model, with the assumption that the interest rate is zero, Bachelier derived the closed formula for pricing a call option. The price C_0 is

$$C_0 = (S - K)N\left(\frac{S - K}{\sigma\sqrt{T}}\right) + \sigma\sqrt{T}N'\left(\frac{K - S}{\sigma\sqrt{T}}\right) \tag{1.2}$$

where K is the strike price, T is the maturity date, and

$$N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{t^2}{2}}dt \tag{1.3a}$$

$$N'(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \tag{1.3b}$$

There are three main drawbacks of Bachelier's model.

(1) Under his model, the stock price may be negative, which is not true in the real financial market.

(2) The option price may be greater than the stock price, which makes the option useless.

(3) There is no discount factor.

Although there are three main drawbacks of his model, Bachelier's work is before his time and it took about sixty years before improvements were found. Case Sprenkle(1961) and Paul Samuelson (1964) improved Louis Bachelier's model respectively . They substituted dS_t with the stock return $\frac{dS_t}{S_t}$, i.e.,

$$\frac{dS_t}{S_t} = \rho dt + \sigma dW_t \tag{1.4}$$

where ρ is the average rate of the growth of a stock price.

By Itô's formula, we can rewrite (1.4) as

$$d\ln S_t = \left(\rho - \frac{\sigma^2}{2}\right) dt + \sigma dW_t \tag{1.5}$$

Although $\ln S_t$ may be negative, S_t is always positive. Furthermore, Case Sprenkle (1964) assumed the investors were risk averse and came up with a closed form formula for the price of a European call option.

$$C_0 = e^{\rho t} S N(d_1) - (1 - A) K N(d_2) \tag{1.6}$$

where A is the degree of the risk aversion, and

$$d_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln\frac{S}{K} + \left(\rho + \frac{\sigma^2}{2}\right)T\right] \tag{1.7a}$$

$$d_2 = d_1 - \sigma\sqrt{T} \tag{1.7b}$$

Based on Case Sprenkle's work, James Boness (1964) improved the formula (1.6) by discounting the payoff at the maturity date. Suppose S_t follows (1.4), then the revised formula for a call option is

$$C_0 = SN(d_1) - e^{-\rho T}KN(d_2) \tag{1.8}$$

where d_1 and d_2 are the same as in (1.7).

Samuelson (1965) improved Boness's work. He suggested that the average growth rate of a call option α was different from ρ, then the formula for a call option will be

$$C_0 = e^{(\rho-\alpha)T}SN(d_1) - e^{-\alpha T}KN(d_2) \tag{1.9}$$

where d_1 and d_2 are the same as in (1.7).

From the development of the option pricing, we can see formula (1.6), (1.8) and (1.9) are more and more close to the Black-Scholes-Merton's formula. The difference is that they are not risk-neutral. They rely on the average growth rate of a stock price ρ and the average growth rate of an option price α. Since different investors may have different expectations for the ρ and the α, the option price may be different according to different investors. Although these formulas were strictly derived, they are not practical in actual financial markets.

In 1973, Fischer Black and Myron Scholes published the breakthrough paper: "The pricing of options and corporate liabilities"[10]. In their paper, there is no ρ and α. They introduced the risk-free interest rate r as the expected return rate. To simplify the problem, they made the following assumptions:

- The market is arbitrage-free, i.e., an immediate risk-free profit is not possible.
- The market is liquid and the trade is possible at any time.
- The risk-free interest rate r is a positive constant.
- There are no transaction costs and taxes.
- Then underlying asset pays no dividends during the life of the option.
- All securities are perfectly divisible (i.e. it is possible to buy any fraction of a share).

Through the risk-neutral hedging strategy, they obtain the risk-neutral process for the underlying asset:

$$dS = rSdt + \sigma SdW \tag{1.10}$$

And the corresponding Black-Scholes equation is

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - rC = 0 \tag{1.11}$$

where C is the price of an option.

When applied to a European call option, i.e., the payoff process $C(T) = (S(T) - K)^+$, we can obtain a closed form solution for the call option price at time 0.

$$C_0 = SN(d_1) - Ke^{-rT}N(d_2) \tag{1.12}$$

where

$$d_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln\frac{S}{K} + \left(r + \frac{\sigma^2}{2}\right)T\right] \tag{1.13a}$$

$$d_2 = d_1 - \sigma\sqrt{T} \tag{1.13b}$$

Since the interest rate r is risk-free and does not depend on the preference of individual investors, the Black-Scholes formula brings all the investors to a risk neutral world and the expected return rate is just the risk-free interest rate. In this way, the option price only depends on the volatility of the stock price, the strike price, the time to the maturity date, the risk-free interest rate, and the underlying stock price. The advantage of the Black-Scholes' formula is that the option price is the same for every investor regardless of their individual risk aversion.

Black and Scholes also mentioned by holding a certain number of the underlying stocks, known as the delta, the risk of the short position can be completely dynamically hedged. This hedging strategy only depends on the stock price, the risk-free interest rate, the time to the maturity date, the strike price and the volatility of stock price. So it is also uniquely determined.

In 1973, Robert Merton extended the Black-Scholes equation to an option with the dividend paying stock[48].

$$\frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} + (r-q)S\frac{\partial F}{\partial S} - rF = 0 \tag{1.14}$$

where the q is the continuous dividend-pay rate. He also gave an closed form of the solution to an European call option.

$$C(S,t) = e^{-q(T-t)}SN(d_1) - e^{-r(T-t)}N(d_2) \tag{1.15}$$

where d_1 and d_2 are defined as

$$\begin{aligned} d_1(S,t) &= \frac{1}{\sqrt{T-t}}\left\{\ln\left(\tfrac{S}{K}\right) + \left(r - q + \tfrac{1}{2}\sigma^2\right)(T-t)\right\} \\ d_2(S,t) &= d_1(S,t) - \sigma\sqrt{T-t} \end{aligned} \tag{1.16}$$

In 1976, Merton extended the Black-Scholes model to the jump-diffusion model[48], which is a model for the stock price that has small continuous movements with large, randomly occurring jumps. When the jumps follow a Poisson process with the rate λ, he derived the closed form of a European call option under the jump-diffusion process.

$$C(S,t) = \sum_{n=0}^{\infty} \frac{1}{n!} e^{-\lambda'(T-t)} \left(\lambda' \left(T-t\right)\right)^n C_{BS}(S,t;\sigma_n,r_n) \tag{1.17}$$

where $C_{BS}(S,t;\sigma_n,r_n)$ is the formula for a standard Black-Scholes Model with the volatility σ_n, the risk-free interest rate r_n, and

$$\begin{aligned} \lambda' &= \lambda(1+k) \\ \sigma_n^2 &= \sigma^2 + \frac{n\sigma'^2}{T-t} \\ r_n &= r - \lambda k + \frac{n\log(1+k)}{T-t} \end{aligned} \tag{1.18}$$

In 1976, Black derived the Black-76 model, which is an application of the Black-Scholes model to a future contract.

Because of their breakthrough work, Myron Scholes and Robert Merton received the Nobel economics prize in 1997 (Black died before 1997, but he was mentioned as a contributor by the Swedish academy.).

There are some other developments of the Black-Scholes model. For example, in 1985, H. E. Leland studied the pricing for the European option when there are transaction costs. In 1993, Steven Heston studied the pricing of a European option under stochastic volatility, and gave a close-form solution for a call option. Interested readers can refer to [42, 31].

1.4 Numerical Solutions to the Option Pricing

Although Black, Scholes, and Merton derived the closed form solutions for European call options under different assumptions, most of the option pricing problems have no analytical solutions. For example, American options are optimal stopping time problems since the option holder can exercise the right at any time prior to the maturity date. As a consequence the holder does not know when to exercise the right *a priori* as a function of the time. Bensoussan (1984) and Karatzas (1988) provided an arbitrage argument of American options, and they showed that the option price F_t at time $t \in [0, T]$ was given by

$$F(S,t) = \text{ess} \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}^Q \left[e^{-r(\tau - t)} \phi(S_\tau) | \mathcal{F}_t \right] \tag{1.19}$$

where $\phi(S_\tau) = (K - S_\tau)^+$ for a put option with the strike price K, and $\mathcal{T}_{t,T}$ is the set of all stopping times in $[t, T]$.

Since the American option gives the holder more opportunities to exercise the right, the price of an American option should be higher than that of the corresponding European option.

$$F^{Am} \geq F^{Eur}$$

For American options, earlier exercises may happen.The possibility of an early exercise leads to a free boundary problem for the pricing of an American option. At each time t, there is a value $S^*(t)$ which marks the boundary between two regions: the exercise region and the continuation region. If $S(t)$ is less than $S^*(t)$, then the option should be exercised at time t; if $S(t)$ is greater than $S^*(t)$, then the option should be held.

There are several numerical methods to solve the free boundary problem for the associated American option. One is the partial differential equation (PDE) approach, see

[2], [24], [28], [32], and [40]. The main idea is that at each time step, we find the exercise boundary which splits the domain into two parts: the exercise region and the continuation region. When the stock price is in the continuation region, the option price satisfies the Black-Scholes equation; when the stock price is in the exercise region, the option should be exercised since it is worth more, then the option price in this region is the instant pay-off value. The numerical solution for an American option can be found once the exercise boundary is identified.

The tree-method is also applied to solve the pricing of European or American options. It follows the idea of the dynamic programming to solve the pricing problem. We will review this method in the chapter 2.

For the simulation method, it is hard to get an unbiased estimation for the pricing of an American option. In 2001, Longstaff and Schwartz derived a method to value an American option by a Least-Square Monte Carlo approach (LSM)[45]. This method gives a quite good simulation result, so it is widely used in the pricing for American-style options. Recently the application of this method has been extended to more general cases, such as swing options.

In this dissertation, we study the pricing of swing options, which are commonly used in the energy market, especially in the power sectors. A swing option is a generalized American style option and the pricing of a swing option is a multiple optimal stopping time problem. Since for a single stopping time problem, the closed-form solution does not exist, for the more complicated multiple stopping time problem, we expect that at best we may find an approximate solution for the swing option by numerical methods or Monte Carlo simulations. In [13], Carmona and Touzi gave a thorough analysis of the optimal multiple stopping problem. They proved the existence of the multiple exercise policies. Under the risk neutral paradigm, they also sketch a general solution strategy for the pricing of swing options. This will be the theoretical basis of our study. Furthermore, in [12] Carmona and Dayanik studied the optimal multiple stopping problem for a standard diffusion process. Recently, Wilhelm and Winter [56] developed an algorithm using the finite element method (FEM) to evaluate a swing option with up to five exercise rights. They compared their results with those obtained by Monte Carlo simulations and a lattice method. They concluded that the FEM performed well.

In the financial and energy markets, it is well known that volatility is not a constant. This phenomenon is substantially more pronounced in the power sector. The constant volatility assumption was used for modeling convenience. It usually yields only approximations to actual prices. In this dissertation, we allow possible volatility as a stochastic process. We first propose an approach based on the Monte Carlo simulations to compute the price of a swing put option under two different stochastic volatility models, then we use the FEM to obtain the numerical solution for the swing option price. The FE approach uses a key idea given in Carmona and Touzi [13], namely, transforming the optimal multiple stopping time problem to a single optimal stopping time problem. Here, we developed an algorithm to solve the swing option under the Stein-Stein's stochastic volatility model.

Chapter 2

Review of the Methods for the Pricing Swing Options

In this chapter, we will review some numerical or simulation methods for the pricing of swing options. In the financial market, the swing option or the swing contract is a financial tool to give the option holder a flexibility in the delivery amount and time, so it is extensively used in the energy market.

2.1 Swing Options

Due to the deregulation of the energy market in the past two decades, energy prices are determined by the free market, not by regulators. The different demands for energy consumption and limited storage facilities lead to widely varying prices, especially in the electricity market. Consumers have to find ways to control their expenses. This leads to the use of financial tools on energy prices to reduce market risk caused by sudden energy product price fluctuations. These financial tools allow investors to transfer the price risk to others who wish to profit from the risk. The most common financial tools are forwards, futures, swaps and options.

Swing options are commonly used in the energy market, particularly in the power sector and the natural gas industry. Since the energy market frequently experiences high volatilities, a swing option gives the option holder the flexibility in delivery with respect to both the timing and the amount of energy delivered. This flexibility can reduce the risks caused by the sudden fluctuations of the underlying asset price, hence the swing option is a useful financial tool for risk management.

Swing options may have different forms since the demand of the flexibility in the delivery time and the amount may be different, but they have similar computational models.

According to [44], a swing option contains a base load agreement. The base load agreement is a set of forward contracts with different expiry dates, t_j, $j = 1, \cdots, N$. Each forward contract f_j is based on a fixed amount of the commodity q_j. At each expiry date, the option holder has the right to purchase an excess amount or decrease the base load volume. This means that the amount of the commodity purchased at a predetermined price (i.e., the strike price) by the holder of the swing option can "swing" within a certain range $(q_j + \Delta_j)$. If the Δ_j is positive (negative), the option exercised by the holder at an opportunity time t_j is called upswing (downswing). Thus, an upswing is a buy and a downswing is a sell. From the above definition, we can see that a swing option has two components: a set of pure forward contracts and a fixed number of exercise rights which could be either a sell or a buy.

For a typical swing option, there usually are further restrictions:

1. The total number of upswings, U, and downswings, D, are limited, i.e. $U \leq N$, $D \leq N$, or $D + U \leq N$, for some fixed $N > 0$.

2. Between any two exercise rights, there is a minimum waiting time requirement, which is called the refraction time.

3. The swing option might include penalties if the overall volume purchased during the life of the contract exceeds a predefined quantity.

In the past twenty years, a number of analyses of swing options have been published. Some literatures focus on the theoretical setting of the swing option. Dahlgren and Korn[19] investigated the swing option on the stock market and they derived a continuous time model for the price of the swing option based on the Black-Scholes framework and dynamic programming. Carmona and Dayanik[12], and Carmona and Touzi[13] developed a mathematical framework for swing options as a sequence of European and American options. In Carmona and Dayanik's work, they include the constraint of a refraction time. Other literatures gave their numerical or simulation methods to evaluate the swing option. Our review focuses on the valuation methods of swing options. There are 4 main approaches to evaluating swing options.

1. The binomial or trinomial tree approach

2. The numerical PDE approach

3. The stochastic programming approach

4. The Monte Carlo simulation

We will review each of these methods in next sections. Finally, we will also introduce a transform method which is called Fourier Space Time-stepping (FST) method[33]. This method has be used to evaluate European/American options. We extend this method to swing options with a constant volatility and use it as a comparison method to the FEM in next chapter.

2.2 The Binomial or Trinomial Tree Approach

Both binomial and trinomial tree approaches are applications of the dynamic programming. These approaches discretize the time domain, and calculate the option price at each time step backward. Lari-Lavassani, Simchi and Ware (2001) suggested a binomial tree approach to evaluate the swing option[41], and Jaillet, Ronn and Tompaidis gave a trinomial tree approach in 2004[34].

The main ideas of binomial and trinomial are similar, i.e., use dynamic programming backward recursion in the discrete time domain. To explain the idea clearly, we suppose that at each time when the right is exercised, a fixed load q is delivered. The swing option price is a function of time, current underlying asset price, and number of exercise rights left.

$$F = F(S, t, u, d)$$

where u (d) is the number of upswing (downswing) rights left, and $0 \leq u \leq U$, $0 \leq d \leq D$.

At time $t_N = T$, the holder can maximize his profit by calculate the payoff function, and decides whether to exercise the upswing right, or the downswing right, if there are still rights left. If the holder chooses to give up the rights or there is no right left, then the option will be worthless. Then at the time T, the swing option value can be written as:

$$F(S, T, u, d) = \max\{q(S-K)^{+}\mathbf{1}_{u>0}, q(K-S)^{+}\mathbf{1}_{d>0}\} \tag{2.1}$$

where $\mathbf{1}_{u>0}$ is an indicator function.

Now we go backward to calculate the time t_i value, for $0 \le i \le N-1$. If at time t_i, there is no upswing or downswing exercise rights left, or at time t_i it is not optimal to exercise the rights, then the time t_i value of the swing option is just the conditional expected value of its discounted price at time t_{i+1}. We define this value as $V_c(S_i, t_i, u, d)$, then

$$V_c(S_i, t_i, u, d) = e^{-r(t_{i+1}-t_i)} \mathbb{E}_{t_i}[F(S_{t_{i+1}}, t_{i+1}, u, d)] \tag{2.2}$$

If there is still exercise rights and the holder find it is optimal to exercise an upswing right, then it will lead to an immediate cash flow $q(S-K)$ and the expected value of its discounted price at time step t_{i+1} with one upswing right less. We define this value as $V_U(S_i, t_i, u, d)$, then

$$V_U(S_i, t_i, u, d) = q(S_i - K)^+ + e^{-r(t_{i+1}-t_i)} \mathbb{E}_{t_i}[F(S_{t_{i+1}}, t_{i+1}, u-1, d)] \tag{2.3}$$

For the downswing case, we can apply the similar process, and we will obtain the corresponding $V_D(S_i, t_i, u, d)$ as

$$V_D(S_i, t_i, u, d) = q(K - S_i)^+ + e^{-r(t_{i+1}-t_i)} \mathbb{E}_{t_i}[F(S_{t_{i+1}}, t_{i+1}, u, d-1)] \tag{2.4}$$

These three values are the possible outcomes at time step t_i. To obtain the optimal profit, the swing option value at this time will be

$$F(S_i, t_i, u, d) = \max\{V_U(S_i, t_i, u, d)\mathbf{1}_{u>0}, V_D(S_i, t_i, u, d)\mathbf{1}_{d>0}, V_c(S_i, t_i, u, d)\} \tag{2.5}$$

Now we introduce the idea of the binomial tree method in [41]. Suppose at time t_i (for $i = 0, \cdots, N-1$), the underlying asset price is S_i, then at time t_{i+1}, S_{i+1} only has two possible outcomes, $S_{i+1} = u_0 S_i$ or $S_{i+1} = d_0 S_i$, where u_0 and d_0 are constants, and $u_0 > 1, 0 < d_0 < 1$. In most applications, we choose $u_0 \cdot d_0 = 1$. Note that the probability distribution in which price goes up or goes down should be risk-neutral.

Starting from the time step t_0, we can generate a tree of underlying asset values that spreads out step by step. At each node, it will be split into two nodes in the next time step. We can set up the corresponding risk-neutral probability distributions of all these spread prices for each time step.

Then we apply the dynamic programming backward recursive algorithm, calculate the swing option value at time T, then go backward to $t = 0$. At each time step t_i, we choose the maximum value of V_c, V_D, and V_U. Notice that for each (u, d) where $0 \leq u \leq U,\ 0 \leq d \leq D$, there is a separate tree, and at each node of the tree where there is an exercise opportunity, we have to decide which tree to swing to. In this sense, the tree for swing options is not just a tree, it is a forest of trees.

[41] studied the binomial tree method for one- and two-factor mean-reverting assets, they also gave some sensitivity and convergence analysis.

The trinomial tree method[34] is similar to the binomial tree method, the difference lies in that it allows for three outcome possibilities at each node, i.e. the current price at time t_i can go up, stay the same, or go down at time t_{i+1}. There is a corresponding risk-neutral probability distribution with these three movements. For each (u, d) there is a corresponding tree. It is also a forest tree method. [34] applied this method to a one-factor mean-reverting asset.

The tree approach is relatively easy to implement. If we use a large number of time steps, the numerical solution for the price of a swing option is accurate. Tree approaches can be easily extend to one- and two-factor models, but the corresponding number of nodes will increase exponentially, which will occupy a huge memories and require extensive CPU time. These disadvantages will make the computation slow, and sometimes the huge memories requirement will crash down the computer operation system.

2.3 The Numerical PDE Approach

Like tree methods, many of the numerical PDE approaches also are based on the dynamic programming backward recursive algorithms. They follow the similar recursive process as in tree approaches. The difference is that they calculate the option value at each time step based on the partial differential equation, not the tree nodes.

Many numerical PDE approaches are based on the finite difference method. The finite difference method is easy to implement and still has good approximate results. Since the corresponding PDE is a generalized heat equation, both the time domain and the spot price domain have to be discretized. Different time schemes have been applied, such as the explicit scheme, the implicit scheme, and the Crank-Nicolson scheme. And different time schemes will lead to different convergence rates.

Wegner[55] applied the finite difference method to calculate the price of a swing option with the underlying asset following a seasonal mean-reverting log-price model. He also explored the behavior of the greeks. The results show that the PDE approach can provide reliable values for the greeks, which is not always true for the tree methods

Kjaer[39] investigated the pricing of swing options using the finite difference method. The underlying asset follows a mean-reverting jump diffusion process. He proved the existence of an optimal exercise strategy and presented a numerical algorithm for the pricing problem. He solved the resulting partial integro-differential equations (PIDEs) by the finite difference method. The numerical results showed that adding jumps to a diffusion process may increase the swing option price.

Dahlgren[18] investigated a swing option on commodities under the additional constraint of a refraction time between two consecutive exercise times. He modeled the pricing problem as a continuous time stochastic impulse control problem. He also investigated the connection between the pricing problem and the Hamilton-Jacobi-Bellman quasi-variational inequalities (HJBQVI) and showed that the price of the option satisfied a system of HJBQVI.

Wilhelm and Winter[56] evaluate the price of a swing option using the finite element method. They based on the Carmona and Touzi's framework[13], which reduced the multiple stopping time problem to a sequence of single stopping time problems. So the pricing of a swing option requires the solving of a sequence of pricing European and American options. This algorithm is different from the previous dynamic programming. We will explain this algorithm in detail in chapter 4. The numerical results showed a smooth and stable behavior. They also compared their approach to the Monte Carlo method and the binomial tree method. The results showed the accuracies of both the finite element method and the tree method are better than that of the Monte Carlo method, and the finite element method and tree method are faster than the Monte Carlo method.

One advantage of the PDE approach is that this approach can calculate the option price for all the initial spot prices, while Monte Carlo methods or tree methods are designed to calculate the option price for only one initial spot price. So the PDE approach is the fastest among these three methods. The PDE approach can also obtain the option price for every time step, which can be used to derive the exercise boundary for the optimal exercise problem.

2.4 The Stochastic Programming Approach

More recently, the stochastic programming approach has been applied to the pricing of swing options in the power market. Haarbrücker and Kuhn[29] investigated the pricing of swing options in an electricity market driven by several exogenous risk factors. The underlying price process is a forward price with two exogenous risk factors. They established an exact pricing scheme and converted this pricing scheme to a computationally tractable stochastic programming based on three approximations: the aggregation of decision stages, the discretization of the probability space, and the reduction of the number of decision variables. Numerical results indicate that this approach achieves a high degree of precision, and can calculate a right lower bound on the option premium.

Their work also indicates that the stochastic programming approach performs well when the price process has several risk factors and state variables, while the Least Squares Monte Carlo method or dynamic programming approaches often require high computational efforts.

Baldick, Kolos and Tompaidis[5] also applied the stochastic programming to evaluate interruptible contracts from the point view of the retailers in the deregulated market. They provided a structural model to calculate the electricity prices based on the stochastic models for both the supply and the demand. Then they applied the stochastic programming method to price the interruptible contracts, and gave an optimal interruptible strategy.

The stochastic programming approach is different from the dynamic programming, or the PDE approach. It transfers the pricing problem to an optimization problem. This approach has some advantages when the price process has several factors. The drawback lies in that the algorithm is complicated and needs more implementation efforts. If the math model is complicated, it may be hard to find the global solution for the corresponding optimization problem.

2.5 The Monte Carlo Approaches

Monte Carlo methods are widely used in the computational finance to evaluate the prices of portfolios and options. The basic idea is to generate the samplings of the underlying asset, then calculate the values for each sampling, and finally obtain the average value. Since Longstaff and Schwartz[45] provided the Least Square method (LSM) to evaluate American options in 2001, Monte Carlo methods were extended to the swing options. Dörr's Master dissertation[21] may be the earliest application of LSM to the pricing of swing options with the two-factor mean reverting underlying assets. He also showed how to derive the exercise strategy for the swing option from the LSM method. We revised Dörr's method to our swing option settings and compared the simulation result with that of the FEM. The detailed algorithm of LSM will be discussed in chapter 3.

Meyer[49] developed Dörr's approach to the two different price processes using the Quasi-Monte Carlo method: the first one is the standard mean-reverting process of the logarithmic prices, and the second price process follows the Barlow model, which exhibits the feature of price strikes.

Figueroa[23] studied the interruptible-swing contracts under a mean-reverting jump-diffusion model with seasonality by the Monte Carlo method. He calculated the swing option based on Dörr's work and obtained the lower and upper bounds of the swing contract. He also provided a semi-analytical formula which is computationally efficient to calculate the lower bound.

Meinshausen and Hambly used the LSM and extended this approach to swing options based on the duality ideas from the pricing of American options[47]. This approach generates two sets of price scenarios from the same price process. They calculated the negative-biased value as well as the positive-biased value, and the difference between these two biased values is below 1.5%.

The Monte Carlo approach is very easy to implement. When the underlying asset price process has several risk factors, it is easy to simulate the price process by the Monte Carlo approach. The drawback lies in the low accuracy and the low computation speed.

2.6 The Transform Method

Jackson, Jaimungal and Surkov [33] described a Fourier Space Time-stepping (FST) method for the option pricing with Lv́vy jumps. This method also works for a mean-reverting process. In this dissertation, we applied this method to a swing option with a constant volatility under the Carmona and Touzi's framework[13]. We use this method as a comparison with the finite element method in chapter 5. So we explain the basic idea of this method in this section.

The Fourier transform method is a powerful tool to solve ordinary differential equations (ODE). Since the Black-Scholes equation is a PDE, Jackson, Jaimungal and Surkov introduced the FST method to convert the PDE problem to an ODE problem. Let $H(S,t)$ is the solution of a Black-Scholes equation for a European put option. Then the $H(S,t)$ satisfies the following PDE:

$$\frac{\partial H}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 H}{\partial S^2} + rS\frac{\partial H}{\partial S} - rH = 0 \tag{2.6}$$

Since the FST method can only be used to solve the PDE with constant coefficients, we have to do some transforms for the equation (2.6) before using the FST method. Define $x = \log S$, and $P(x,t) = H(e^x, t)$,, then $S\frac{\partial H}{\partial S} = \frac{\partial P}{\partial x}$ and $S^2\frac{\partial^2 H}{\partial S^2} = \frac{\partial^2 P}{\partial x^2} - \frac{\partial P}{\partial x}$, then the equation (2.6) becomes

$$\begin{aligned} &\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2\frac{\partial^2 P}{\partial x^2} + \left(r - \frac{\sigma^2}{2}\right)\frac{\partial P}{\partial x} - rP = 0 \quad && in\ \mathbb{R}\times(0,T] \\ &P(x,T) = \varphi(x) = (e^x - K)^+, && t = T \end{aligned} \tag{2.7}$$

We can rewrite the equation (2.7) as

$$(\partial_t + \mathcal{L})P = 0 \tag{2.8}$$

where $\mathcal{L}P = \frac{1}{2}\sigma^2\frac{\partial^2 P}{\partial x^2} + \left(r - \frac{\sigma^2}{2}\right)\frac{\partial P}{\partial x} - rP$, here $\mathcal{L}$ is called as infinitesimal generator.

Applying the Fourier transform to $\mathcal{L}P$ with respect to x, we obtain

$$\begin{aligned} \mathcal{F}[\mathcal{L}P](t,\omega) &= \left(i(r - \frac{\sigma^2}{2})\omega - \frac{\sigma^2\omega^2}{2} - r\right)\mathcal{F}[P](t,\omega) \\ &= \Psi(\omega)\mathcal{F}[P](t,\omega) \end{aligned}$$

where $\Psi(\omega)$ is the characteristic exponent.

After applying the Fourier transform to the equation (2.8), we obtain the following equation in the frequency domain

$$\begin{cases} \partial_t\mathcal{F}[p](t,\omega) + \Psi(\omega)\mathcal{F}[P](t,\omega) = 0 \\ \mathcal{F}[P](T,\omega) = \mathcal{F}[\varphi](\omega) \end{cases} \tag{2.9}$$

The equation (2.9) is an ODE problem with the initial boundary condition parameterized by ω. Given the value of $\mathcal{F}[P](T,\omega)$, the system is easily solved to find the value at any time $t < T$:

$$\mathcal{F}[P](t,\omega) = \mathcal{F}[P](T,\omega)e^{\Psi(\omega)(T-t)} \tag{2.10}$$

From the equation (2.10), we can get the value of $P(x,t)$ by the inverse Fourier transform

$$P(x,t) = \mathcal{F}^{-1}\left\{\mathcal{F}[P](T,\omega)e^{\Psi(\omega)(T-t)}\right\}(x) \tag{2.11}$$

Since a European option is path-independent, the price can be obtained in one step by directly applying the equation (2.11), so the numerical algorithm for a European option is very straightforward.

To solve an American option, at each time step t_i, we enforce the constraint $P(x,t) \geq P(x,T)$. Consider a partition of the time interval $[0,T]$ into a finite mesh of time steps $\{t_m | m = 0, \ldots, M\}$, where $t_m = m\triangle t$, and $\triangle t = T/M$. Define $P^m = P(x, t_m)$. We first calculate the price of P^M, then go backwards. For each time step t_m, $0 \leq m \leq M$, we do the following calculations:

$$\begin{aligned} P^{m-1} &= \mathcal{F}^{-1}\left[\mathcal{F}[P^m]e^{\Psi\triangle t}\right] \\ P^{m-1} &= \max(P^{m-1}, P^M) \end{aligned} \tag{2.12}$$

Notice that the American option is path-dependent, so we have to calculate the price for each time step before we obtain the initial time price for the American option.

Once we solve the European option and the American option respectively, using the framework by Cormona and Touzi, we can extend this method to a swing option with a constant volatility. The FST method is easy to implement and the numerical result is accurate. The drawback is that it is limited to the PDE with constant coefficients. If we cannot convert the original PDE to this form, for example the swing option under a stochastic volatility model, we cannot directly apply this method.

Chapter 3

Monte Carlo Approaches for Pricing Swing Options

3.1 Introduction

The Monte Carlo Method is a class of stochastic techniques used in the scientific computing. It is based on using repeated random sampling experiments to provide approximate solutions to a variety of mathematical problems. The approximation is usually given as the average value of the samples whose mathematical expectation is equal to the exact value. This method is especially suited for the calculation on the computer. Compared with other computational methods, the Monte Carlo method has several advantages. First, it is often used when other methods are hard or more costly to compute an exact result. Second, it is conceptually very simple and is easy to implement on the computer. And third, its convergence rate is independent of the dimension d of the underlying random variables.

The main disadvantage of the Monte Carlo methods is the slowness of convergence, especially in the low dimensions. To obtain one more decimal digit of the precision, this method needs 100 times the sample size. Thanks to the fast development of the computer technology, we can do a large number of random sampling experiments in a short time and get a good approximate solution. Nowadays, Monte Carlo methods are widely used in statistics, physics, economics, mathematics, and finance

In computational finance, a typical problem is to estimate the price of a certain option, or evaluate the sensitivities. These problems can finally be converted to an expectation of a certain random variable. In most cases, the distribution of this random variable is very complicated and it is hard to compute the expectation using traditional numerical methods. Using Monte Carlo methods, we generate the random sampling experiments from the certain specified probability distribution of the random variable, calculate the value of the payoff function for each sampling experiment and compute the average value over the range of the payoff outcomes to obtain the final result.

In the following section, we will introduce pricing European options using Monte Carlo methods, then we introduce the Least-Squares Monte Carlo approach to evaluate American options [45]. After that, we review the Monte Carlo algorithm for swing options provided by Dörr[21] and improve this algorithm to fit for the swing option under two different stochastic volatility models: the Stein-Stein's model and the Heston's model. We study the behavior of these two models.

3.2 Monte Carlo Method for Option Pricing

3.2.1 European Options

The Monte Carlo algorithm for pricing European options is a typical application. Let $S(t)$ be the stock price at time t. We consider a European put option with the strike price K and the maturity date T. The current time is $t = 0$ and the current stock price $S(0)$ is known. Suppose $S(t)$ follows the Geometric Brownian Motion process under the risk-neutral measure Q

$$dS(t) = rS(t)dt + \sigma S(t)dW(t) \tag{3.1}$$

where $W(t)$ is a Brownian Motion process.

The solution of the above stochastic differential equation is

$$S(T) = S(0)\text{exp}\left((r - \frac{1}{2}\sigma^2)T + \sigma W(T)\right) \tag{3.2}$$

More generally, for any $0 \leq t \leq T$

$$S(t) = S(0)\text{exp}\left((r - \frac{1}{2}\sigma^2)t + \sigma W(t)\right) \tag{3.3}$$

Since $W(t)$ is normally distributed with mean 0 and variance t, we can substitute it with $\sqrt{t}Z$, where Z is normally distributed with mean 0 and variance 1.

Let $\mathcal{T}$ be a partition of the time domain such that $0 \leq t_0 < t_1 < \cdots < t_n = T$. Since the increments of W are independent and normally distributed, we can derive a procedure to simulate the values of S at t_i for $i = 1, 2, \cdots, n$.

$$S(t_i) = S(t_{i-1})exp\left((r - \frac{1}{2}\sigma^2)(t_i - t_{i-1}) + \sigma\sqrt{t_i - t_{i-1}}Z_i\right) \tag{3.4}$$

Let $F(S,t)$ be the price of a European Put option, then $F(S,t)$ satisfies the Black-Scholes equation

$$\frac{\partial F(S,t)}{\partial t} + rS\frac{\partial F(S,t)}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 F(S,t)}{\partial S^2} - rF(S,t) = 0 \tag{3.5a}$$

$$F(S,T) = \max(K - S(T), 0) \tag{3.5b}$$

The above Black-Scholes equation is a parabolic equation, so the solution to (3.5) is

$$F(S,t) = E^Q_{S,t}\left[e^{-r(T-t)}(S(T) - K)^+\right] \tag{3.6}$$

Suppose there are m sample paths. Based on (3.6), we can calculate the option price F_i for $1 \leq i \leq m$, then the mean value $\hat{F}$ of all these option prices is the value for the European option. Note that $\hat{F}$ is an unbiased estimation of $F(S,0)$, and it is also a consistent estimation, i.e., as $m \to \infty$,$\hat{F} \to F(S,0)$ with probability 1.

Since for a one-dimension European call or put option we can obtain the exact solution, the Monte Carlo method is not a competitive method for one-dimension European options. It will have advantages for multi-dimension European options, especially when these underlying assets are correlated.

Here we give an example of the European put option with parameters as following:$K = 100$, $S(0) = 100$, $T = 1$, $r = 0.05$, $\sigma = 0.3$. The exact price of the option is 9.3542. Using the Monte Carlo method, we choose three different numbers for the sample paths, the simulation results are in table 3.1.

From this table, we can see that the convergence rate of the Monte Carlo method is not fast for the one-dimension problem.

Table 3.1: Monte Carlo methods for European put option

number of sampling	option price	error
100	10.1391	0.7849
10000	9.2548	0.0994
1000000	9.3558	0.0071

3.2.2 American Options

American options are path-dependent options with one early exercise right. We can maximize the value of an American option by exercising this right optimally. There is a difficulty for the Monte Carlo method. Since the determination of the optimal exercise time depends on an average over the future events, the Monte Carlo simulation for an American option has a "Monte Carlo on Monte Carlo" feature that makes it computationally complicated[11].

There are some Monte Carlo methods for pricing American options. Among them, the most commonly used algorithm is the Least Squares Monte Carlo method (LSM) derived by Longstaff and Schwartz[45] in 2001. We explain the LSM briefly here. For details, readers can refer to Longstaff and Schwartz's paper. We use the Bermudan option to approximate the American option since early exercise is only allowed at discrete times $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = T$. Applying the idea of dynamic programming, beginning from t_n to t_0, at each time t_k, we need to compare the payoff from the immediate exercise with the continuation value, which is the conditional expectation of the option payoff with respect to the risk-neutral pricing measure Q. The basic idea of the LSM is to use the least squares regression on a finite set of basis functions to approximate the continuation values.

The steps of a LSM algorithm is as following:

Step1: Generate a certain number of the sample paths, store the stock prices and exercise payoff values at each time step. At time $t_n = T$, for each path, set the cash value as the corresponding exercise payoff value.

Step2: At each time step t_k, where $1 \leq k < n$, for each path whose early exercise payoff is great than 0, i.e., when it is in the money, we calculate the sum of the discounted cash value from t_{k+1} to t_n, perform a least square regression of the sum on a finite set of basis functions. We can get the coefficients for the basis functions.

Step3: Using these coefficients, we can calculate the continuation value at t_k for each path where the early exercise payoff is greater than 0.

Step4: For each path, compare the continuation value with early exercise payoff. If the early exercise payoff is larger, then it is optimal to exercise at t_k, and the cash value at t_k is the early exercise payoff value. At the same time, set all cash values at t_i zero, where $k+1 \leq i \leq n$. If the continuation value is larger, it is not optimal to exercise at this moment, and the cash value at t_k is set to zero.

Step5: At time $t_0 = 0$, calculate the discounted cash value for each path, then find the mean value for all sample paths. This mean value is the estimation of the price for the American option.

Clement, Lamberton and Protter[17] proved the convergence of the LSM. Since the convergence rate of the Monte Carlo method is slow, we should use a large number of sample paths to get a good approximation. The accuracy of the LSM also depends on the choice of basis functions. Polynomials of $1, S, S^2, \cdots, S^m$ for some small value of m are a popular choice.

Here we use the LSM to evaluate an American put option. In our simulation, we partition the time domain into 10 subintervals. We use 10 different seeds and for each seed, we use 4,000 simulations. The basis functions are 1, S, S^2. We compare with the FST method, in which there are 400 mesh points in the frequency domain and 1000 time steps.

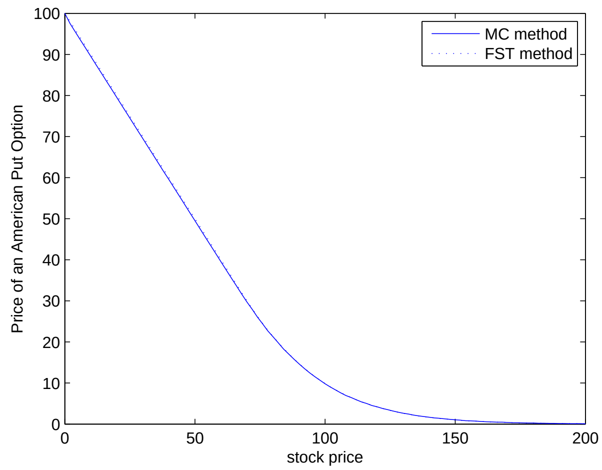

Figure 3.1: The price of the American put option

From the comparison, we see that the LSM provides quite good simulation results. It is relatively easy to implement. The main disadvantage of the LSM is that the least square technique makes the LSM slower than the FST method. In our simulation, it took the FST method 4.96 seconds while for the Monte Carlo method, it took 65 seconds.

3.3 Monte Carlo Methods for Swing Options with a Constant Volatility

Swing options are a kind of American-style options, so we can use the idea of the LSM to evaluate the swing option. In his Master dissertation, Dörr[21] provided an extension of the LSM to calculate the swing option and find the exercise strategy. He applied this approach to the one-factor and the two-factor mean reverting price processes.

The difference between swing options and American options is the number of early exercise rights. This will make the LSM for swing options more complicated. The difficulties lie in the calculation of the immediate exercise values and the rearranging of the cash flow values. The immediate exercise value is not just the payoff function value, but the sum of the payoff and the swing option value with one less exercise rights. The rearranging of the cash flow also requires the information of the cash flow matrix of the swing option with one less exercise right[21]. For the detailed algorithm, readers can refer to Dörr's dissertation.

Here we follow Dörr's extended LSM. We modify this extended LSM to fit for our models. First, we consider a swing option with a constant volatility. Suppose S_t follows model (3.1). Choose $K = 100$, $r = 0.05$, $\sigma = 0.3$, $\delta = 0.1$, and $T = 1$. We simulate the swing put option at the money with exercise rights from 1 to 3. In our simulation, there are 10 time steps. We use 10 different seeds and for each seed, we use 2,000 simulations. The basis functions are 1, S, S^2. We compare the simulation results with the numerical results of the FST method, where the frequency steps are 1000, and the time steps are 400.

From Table 3.2, we can see that the extended LSM works well for the swing option with a constant volatility when at the money. We also compare the computing time for each method. When there are 2 exercise rights, it took the FST method 0.2652 seconds

Table 3.2: Swing put option prices at the money

number of exercise rights	FST	Monte Carlo [stand.dev]
$p = 1$	9.8594	9.8513 [0.13]
$p = 2$	19.2533	19.2296 [0.27]
$p = 3$	28.1559	28.1584 [0.33]

to get the numerical result for a single stock price point, while for the extended LSM, it took 1.7 seconds to get a single stock price point simulation result. So the extended LSM is slower than the FST method.

We also study the behavior of the extended LSM at other stock price values, and compared the results with those of the FST method. In these cases the number of exercise rights is 3. From the table 3.3, we can see the extended LSM for the swing option provides a good approximation solution.

Finally we study the convergence behavior for this Monte Carlo method when the stock price is at the money. We use the numerical result in [56] as a benchmark, which uses 4000 mesh points for the stock prices and 1000 time steps. These swing option prices are $F^{(1)}(100, 0, 0)) = 9.8700$, $F^{(2)}(100, 0, 0)) = 19.2550$, and $F^{(3)}(100, 0, 0)) = 28.1265$. Let M be the number of simulation paths. The unit of computing time is the second.

Table 3.3 shows the simulation behavior of the Monte Carlo method for pricing a swing option under the constant volatility. The computing time in Table 3.4 is the time needed to calculate the price of a single spot price. From this table, we can see that as the number of the sample paths increases, the differences between the simulation results and the benchmark values will decrease to 0. In the next chapter, we will also see that compared with the FEM, the computing speed of the Monte Carlo method is much slower.

Table 3.3: Prices of swing option with a constant volatility

Stock Price	Volatility	FST	Monte Carlo [stand.dev]
80	0.16	58.5950	57.2635 [0.10]
90	0.16	30.7382	30.3602 [0.18]
100	0.16	13.0872	12.9357 [0.20]
110	0.16	4.7592	4.7406 [0.10]
120	0.16	1.5027	1.4646 [0.11]
80	0.40	70.2757	70.1338 [0.44]
90	0.40	52.6668	52.5678 [0.31]
100	0.40	38.9997	38.8949 [0.34]
110	0.40	28.6099	28.5449 [0.65]
120	0.40	20.8502	20.6345 [0.53]

Table 3.4: Absolute errors and the computing time using the Monte Carlo simulation for a swing option under the constant volatility

	$M = 2000$		$M = 4000$		$M = 8000$	
Rights	Error(std)	Time	Error(std)	Time	Error(std)	Time
$p = 1$	0.0605(0.1276)	1.48	0.0452(0.0959)	2.25	0.0283(0.0838)	2.91
$p = 2$	0.1132(0.2843)	1.77	0.0906(0.2190)	3.16	0.0490(0.1032)	5.38
$p = 3$	0.1362(0.3888)	3.24	0.0967(0.2621)	4.88	0.0647(0.1554)	7.32

3.4 Monte Carlo Methods for Swing Options under Stochastic Volatility

Now we consider a swing option under stochastic volatility, which has two sources of randomness. This is a two-dimension problem. One advantage of the Monte Carlo methods is that it is easy to implement for the multi-dimension model. The computational complexity increases almost linearly in the number of the dimension. In this section, we study the behavior of the swing option under two different stochastic volatility models.

3.4.1 The Stochastic Volatility Model

Of all the parameters in the Black-Scholes model for the option pricing, the volatility is the only parameter that cannot be directly observed from the market. In the Black-Scholes formula, the volatility is assumed to be a constant. The historic volatility or the implied volatility is typically used as an approximation. The historic volatility gives an average volatility for the given time interval. It does not reflect the future volatility movement. It is well known that the implied volatility exhibits the 'smile' effect, i.e., at-the-money options tend to have a lower implied volatility than in-the-money or out-of-the-money options. In assessing the volatility of underlying assets for the option pricing, traders almost always adjust the volatility value according to their own experiences and expectations about the market. This process is nevertheless ad-hoc. Taking the time varying nature of the volatility change in a formal framework invariably renders the model more realistic.

There are several ways to model the change of the volatility value over time. The GARCH model and its variants are used by many practitioners. Another choice is the stochastic volatility model. In a stochastic volatility model, it is commonly assumed that the volatility follows a mean-reverting Brownian Motion Process. In [20], Danielsson compared stochastic volatility models with GARCH models and found that stochastic volatility models provide a better estimation than GARCH models and observed that stochastic volatility models could capture the market behavior more accurately than GARCH models. So in our study, we assume the swing option is under the stochastic volatility paradigm.

Under the risk neutral measure Q, the price process S_t of the underlying asset and the volatility process σ_t satisfy the following SDEs:

$$dS_t = rS_t dt + \sigma_t S_t dW_{1t} \tag{3.7}$$

$$\sigma_t = f(Y_t) \tag{3.8}$$

$$dY_t = \mu(t, Y_t)dt + \hat{\sigma}(t, Y_t)d\hat{W}_t \tag{3.9}$$

where $(\hat{W}_t)$ is a Brownian Motion which may be correlated with W_{1t} with a correlation coefficient ρ. Thus $\hat{W}_t$ can be written as a linear combination of W_{1t} and another independent Brownian motion W_{2t}

$$\hat{W}_t = \rho W_{1t} + \sqrt{1-\rho^2}W_{2t} \tag{3.10}$$

Stochastic volatility models have appeared in the literature for more than twenty years. In Table 3.5, we summarize the parameter specifications for (3.8) and (3.9) used in several commonly cited models.

The Stein-Stein's model and the Heston's model are many times studied in the literatures. So we will study the behaviors of the swing option under these two models using the Monte Carlo method.

Table 3.5: Stochastic volatility models

	$f(y)$	$\mu(t,y)$	$\hat{\sigma}(t,y)$	ρ
Ball and Roma (1994)	$\sqrt{y}$	$\alpha(m-y)$	$\beta\sqrt{y}$	$\rho=0$
Heston(1993)	$\sqrt{y}$	$\alpha(m-y)$	$\beta\sqrt{y}$	$\rho\neq 0$
Stein and Stein(1991)	$\lvert y\rvert$	$\alpha(m-y)$	β	$\rho=0$
Scott(1987)	e^y	$\alpha(m-y)$	β	$\rho=0$
Hull and White(1987)	$\sqrt{y}$	μy	βy	$\rho=0$

3.4.2 The Stein-Stein's Model

The Stein-Stein's stochastic volatility model has the following dynamics:

$$\begin{aligned} dS_t &= rS_t dt + \sigma_t S_t dW_{1t} \\ \sigma_t &= |Y_t| \\ dY_t &= \alpha(m - Y_t)dt + \beta dW_{2t} \end{aligned} \tag{3.11}$$

where W_{1t} and W_{2t} are two independent Brownian motions.

We set the parameters as follows: the risk free rate of interest $r = 0.05$, the strike price $K = 100$, the maturity date $T = 1$, $\alpha = 1$, $m = 0.16$, and $\beta = \frac{\sqrt{2}}{2}$.

In Figure 3.2, we plot the spot price scenarios for a constant volatility model and a stochastic volatility model. For the stochastic volatility model, we choose the starting spot value $S_0 = 100$, the starting volatility $\sigma_0 = 0.4$. For the constant volatility, $S_0 = 100$, $\sigma_0 = 0.4$. For these two models, they share the same randomness for W_{1t}. Notice that for the stochastic volatility model, there is another independent randomness, i.e., W_{2t}.

From Figure 3.2, we can see that this added randomness makes the spot price more volatile.

Following the Stein-Stein's model, we can simulate the stock price for this model at t_i for $1 \leq i \leq n$

$$Y(t_i) = Y(t_{i-1}) + \alpha(m - Y(t_{i-1}))(t_i - t_{i-1}) + \beta\sqrt{t_i - t_{i-1}}Z_{1i} \tag{3.12a}$$

$$\sigma_i = |Y(t_i)| \tag{3.12b}$$

$$S(t_i) = S(t_{i-1})exp\left((r - \frac{1}{2}\sigma_i^2)(t_i - t_{i-1}) + \sigma_i\sqrt{t_i - t_{i-1}}Z_{2i}\right) \tag{3.12c}$$

where Z_{1i} and Z_{2i} are two independent random variables following the standard normal distribution.

Once we obtain the sample path for the stock price, the rest process is the same as for constant volatility case. So its computational complexity does not increase too much.

In Table 3.6, we show some simulation results for the swing option with 3 exercise rights

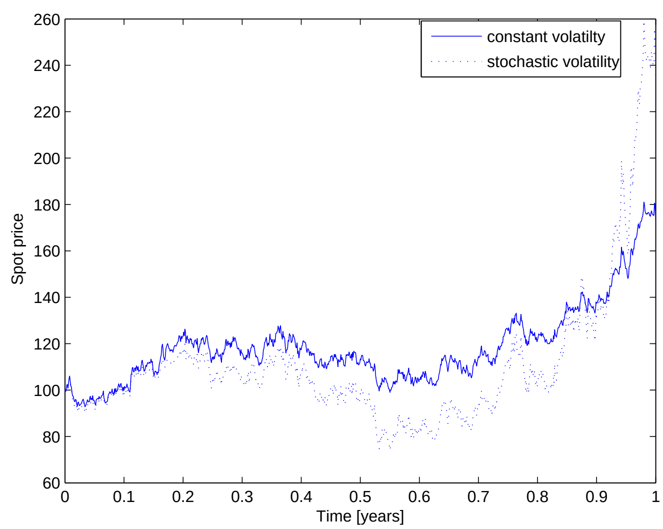

Figure 3.2: Spot price scenario of the model (3.1) and (3.7)

under stochastic volatility. We apply the same parameters as those in the American option under stochastic volatility case in the previous section. We use 10 different seeds. For each seed, there are 2000 sample paths, and for each sample path, there are 10 time steps.

Table 3.6: Prices of the swing option under stochastic volatility with 3 exercise rights

Stock Price	Volatility	Monte Carlo [stand.dev]
90	0.16	49.3582 [0.70]
100	0.16	35.9164 [0.40]
110	0.16	26.6868 [0.82]
90	0.40	57.2476 [0.51]
100	0.40	44.1738 [0.85]
110	0.40	34.7105 [0.57]

We will compare these simulation results with those from the FEM in the next chapter.

3.4.3 The Heston's Model

In the Stein-Stein model, the correlation coefficient $\rho = 0$, i.e., the two randomness sources are uncorrelated. In some applications, we need consider the case when the two randomness sources are correlated. Since the Heston's stochastic volatility model can deal with the case when $\rho \neq 0$, here we extend our study to the swing option under the Heston's model. The Heston's model is defined as following:

$$dS = rSdt + \sigma S dW_1 \tag{3.13a}$$

$$\sigma = \sqrt{Y} \tag{3.13b}$$

$$dY = \alpha(m - Y)dt + \beta\sqrt{Y}dW_2 \tag{3.13c}$$

where W_1 and W_2 are correlated with the correlation coefficient ρ.

The Monte Carlo algorithm for this model is similar to that of the Stein-Stein's model:

$$Y(t_i) = Y(t_{i-1}) + \alpha(m - Y(t_{i-1}))(t_i - t_{i-1}) + \beta\sqrt{Y(t_{i-1})(t_i - t_{i-1})}(\sqrt{1-\rho^2}Z_{1i} + \rho Z_{2i}) \tag{3.14a}$$

$$\sigma_i = \sqrt{Y(t_i)} \tag{3.14b}$$

$$S(t_i) = S(t_{i-1})exp\left((r - \frac{1}{2}\sigma_i^2)(t_i - t_{i-1}) + \sigma_i\sqrt{t_i - t_{i-1}}Z_{2i}\right) \tag{3.14c}$$

where Z_{1i} and Z_{2i} are two independent random variables following the standard normal distribution.

To explore the impact of the Heston's model, we study two cases. Firstly, when $\rho = 0$, we compare the behaviors of the Stein-Stein's model and the Heston's model. We use the same values for m, α, and β as above, and do simulations on the same random paths. In Figure 3.3, we plot the price for $\sigma = 0.16$ and $n = 3$.

Figure 3.3 shows that in the exercise region, the results of these two simulations agree well, but in the continuation region, the simulation result of the Stein-Stein's model is a little larger. So different stochastic volatility models do affect the pricing process. We have to choose the optimal model according the data behavior.

Secondly, we study the behavior of the Heston's model under different ρ values. We use the same parameters as above and simulate on the same random paths.

From the simulation results in Figure 3.4, we can see that the ρ does effect the pricing process. When $\rho > 0$, the price is less than that of $\rho = 0$. Furthermore, for the simulation results from $\rho = 0.5$ and $\rho = -0.5$, they are almost symmetric around the result from $\rho = 0$.

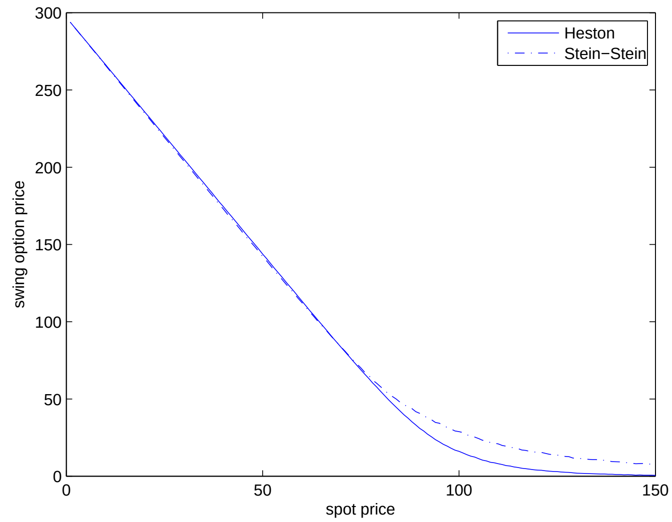

Figure 3.3: The behaviors of two different stochastic volatility models when $\rho = 0$

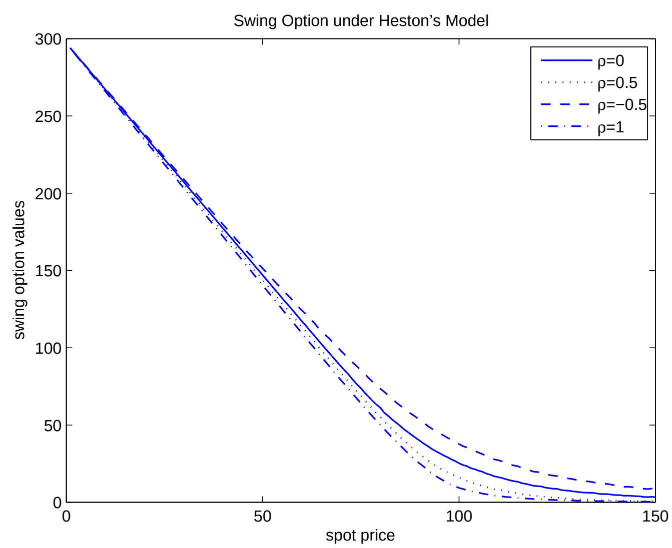

Figure 3.4: Swing option under Heston's model for different ρ

Chapter 4

Finite Element Method for Swing Options under Stochastic Volatility

In this chapter, we will introduce the basic theory about swing options based on the Carmona and Touzi's framework[13], i.e., a swing option can be converted to a sequence of single-optimal stopping time problems. We will also study the swing option under stochastic volatility.

4.1 Multiple Stopping Time Problem

Before we introduce the framework of the swing option, we give a strict definition of a swing option.

Definition 4.1: A swing option is a contract that gives the option holder the right to exercise up to p times at some epochs during the life of the option, where $p \in \mathbb{N}$ is a prespecified number. Between any two consecutive exercises, the delivery waiting time must be greater than a prespecified number δ, called the refraction time for a swing option. After each exercise, the option holder may receive a gain based on the specification of the payoff function.

In the commodity and energy markets, the requirement for the refraction time is an important contract constraint, since it prevents the holder from exercising all the rights at the same time, i.e., it prevents the case of a single optimal exercise when $p \geq 2$. Since an American option is a single-optimal stopping time problem, a swing option is a multi-optimal stopping time problem. In this sense, a swing option is a generalized American option.

The option holder may choose to exercise up to p times, but not obligate to exercise them at all. The holder may choose to exercise less than p times. Depending on the price movement of the underlying asset, the holder can manage the risk as well as maximize the gain.

In this section, we introduce the pricing of the standard swing option based on the work of [13, 56].

Let $(\Omega,\mathcal{F},P)$ be a complete probability space. and $\mathbb{F}=\{\mathcal{F}_t\}_{t\geq 0}$ be a filtration generated by a standard Brownian motion $(\hat{W}_t)_{t\geq 0}$. $\mathbb{F}$ is an increasing continuous family of the σ-algebras of $\mathcal{F}_t$. Let $S = \{S_t\}_{t\geq 0}$ be the risky underlying asset price which is adapted to the $\mathbb{F}$ filtration. It is the solution of the following stochastic differential equation:

$$dS_t = \mu(S_t, t)S_t dt + \sigma(S_t, t)S_t d\hat{W}_t \tag{4.1}$$

with initial value $S_0 = s$

Let the bank account process B_t be the price of a risk free asset such that

$$dB_t = r_t B_t dt, \qquad B_0 = 1$$

where r_t is an adapted process.

For this model there exists a risk-neutral probability measure Q, such that Q is equivalent to the probability measure P. Under the risk-neutral measure Q, the discounted price process $\tilde{S}_t = S_t/B_t$ is a martingale. Applying Girsanov's theorem, we get

$$W_t = \frac{\mu(S_t,t) - r_t}{\sigma(S_t,t}t + \hat{W}_t$$

W_t is a standard Brownian motion in (Ω,$\mathcal{F}$,Q), then S_t satisfies the following stochastic differential equation:

$$dS_t = r_t S_t dt + \sigma(S_t,t))S_t dW_t \tag{4.2}$$

Assuming the contract originates from time t, the swing option expires at time T. Let $\mathcal{T}_t^{(p)}$ be the sequence of an admissible stopping time for the swing option with up to $p \in \mathbb{N}$ exercise rights. Let the refraction time be $\delta > 0$. Using the definition in [56], the admissible stopping time set is defined as follows:

$$\begin{aligned}\mathcal{T}_t^{(p)} := \{\tau^{(p)} = (\tau_1, \tau_2, \cdots, \tau_p) \,|\tau_i \geq t \ for \ i = 1, \cdots, p \\ \tau_1 \leq T \quad a.s. \quad and \quad \tau_{i+1} - \tau_i \geq \delta \ for \ i = 1, \cdots, p-1\}.\end{aligned} \tag{4.3}$$

Assuming the payoff process of the swing option $\phi(S) : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies the integrability condition:

$$\mathbb{E}\{\phi(\bar{S})^\alpha\} < \infty \qquad \text{for some } \alpha \geq 1 \tag{4.4}$$

where $\phi(\bar{S}) = \sup_{t\geq 0} \phi(S_t)$ and $\phi(S_t) = 0$ for $t > T$.

Let $F^{(p)}(t,s)$ be the value of a swing option with up to p exercise rights, which starts at time t, with the starting asset value s, and the maturity date T. Under the risk-neutral measure Q, $F^{(p)}(t,s)$ is the supremum of the expected discounted payoff at each stopping time, i.e.

$$F^{(p)}(t,s) = \sup_{\tau^{(p)} \in \mathcal{T}_t^{(p)}} \mathbb{E}^Q \left[\sum_{i=1}^{p} e^{-r(\tau_i - t)} \phi\left(S_{\tau_i}\right) | S_t = s \right] \tag{4.5}$$

for all $t \in [0,T]$, and S_t satisfies the dynamics in (4.2).

Carmona and Touzi [13] proved the following existence of an optimal stopping time for the pricing process of a swing option.

Theorem 4.2 *Assume the filtration $\mathbb{F}$ is left continuous and every $\mathbb{F}$-adapted martingale has continuous sample paths. If the payoff process of the swing option $\phi(S_t)$ is continuous almost surely, and (4.4) holds, then for any $p \in \mathbb{N}$, there exists $\tau^* = (\tau_1^*, \cdots, \tau_p^*) \in \mathcal{T}_t^{(p)}$ such that*

$$F^{(p)}(t,s) = \mathbb{E}^Q \left[\sum_{i=1}^{p} e^{-r\left(\tau_i^* - t\right)} \phi(S_{\tau_i^*}) | S_t = s \right] \tag{4.6}$$

Proof: See Carmona and Touzi[13].

Applying the result of the Theorem 4.2, Carmona and Touzi reduced the optimal multiple-stopping time problem to a sequence of the optimal single stopping time problems.

Corollary 4.3 *For any $p \in \mathbb{N}$, $s \in \mathbb{R}^+$ and $t \in [0,T]$:*

$$F^{(p)}(t,s) = \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}^Q \left[e^{-r(\tau - t)} \Phi^{(p)}(\tau, S_\tau) | S_t = s \right], \tag{4.7}$$

with

$$\Phi^{(p)}(t,s) := \begin{cases} \phi(s) + e^{-r\delta} \mathbb{E}\left[F^{(p-1)}(t+\delta, S_{t+\delta}) | S_t = s\right] & if\ t \leq T - \delta \\ \phi(s) & if\ t \in (T-\delta, T] \end{cases} \tag{4.8}$$

When $p = 0$, there is no exercise right remaining, it follows $F^{(0)}(t,s) := 0$.

Proof: See Carmona and Touzi[13].

When the number of the exercise rights is 1, i.e., it is a single stopping time problem, the Corollary 4.3 gives the standard formula for an American option. For $p > 1$, the Corollary 4.3 states that the swing is an American option with a specific payoff function which is the value of an optimal stopping time problem with $p-1$ exercise rights. Notice that the refraction time limits the number of exercise rights until the maturity date T, we can get the following relationship for $p \geq 2$

$$F^{(p)}(s,t) = F^{(p-1)}(s,t) \quad \text{for } t \in (T-(p-1)\delta, T],\ s \in \mathbb{R}^+ \tag{4.9}$$

In [56] Wilhelm and Winter proved that the only price of a swing option with p exercise rights which is arbitrage free is given by (4.5).

Corollary 4.4 *The only price of a swing option with $p \in \mathbb{N}$ exercise rights, the payoff function ϕ and the maturity date T that does not create any arbitrage opportunities is given by:*

$$F^{(p)}(t,s) = \sup_{\tau^{(p)} \in \mathcal{T}_t^{(p)}} \mathbb{E}^Q \left[\sum_{i=1}^{p} e^{-r(\tau_i - t)} \phi\left(S_{\tau_i}\right) | S_t = s \right] \tag{4.10}$$

for all $(s,t) \in \mathbb{R}^+ \times [0,T]$

Proof: See Wilhelm and Winter[56].

Thus the arbitrage free price of a swing option can be determined by a sequence of the single optimal stopping time problems. Now we elaborate on the solution procedure. To begin with, in (4.7) we see that the value of the swing option with p exercise rights is the value of an American option with the payoff process $\Phi^{(p)}(\tau, S_\tau)$. Then (4.8) shows that the payoff process $\Phi^{(p)}(\tau, S_\tau)$ is the sum of a swing option payoff process and the value of a European option (in the parlance of dynamic programming, the two terms correspond to the immediate payoff and the value of the optimal return function in the subsequent stage). With regard to this European option, the payoff function is none other than the value of the swing option with $p-1$ exercise rights following the refraction time δ.

Based on the above analysis, we are able to compute the value of swing option with p exercise rights recursively. The algorithm is summarized below:

Assuming that the price of a swing option under the stochastic volatility model with m exercise rights has been calculated.

Step1: calculate the value of the corresponding European option with the payoff process defined by the price of the swing option with m exercise rights;

Step2: calculate the payoff process for $\Phi^{(m+1)}(\tau, S_\tau)$ using (4.8);

Step3: calculate the swing option with $m+1$ exercise rights using (4.7), and let $m = m+1$, stop if $m = p$; else go to Step 1.

4.2 Swing Options under Stochastic Volatility

In the previous section, we have shown that swing option can be reduced to a sequence of single optimal stopping time problems. We can calculate the value of swing options by recursively calculating the corresponding European option values and American option values. When we plan to determine the price of a swing option under stochastic volatility, we can use the similar process: calculate the corresponding European options under stochastic volatility and American options under stochastic volatility, we can then use (4.7) and (4.8) to compute the corresponding swing option under stochastic volatility.

Consider a European option under stochastic volatility with dynamics (3.7),(3.8) and (3.9). Suppose the maturity date is T and the payoff function is $g(S_T)$. At time t, let $F(S_t, Y_t, t)$ denote the price of the swing option when the price of the underlying asset is S_t

and the volatility process is at a level Y_t. There are two independent sources of randomness, i.e., W_{1t} and W_{2t}. To find an arbitrage-free price, we need to introduce another option with the maturity date T_1. Then we set up a self-financing hedged portfolio containing a_t shares of the underlying asset, and b_t options with the maturity date T_1

$$\Pi_t = F - a_t S_t - b_t F_1 \tag{4.11}$$

where S_t is the price of the underlying asset, and F_1 is the value of the corresponding option with the maturity date T_1.

To obtain a non-arbitrage price for the swing option, the dynamics of this self-financing portfolio should satisfy $d\Pi_t = r\Pi_t dt$, where r is the risk-free interest rate. For simplicity, assume that r is a constant.

Applying $It\hat{o}'s$ Lemma and using (3.7) and (3.9), we obtain

$$\begin{aligned}
d\Pi_t &= dF - a_t dS_t - b_t dF_1 \\
&= \left[\frac{\partial F}{\partial t} + \frac{1}{2}f^2(Y)S^2\frac{\partial^2 F}{\partial S^2} + \rho f(Y)\hat{\sigma}S\frac{\partial^2 F}{\partial S \partial Y} + \frac{1}{2}\hat{\sigma}^2\frac{\partial^2 F}{\partial Y^2}\right] dt \\
&\quad - b_t\left[\frac{\partial F_1}{\partial t} + \frac{1}{2}f^2(Y)S^2\frac{\partial^2 F_1}{\partial S^2} + \rho f(Y)\hat{\sigma}S\frac{\partial^2 F_1}{\partial S \partial Y} + \frac{1}{2}\hat{\sigma}\frac{\partial^2 F_1}{\partial Y^2}\right] dt \\
&\qquad + \left(\frac{\partial F}{\partial S} - b_t\frac{\partial F_1}{\partial S} - a_t\right) dS_t + \left(\frac{\partial F}{\partial Y} - b_t\frac{\partial F_1}{\partial Y}\right) dY_t
\end{aligned} \tag{4.12}$$

To remove the randomness induced by the diffusions so that $d\Pi_t$ only has dt term, we choose a_t, b_t such that

$$\frac{\partial F}{\partial S} - b_t\frac{\partial F_1}{\partial S} - a_t = 0$$

$$\frac{\partial F}{\partial Y} - b_t\frac{\partial F_1}{\partial Y} = 0$$

Since $d\Pi_t$ is driftless, we have

$$d\Pi_t = r\Pi_t dt = r(F - a_t S_t - b_t F_1)dt \tag{4.13}$$

Substituting the values of a_t and b_t into the equation (4.12) and (4.13), we obtain that

$$\begin{aligned}
&\frac{1}{\frac{\partial F}{\partial Y}}\left[\frac{\partial F}{\partial t} + \frac{1}{2}f^2(Y)S^2\frac{\partial^2 F}{\partial S^2} + \rho f(Y)\hat{\sigma}S\frac{\partial^2 F}{\partial S\partial Y} + \frac{1}{2}\hat{\sigma}^2\frac{\partial^2 F}{\partial Y^2} + rS\frac{\partial F}{\partial S} - rF\right] \\
&= \frac{1}{\frac{\partial F_1}{\partial Y}}\left[\frac{\partial F_1}{\partial t} + \frac{1}{2}f^2(Y)S^2\frac{\partial^2 F_1}{\partial S^2} + \rho f(Y)\hat{\sigma}S\frac{\partial^2 F_1}{\partial S\partial Y} + \frac{1}{2}\hat{\sigma}^2\frac{\partial^2 F_1}{\partial Y^2} + rS\frac{\partial F_1}{\partial S} - rF_1\right]
\end{aligned} \tag{4.14}$$

In the equation (4.14), the left hand side does not depend on T, and the right hand side does not depend on T_1, so the value of each side depends only on S, Y and t. Define the right hand side as $h(S, Y, t)$, then

$$\begin{aligned}
&\frac{1}{\frac{\partial F}{\partial Y}}\left[\frac{\partial F}{\partial t} + \frac{1}{2}f^2(Y)S^2\frac{\partial^2 F}{\partial S^2} + \rho f(Y)\hat{\sigma}S\frac{\partial^2 F}{\partial S\partial Y} + \frac{1}{2}\hat{\sigma}^2\frac{\partial^2 F}{\partial Y^2} + rS\frac{\partial F}{\partial S} - rF\right] \\
&\qquad = h(S, Y, t)
\end{aligned}$$

In most applications, we let $h(S, Y, t) = -(\mu(t, Y) - \Lambda(S, Y, t)\hat{\sigma}(Y, t))$, then we find the partial differential equation for a European option under the stochastic volatility model as following

$$\begin{aligned}
&\frac{\partial F}{\partial t} + \frac{1}{2}f^2(Y)S^2\tfrac{\partial^2 F}{\partial S^2} + \rho f(Y)\hat{\sigma}S\frac{\partial^2 F}{\partial S\partial Y} \\
&+\frac{1}{2}\hat{\sigma}^2\frac{\partial^2 F}{\partial Y^2} + rS\frac{\partial F}{\partial S} + (\mu - \Lambda\hat{\sigma})\frac{\partial F}{\partial Y} - rF = 0 \quad (0 \le t < T, S > 0, Y \in \mathbb{R}) \\
&F(S, Y, T) = g(S_T) \quad (t = T, S > 0, Y \in \mathbb{R})
\end{aligned} \tag{4.15}$$

where the function $g(S_T)$ is the initial condition, and $\Lambda(S, Y, t)$ represents the market price of the volatility risk. Sometimes it is also called the volatility risk premium.

Comparing with European options under stochastic volatility, the American options under stochastic volatility share the same partial differential equation and the same maturity date payoff process. The only difference is that for an American option, the exercise is permitted at any time during the life of the option. The early exercise possibility results in a free boundary problem for American-style options (e.g., see Peskir and Shiryaev [50]). The free boundary splits the whole region into two parts - the exercise region and the continuation region. When S_t is in the continuation region, the price $F(S, Y, t)$ satisfies the partial differential equation (4.15). When S_t is in the exercise region, the option should be exercised since it is worth more, so the price $F(S, Y, t)$ is just the payoff value.

Define the generalized Black-Scholes operator $\mathcal{A}$ as

$$\begin{aligned}\mathcal{A}F = \frac{1}{2}f^2(Y)S^2\frac{\partial^2 F}{\partial S^2} + \rho f(Y)\hat{\sigma}S\frac{\partial^2 F}{\partial S \partial Y} + \frac{1}{2}\hat{\sigma}^2\frac{\partial^2 F}{\partial Y^2} \\ + rS\frac{\partial F}{\partial S} + (\mu - \Lambda\hat{\sigma})\frac{\partial F}{\partial Y} - rF\end{aligned} \quad (4.16)$$

Then the American option under stochastic volatility can be characterized as

$$\begin{aligned}\frac{\partial F}{\partial t} + \mathcal{A}F \leq 0 \quad & (0 \leq t < T, S > 0, Y \in \mathbb{R}) \\ F \geq g \quad & (0 \leq t < T, S > 0, Y \in \mathbb{R}) \\ (\frac{\partial F}{\partial t} + \mathcal{A}F)(F - g) = 0 \quad & (0 \leq t < T, S > 0, Y \in \mathbb{R})\end{aligned} \quad (4.17)$$

with the initial condition

$$F|_{t=T} = g(S_T)$$

The asymptotic behavior of $F(S, Y, t)$ depends on the payoff process $g(S)$. For example, for a put option, i.e., $g(S) = (K - S)^+$, where K is the strike price, $F(S, Y, t)$ should satisfy the following conditions:

$$\lim_{S\to\infty} \frac{\partial F(S, Y, t)}{\partial S} = 0 \quad (4.18)$$

and

$$\lim_{Y\to\infty} \frac{\partial F(S, Y, t)}{\partial Y} = 0 \quad (4.19)$$

If we denote the free boundary by the critical curve $S^* = S^*(t)$ for $t \in [0, T]$, then we can identify the behavior of $F(S, Y, t)$ for a put option when the underlying asset price approaches $S^*(t)$

$$\lim_{S \to S^*(t)} F(S, Y, t) = K - S^*(t) \tag{4.20}$$

and the so-called 'smooth-pasting condition'

$$\lim_{S \to S^*(t)} \frac{\partial F(S, Y, t)}{\partial S} = -1 \tag{4.21}$$

The pricing of swing option under stochastic volatility can be described as a sequence of solving European options under stochastic volatility and American option under stochastic volatility. Once we solve the European/American option under stochastic volatility, based on (4.7) and (4.8), we can calculate the price the swing option under stochastic volatility. In the following sections, we will describe the numerical algorithm to solve the pricing of a swing option under stochastic volatility. There are several alternative approaches can be considered to tackle the problem (e.g., the finite-difference method, a Fourier transform-based method, or Monte Carlo simulations). In this chapter, we choose the finite element (FE) method. Our choice is based on the degree of the precision and the computation time needed for solving the problem.

4.3 A Brief Review of the Finite Element Method

4.3.1 Basic Idea of the FEM

From above, we know that pricing of a swing option under stochastic volatility can be reduced to solve a sequence of PDE problems. Since there is no analytical solution to these PDE problems, we resort to numerical methods to find the approximate solutions. Most commonly used numerical methods for PDE are the finite difference method and the finite element method. The finite difference (FD) method is based on the local Taylor expansion to approximate the differential equations. The FD method with equidistant grids is easy to understand and straightforward to implement.

The finite element method is a numerical method for finding the approximate solutions for PDE problems. It is based on using variational methods and/or weak formulations. Instead of using finite differences to approximate the derivatives, the FEM converts the PDE into an integrated form. The use of the integrated form is advantageous since it provides a reasonable treatment of Neumann boundary conditions and also of possible discontinuous source terms that may reduce the requirements on the regularity of the solution. The FEM may be used on domains of computation that are not rectangular. This is especially useful in multi-dimensional problems. Options with several underlying assets or randomness sources may lead to domains with complex geometry. For such situations, the FEM is often better than the FD method and is highly recommended.

The basic idea of the FEM is the same for any dimension of the space: for a weak formulation problem in an infinite-dimensional functional space V, we can choose a finite-dimensional space $V_h \subset V$ and solve the problem with basis and test functions in V_h instead of V, thus we will obtain an approximation solution. To explain this idea, we discuss the simple case of the differential equation:

$$Lu = f \tag{4.22}$$

where L is a linear differential operator, for example

$$Lu = -\Delta u \quad \text{for } u = u(x, y) \tag{4.23}$$

A two-dimension domain can be partitioned into either rectangles or triangles. Although there is a difference in implementation between these two partitions, there is no essential difference between these two. In this dissertation, we choose the triangular partition.

Suppose the domain of $u(x, y)$ is $\mathcal{D} \subseteq \mathbb{R}^2$. The domain is partitioned into non-overlapping triangles, i.e.,

$$\begin{aligned} &\bar{\mathcal{D}} = \textstyle\bigcup_k \mathcal{D}_k \\ &\mathcal{D}_i \cap \mathcal{D}_j = \emptyset \quad \text{for } i \neq j \end{aligned} \tag{4.24}$$

where all boundaries are included in $\bar{\mathcal{D}}$, not in $\mathcal{D}$. For simplicity, suppose there is no Neumann boundary, and u is 0 on the Dirichlet boundary, i.e.,

$$u = 0 \quad \text{on } \Gamma_D \tag{4.25}$$

Let $\{\varphi_i\}_{i=1}^N$ be a basis of V_h. We seek a function u_0 that approximates the solution of $u(x, y)$.

$$u_0 := \sum_{i=1}^{N} c_i \varphi_i \tag{4.26}$$

where $c_1, \cdots, c_N$ are unknown and need to be determined such that $u_0 \approx u$.

To find the value of $c_1, \cdots, c_N$, we introduce the test functions $\psi_1, \cdots, \psi_N$, such that ψ_i vanishes on the Dirichlet boundary of the domain, for $i = 1, \cdots, N$. Then we do the following integration

$$\int_{\mathcal{D}} L u_0 \psi_j = \int_{\mathcal{D}} f \psi_j \quad \text{for } j = 1, \cdots, N \tag{4.27}$$

Applying the integration by part using a form of Green's identities, we can obtain the weak formulation:

$$\int_{\mathcal{D}} \nabla u_0 \cdot \nabla \psi_j = \int_{\mathcal{D}} f \psi_j \quad \text{for } j = 1, \cdots, N \tag{4.28}$$

where ∇ denotes the gradient and $\cdot$ the dot product in the two-dimensional plane.

This is a system of N equations for the N unknowns $c_1, \cdots, c_N$. To solve the unknowns, for the left hand side of (4.26)

$$\int_{\mathcal{D}} \nabla u_0 \cdot \nabla \psi_j = \int \left(\sum_i c_i \nabla \varphi_i \right) \cdot \nabla \psi_j = \sum_i c_i \int_{\mathcal{D}} \nabla \varphi_i \cdot \nabla \psi_j$$

Define $a_{ij} := \int_{\mathcal{D}} \nabla \varphi_i \cdot \nabla \psi_j$, then all the a_{ij} constitute the stiffness matrix A. For the right hand side of (4.26), Let $b_i := \int_{\mathcal{D}} f \psi_i$, which will constitute a vector $b = (b_1, \cdots, b_N)'$. Then we can rewrite the system of equations (4.27) in a simple form:

$$Ac = B \tag{4.29}$$

where $c = (c_1, \cdots, c_N)'$.

4.3.2 The Basis Functions

There are many possible choices of the subspace V_h. For the FEM, a space of piecewise linear functions is widely used. For such spaces, piecewise linear functions are used as the basis functions. For the two-dimensional case, we choose one basis function φ_i for every node of the triangulation of the domain. The function $\varphi_i(x, y)$ is a pyramid-shaped linear function in V_h. And it takes the value 1 at the ith node and it vanishes at other nodes. Each $\varphi_i(x, y)$ satisfies

$$\varphi_i(x_j, y_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

where (x_j, y_j) is the coordinates of the jth node.

For a triangular element $\mathcal{D}_k$, let (x_1, y_1),(x_2, y_2) and (x_3, y_3) be the three vertices and

φ_1, φ_2 and φ_3 be the corresponding basis functions, then

$$\varphi_i(x,y) = \det\begin{pmatrix} 1 & x & y \\ 1 & x_{i+1} & y_{i+1} \\ 1 & x_{i+2} & y_{i+2} \end{pmatrix} / \det\begin{pmatrix} 1 & x_i & y_i \\ 1 & x_{i+1} & y_{i+1} \\ 1 & x_{i+2} & y_{i+2} \end{pmatrix} \tag{4.30}$$

Based on (4.30), we can calculate

$$\nabla\varphi_i(x,y) = \frac{1}{2|D|}\begin{pmatrix} y_{i+1} - y_{i+2} \\ x_{i+2} - x_{i+1} \end{pmatrix} \tag{4.31}$$

where $|D|$ is the area of $\mathcal{D}_k$, i.e.

$$|D| = \frac{1}{2}\det\begin{pmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{pmatrix}$$

When we choose $\psi_i = \varphi_i$ for $i = 1, \cdots, N$, it is known as the Bubnov-Galerkin method. Then the corresponding entry of the stiffness matrix is

$$a_{ij} = \int_{\mathcal{D}_k} (\nabla\varphi_i)' \cdot \nabla\varphi_j = \frac{1}{4|D|^2}(y_{i+1} - y_{i+2}, x_{i+2} - x_{i+1})\begin{pmatrix} y_{j+1} - y_{j+2} \\ x_{j+2} - x_{j+1} \end{pmatrix} \tag{4.32}$$

with indices modulo 3.

Base on (4.30), we can also calculate the right-hand side component of (4.28).

$$b_j = \int_{\mathcal{D}_k} f\varphi_j \approx \frac{1}{6}\det\begin{pmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{pmatrix} f(x_s, y_s) \tag{4.33}$$

where $f(x_s, y_s)$ is the value of f in the center of gravity (x_s, y_s) of $\mathcal{D}_k$.

In most applications, we need not only the stiffness matrix A, but also the mass matrix M which results from the integration $\int_{\mathcal{D}} u_0 \cdot \psi_j dx$. Based on the basis function defined in (4.30), we can obtain the entry of the mass matrix m_{ij}

$$m_{ij} = \int_{\mathcal{D}_k} \varphi_i\varphi_j dx = \frac{1}{24}\det\begin{pmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{pmatrix}(1 + \delta_{ij}) \tag{4.34}$$

where

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

The following integration formula will be very useful for some complex numerical implementations of the finite element method[2].

Proposition 4.5 *Let φ_i,$i = 1, 2, 3$ be the basis functions of the triangular D, and v_1, v_2, v_3 are three nonnegative integers, and $|D|$ the measure of D, then*

$$\int_D (\varphi_1)^{v_1} (\varphi_2)^{v_2} (\varphi_3)^{v_3} = 2|D| \frac{v_1! v_2! v_3!}{(v_1 + v_2 + v_3 + 2)!} \tag{4.35}$$

We can find that (4.34) is a special application of (4.35) when $v_1 + v_2 + v_3 = 2$, so (4.35) is a more general integration formula and we will use it in our implementations for the numerical solution of a swing option under stochastic volatility.

4.4 Numerical Algorithm for Swing Options under Stochastic Volatility

Before applying the FEM, we first make some assumptions about the swing option under stochastic volatility. To be specific, we consider a swing put option under a standard Stein-Stein's stochastic volatility model in which the volatility is a function of a mean reverting Orstein-Uhlenbeck process,

$$\begin{aligned} dS_t &= rS_t dt + \sigma_t S_t dW_{1t} \\ \sigma_t &= |Y_t| \\ dY_t &= \alpha(m - Y_t)dt + \beta d\hat{W}_t \end{aligned} \tag{4.36}$$

where α, m, and β are positive numbers. The parameter α is the rate of the mean reversion, m is the long-term mean variance level, and the ratio $\frac{\beta^2}{\alpha}$ is the long-term behavior of the variance of Y_t. In the Stein-Stein's stochastic volatility model, the correlation coefficient ρ between the two Brownian motions is assumed to be 0.

Let t denote the time to maturity, i.e., $t = T - \tau$, where τ is the current time. We transform the backward PDE problem to a forward PDE problem. For simplicity, we assume the market price of the volatility risk is zero, i.e., we set $\Lambda(S, Y, t) = 0$. Let $F(S, Y, t)$ be the price of a swing option under stochastic volatility. Define the generalized Black-Scholes operator $\mathcal{A}$ as

$$\mathcal{A}F = -\frac{1}{2}Y^2S^2\frac{\partial^2 F}{\partial S^2} - \rho\beta S|Y|\frac{\partial^2 F}{\partial S\partial Y} - \frac{1}{2}\beta^2\frac{\partial^2 F}{\partial Y^2} - rS\frac{\partial F}{\partial S} - \alpha(m - Y)\frac{\partial F}{\partial Y} + rF \quad (4.37)$$

The payoff process $g(S, t)$ is now defined by

$$g(S, t) = (K - S_t)^+ = \max(K - S_t, 0) \quad (4.38)$$

Before developing the algorithm for the swing put option under stochastic volatility, we use the FEM to solve the pricing problems for European and American put options under stochastic volatility.

4.4.1 FEM for European Options under Stochastic Volatility

Following the development in the last chapter, a European put option under stochastic volatility can be written as

$$\begin{aligned} &\frac{\partial F}{\partial t} + \mathcal{A}F = 0 && \text{in } \Omega \times (0, T] \\ &F(S, Y, 0) = g(S, 0) && \text{in } \Omega \end{aligned} \quad (4.39)$$

where $g(S, t) = (K - S_t)^+$, and $\Omega = \{S > 0, Y \in \mathbb{R}\}$.

There is no need to impose a boundary condition on $S = 0$ because of the degeneracy of the equation and for $S \to \infty$, or $Y \to \infty$

$$\lim_{S \to \infty} \frac{\partial F(S, Y, t)}{\partial S} = 0$$

and

$$\lim_{Y \to \infty} \frac{\partial F(S, Y, t)}{\partial Y} = 0$$

In [1], Achdou, Franchi and Tchou proved the existence of a unique solution to (4.39). Using this observation, we propose an algorithm based on the finite element method and apply the Galerkin scheme to obtain the numerical solution.

Rewrite (4.39) in a variational form, $\forall v \in W$

$$\begin{aligned} &\left(\frac{\partial F}{\partial t}, v\right) + (\mathcal{A}F, v) = 0 \quad \text{in } \Omega \times (0, T] \\ &F(S, Y, 0) = g(S, 0) \qquad\quad in\ \Omega \end{aligned} \tag{4.40}$$

where W is the weighted Sobolev space:

$$W = \left\{ v : \left(\sqrt{1 + Y^2} v, \frac{\partial v}{\partial Y}, S|Y| \frac{\partial v}{\partial S} \right) \in (L^2(\Omega))^3 \right\} \tag{4.41}$$

with the norm

$$||v||_W = \left(\int_\Omega (1 + Y^2) v^2 + \left(\frac{\partial v}{\partial Y} \right)^2 + S^2 Y^2 \left(\frac{\partial v}{\partial S} \right)^2 \right)^{\frac{1}{2}} \tag{4.42}$$

Define the space V as a closed subspace of W which vanishes on the Dirichlet boundary, i.e.,

$$V = \{ v \in W : v|_{\Gamma d} = 0 \} \tag{4.43}$$

where Γd is the domain for the Dirichlet condition.

Since (4.39) is a time-dependent problem, for the time domain, we use the time-difference method. We partition the time interval $[0, T]$ into subintervals $[t_{m-1}, t_m]$, $1 \leq m \leq M$, such that $0 = t_0 < t_1 < \cdots < t_M = T$. Define $\Delta t_i = t_i - t_{i-1}$. Denote the numerical solution at time t_m as F^m.

A variety of techniques for the numerical solution to (4.40) can be employed. Here we write (4.40) in a generalized weighted implicit form with the parameter θ.

$$\left(\frac{F^m - F^{m-1}}{\Delta t_m}, v\right) + \theta(\mathcal{A}F^m, v) + (1-\theta)(\mathcal{A}F^{m-1}, v) = 0 \tag{4.44}$$

When $\theta = 0$, this is an explicit time scheme, whereas when $\theta = 1$, it becomes an implicit time scheme. In particular, when $\theta = \frac{1}{2}$, it is the well-known Crank-Nicolson(CN) scheme. In this dissertation, we choose the CN scheme.

To discretize the S-Y domain, we use the standard triangular partition. Let $\mathcal{T}_h$ is the set of non-overlapping closed triangles forming the partition of $\Omega := (0, S_{max}) \times (-Y_{max}, Y_{max})$. Let N_T be the number of these triangles and E_i be the triangular element of this set, where $1 \leq i \leq N_T$, then $\bar{\Omega} = \bigcup_{i=1}^{N_T} E_i$. Here h is a discretization parameter and $h = \max_{E_i \in \mathcal{T}_h} \text{diameter}(E_i)$

Assuming the number of the vertices is N_V, and the number of vertices lying in the open domain Ω is N_{Vo}. We introduce two spaces of finite dimensions, W_h and V_h. We use piecewise linear functions for the FEM implementation, then

$$W_h = \{v \in \mathcal{C}^0(\bar{\Omega}) : v \text{ is linear on any } E_i \in \mathcal{T}_h, 1 \leq i \leq N_T\} \tag{4.45}$$

and

$$V_h = \{v \in W_h : v|_{\Gamma d} = 0\} \tag{4.46}$$

Hence $W_h \subset W$, $V_h \subset V$, and $V_h \subset W_h$.

Use the basis functions defined in the previous section, we see that $\{\phi_i\}_{i=1}^{N_V}$ form the basis functions of W_h, i.e., $W_h = \text{span}\{\phi_1, \cdots, \phi_{N_V}\}$.

The solution $F(t_m)$ to the swing put option under stochastic volatility can be approximated by a function $F_h^m \in W_h$

$$F(t_m, \cdot) \approx F_h^m(\cdot) = \sum_{i=1}^{N_V} F_i^m \phi_i(\cdot) \qquad m = 0, 1, \cdots, M \tag{4.47}$$

where F_h^m is the numerical solution at time t_m and the F_i^ms are undetermined values.

Substituting F_h^m into the variational form (4.44), applying the CN time scheme, we obtain the discretization form: $\forall v \in V_h$

$$\left(\frac{F_h^m - F_h^{m-1}}{\Delta t_m}, v\right) + \frac{1}{2}(\mathcal{A}F_h^m, v) + \frac{1}{2}(\mathcal{A}F_h^{m-1}, v) = 0 \tag{4.48}$$

After some calculations, we will obtain a linear system like $AF_h^m = b$ for $m = 0, \cdots, M$. The linear system has to be solved for each time step to obtain the value of a European option under stochastic volatility at $t = M$. We use the LU decomposition method to solve this linear system. We evenly divide the time domain into M subintervals, then for each time step, the matrix A is the same. In this way, for the M time steps problem, we only need do the LU decomposition for the first time step. For the following time steps, we use the result of the LU decomposition from the first time step. This will save a lot of computing time.

4.4.2 FEM for American Options under Stochastic Volatility

In contrast to a European option, an American-type option can be exercised at any time prior to maturity. This is an optimal stopping time problem and the arbitrage free price of an American-type option with the payoff process $g(t, S_t)$ is given by:

$$F(t, s, y) = \text{ess} \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}^Q[e^{-r(\tau - t)} g(\tau, S_\tau) | S_t = s, Y_t = y] \tag{4.49}$$

There are several approaches for handling American options under stochastic volatility. In [32], Ikonen and Toivanen discussed five efficient methods for dealing with this time dependent LCP problem. These approaches include the projected SOR method, a projected multigrid method, an operator splitting method, a penalty method, and a componentwise splitting method. The last one is a direct method, while the other four methods are iterative methods. Most of these existing methods share the similar idea: the value of an American option is always no less than the payoff process. At each time step t_m, after solving the variational problem for a corresponding European option, the condition $F^m(S, Y, t) \geq g(S, t)$ is to be enforced. In their paper, Ikonen and Toivanen show that these five methods have the similar accuracies, while for the speed comparison, the direct method is the fastest one. So we choose the LU decomposition method as in the case of European options.

The procedure for the discretization of an American option under stochastic volatility is similar to that used in the valuation of its European counterpart under stochastic volatility. We use the same time scheme for American options under stochastic volatility and the same S-Y domain discretization. We therefore obtain the same discretization form shown as (4.48). By solving the problem for the European option under stochastic volatility and enforcing the payoff condition, the value of the American option under stochastic volatility at each discrete point (S_i, Y_j, t_m) is obtained accordingly. In other words, we find

$$F^m(S_i, Y_j, t_m) = \max(F_e^m(S_i, Y_j, t_m), g(S_i, t_m)) \tag{4.50}$$

where F_e^m is the numerical solution for the corresponding European options.

At each time step t_m , after F^m is computed, we can also capture the information about the optimal exercise boundary. Thus the latter is obtained as a byproduct.

4.4.3 Algorithm for Swing Options under Stochastic Volatility

Now we are ready to develop an algorithm for the valuation of a swing put option under stochastic volatility. Let $F^{(n)}(S, Y, t)$ be the value of a swing put option under stochastic volatility with the payoff process $g(S, t)$, where $n \in \mathbb{N}$ is the number of exercise rights remaining, $t \in [0, T]$ is the time to maturity, and $g(S, t) = \max(K - S_t, 0)$. Following (4.7), the swing option price can be determined as a price of an American option whose pricing function $\Psi(S, Y, t)$ is characterized by

$$\begin{aligned} \frac{\partial F^{(n)}}{\partial t} + \mathcal{A}F^{(n)} &\geq 0 \quad \text{in } \Omega \times (0, T] \\ F^{(n)} &\geq \Psi^{(n)} \quad \text{in } \Omega \times (0, T] \\ \left(F^{(n)} - \Psi^{(n)}\right)\left(\frac{\partial F^{(n)}}{\partial t} + \mathcal{A}F^{(n)}\right) &= 0 \quad \text{in } \Omega \times (0, T] \\ F^{(n)}(S, Y, 0) &= \Psi^{(n)}(S_0, 0) \quad \text{in } \Omega \end{aligned} \tag{4.51}$$

According to (4.8), the n^{th} payoff process can be obtained by

$$\begin{aligned} \Psi^{(n)}(S, Y, t) &:= \begin{cases} g(S, t) + F_e^{(n)}(S, Y, t, \delta) & \text{for } t \in [\delta, T) \\ g(S, t) & \text{for } t \in [0, \delta) \end{cases} \\ \Psi^{(0)}(S, Y, t) &:= 0 \end{aligned} \tag{4.52}$$

where $F_e^{(n)}$ is the price of a European put option under stochastic volatility satisfying the following PDE

$$\begin{aligned} \frac{\partial F_e^{(n)}}{\partial t} + \mathcal{A}F_e^{(n)} &= 0 \quad \text{in } \Omega \times (0, \delta) \\ F_e^{(n)}(S, Y, 0, \delta) &= F^{(n-1)}(S, Y, t-\delta) \quad \text{in } \Omega \end{aligned} \tag{4.53}$$

The discretization of the time and the S-Y domain is the same as we have done for the European/American put option under stochastic volatility. There is only one more requirement for the refraction time δ such that $\delta/\triangle t \in \mathbb{N}$.

For each iteration when the exercise number is $i, i = 1, 2, \cdots, n$, the American option under stochastic volatility is calculated for the complete time domain, i.e., t from 0 to T, whereas for the European option under stochastic volatility, it is calculated only for the time domain where $t \in (0, \delta)$.

Using (4.51),(4.52) and (4.53), we present an algorithm for pricing the swing put option under stochastic volatility. We summarize the solution procedure as follows:

for $l = 1 : n$

for $t = 0 : \triangle t : \delta - 1$

$\Psi^{(l)}(S, Y, t) = g(S, t)$

end

for $t = \delta : \triangle t : T$

if $l > 1$, calculate $F_e^{(l)}(S, Y, \tau)$ using

$$\begin{aligned} \frac{\partial F_e^{(l)}}{\partial \tau} + \mathcal{A}F_e^{(l)} &= 0 \quad \tau \in (0, \delta) \\ F_e^{(l)}(S, Y, 0) &= F^{(l-1)}(S, Y, t-\delta) \quad in\ \Omega \end{aligned}$$

else

$$F_e^{(l)}(S, Y, \delta) = 0$$

end if

$$\Psi^{(l)}(S, Y, t) = g(S, t) + F_e^{(l)}(S, Y, \delta) \qquad \forall t \in (\delta, T]$$

end

Calculate $F^{(l)}(S, Y, t)$ with boundary condition $\Psi^{(l)}(S, Y, t)$

$$\begin{aligned}
\frac{\partial F^{(l)}}{\partial t} + \mathcal{A}F^{(l)} \geq 0 \quad & \text{in } \Omega \times (0, T] \\
F^{(l)} \geq \Psi^{(l)} \quad & \text{in } \Omega \times (0, T] \\
\left(F^{(l)} - \Psi^{(l)}\right)\left(\frac{\partial F(l)}{\partial t} + \mathcal{A}F^{(l)}\right) = 0 \quad & \text{in } \Omega \times (0, T] \\
F^{(l)}(S, Y, 0) = \Psi^{(l)}(S, 0) \quad & \text{in } \Omega
\end{aligned}$$

end

Chapter 5

Numerical Results

In this chapter, we will present numerical results to demonstrate the applications of the algorithm introduced in chapter 4. To check the adequacy of our general-purposed swing option algorithm under stochastic volatility, we first consider the two special cases. One is where the number of exercise opportunity is one. In this case, this model is reduced to an American option under stochastic volatility. The second case is a swing put option with a constant volatility. We compare the results obtained from using our approach with those reported in [56] and the results from the FST method and Monte Carlo simulations. In both cases, we find our algorithm performs satisfactorily. Finally, for the case of stochastic volatility where the swing option has more than one exercise right, we use the algorithm in chapter 4 to calculate the numerical solution. We compare the numerical solution with the results from the Monte Carlo simulations.

5.1 American Options under Stochastic Volatility

Notice that when $n = 1$, the swing option under stochastic volatility is reduced to an American option under stochastic volatility. We set the parameters for the Black-Scholes equation as following: the risk free rate of interest $r = 0.05$, the strike price $K = 100$ and the time to maturity $T = 1$. We consider the Stein-Stein's stochastic volatility model with $\alpha = 1$, $m = 0.16$, the correlation coefficient $\rho = 0$, and $\beta = \frac{\sqrt{2}}{2}$. For simplicity, we assume the market price of volatility risk $\Lambda = 0$. The mesh size for the (S, Y) domain is 100×100, and we use 100 time steps.

Figure 5.1 plots the price of an American put option with one year to maturity.

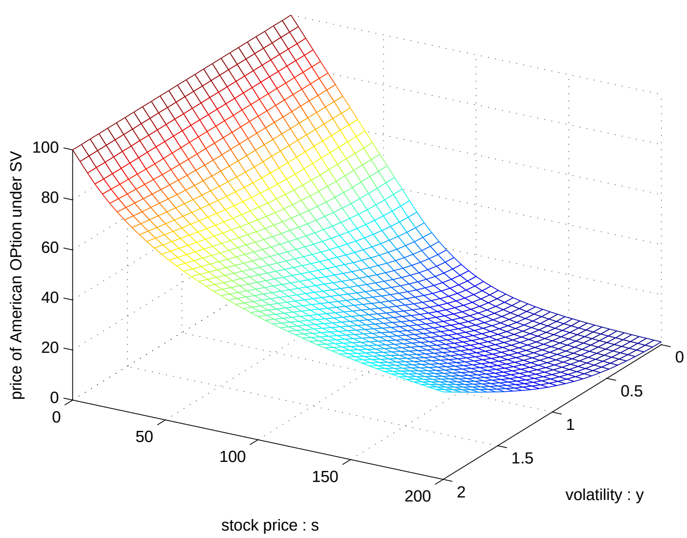

Figure 5.1: Numerical solution for American put option

As mentioned in the previous section, once we find the price of American put option under stochastic volatility, we can also capture the information for the optimal exercise boundary. Figure 5.2 plots the optimal exercise boundary.

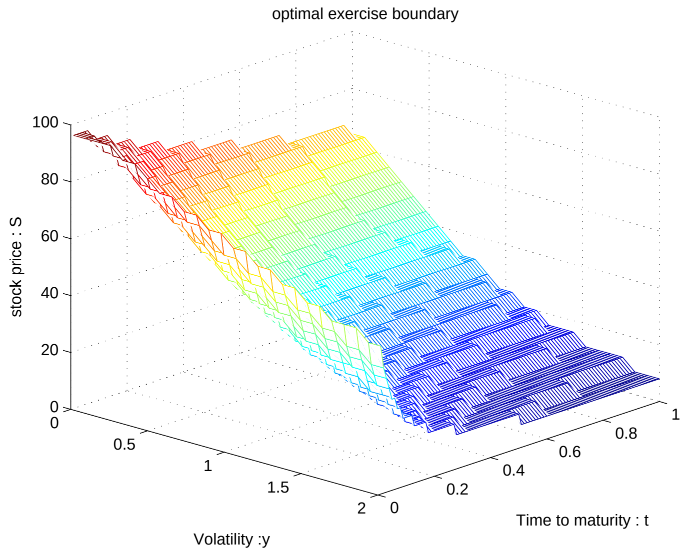

Figure 5.2: Optimal exercise boundary for American put option

We compare the results from the FEM with that from the Monte Carlo method. When $\sigma = 0.4$, we plot the behaviors of these two methods. For the Monte Carlo simulation, we use 10 seeds, 10 time steps and 4000 simulation paths. We choose $1, x, x^2$ as the basis functions.

To see more clearly, we list some price at some specific points.

From Table 5.1 and Figure 5.3, we can see that these two method agree well for pricing the American option under stochastic volatility.

In Figure 5.4 and 5.5, we compare the American option under stochastic volatility and American option with constant volatility. We explore the price difference at two specific σ values when $T = 1$.

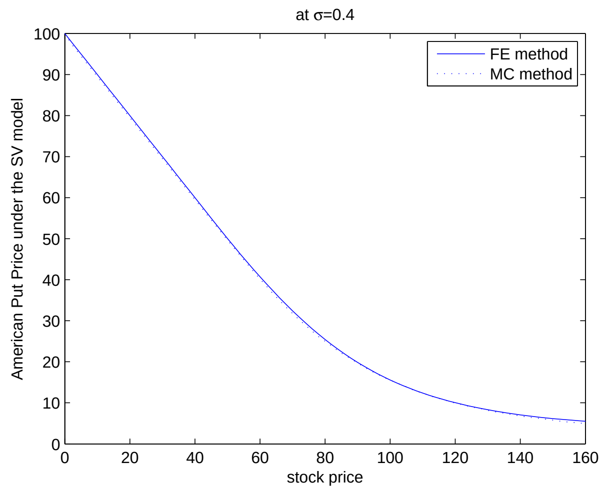

Figure 5.3: The prices of American put option when $\sigma = 0.4$

Table 5.1: The prices of American options under stochastic volatility

Stock Price	Volatility	the FEM	Monte Carlo [stand.dev]
80	0.16	22.9124	22.9249 [0.24]
80	0.40	25.4355	25.2324 [0.26]
90	0.16	16.8695	17.2265 [0.25]
90	0.40	19.8516	19.8874 [0.26]
100	0.16	12.4061	12.9463 [0.36]
100	0.40	15.5671	15.7207 [0.31]
110	0.16	9.26419	9.9865 [0.27]
110	0.40	12.3741	12.3188 [0.19]

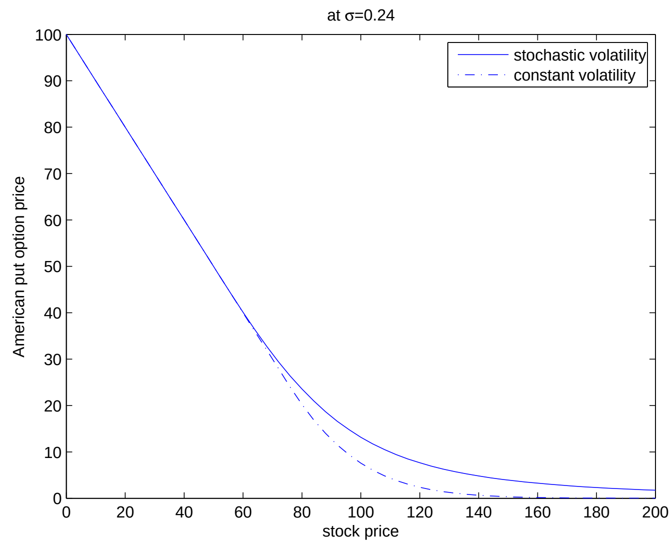

Figure 5.4: The prices of American put option when $\sigma = 0.24$

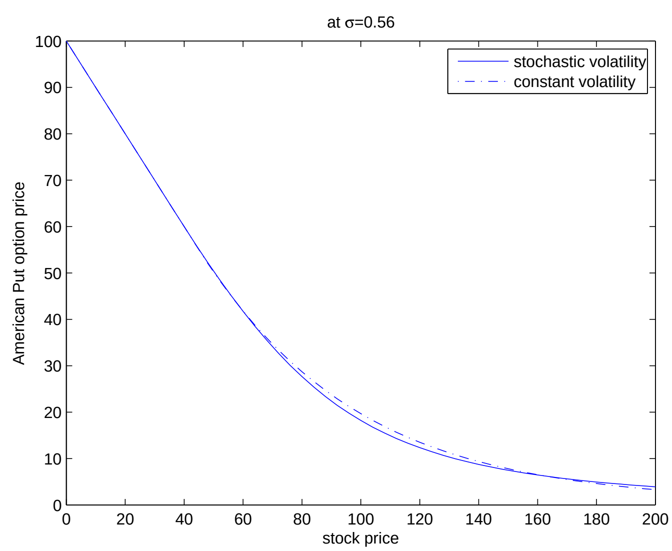

Figure 5.5: The prices of American put option when $\sigma = 0.56$

From Figure 5.4 and 5.5, we find that when in the optimal exercise region, the prices of these two models are the identical. Outside this region, the prices are different. The prices of the constant volatility model could be underpriced when $\sigma = 0.24$, and be a little overpriced when $\sigma = 0.56$.

5.2 Swing Options under Constant Volatility

When $\alpha = 0$, $m = 0$, and $\beta = 0$. the model is reduced to a swing put option with a constant volatility. Let the number of exercise rights $n = 3$. We first use this reduced model to obtain the numerical solution using our algorithm. We then develop an algorithm using the Fourier Space Time-stepping method (FST) described in [33] to compute the solution under the same setting.

In this experiment, we choose $K = 100$, $r = 0.05$, $\sigma = 0.3$, $\delta = 0.1$, $T = 1$. For the FEM, we choose 400 mesh points and 200 time steps, while for the FST method, we use 1000 time steps and 400 frequency points. Figure 5.6 plots the numerical solutions obtained from these two approaches.

In Figure 5.6, we observe that the results obtained from the FEM and FST match well. The price behavior is similar to that of an American option.

We also study the convergence behaviors of this reduced model, the FST method, and the Monte Carlo simulation when the spot price is at the money. We use the numerical result in [56] as a benchmark, which uses 4000 mesh points and 1000 time steps. These swing option prices are $F^{(1)}(100, 0, 0)) = 9.8700$, $F^{(2)}(100, 0, 0)) = 19.2550$, and $F^{(3)}(100, 0, 0)) = 28.1265$. Let N_t be the number of time steps, and N be the number of frequency points. The unit of computing time is the second.

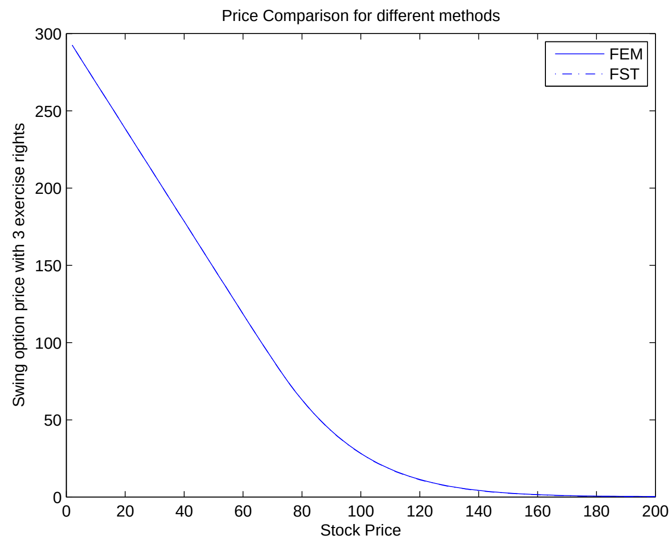

Figure 5.6: Swing put prices from the FEM and the FST method

Table 5.2: Absolute errors and the computing time using the FEM for a swing option under the constant volatility with 400 mesh points

	$N_t = 100$		$N_t = 200$		$N_t = 400$		$N_t = 800$	
Rights	Error	Time	Error	Time	Error	Time	Error	Time
$p = 1$	0.0216	0.134	0.0111	0.279	5.64e-03	0.422	2.86e-03	0.858
$p = 2$	0.0193	0.166	9.9e-03	0.369	4.8e-03	0.658	2.2e-03	1.725
$p = 3$	0.0122	0.288	5.7e-03	0.442	1.7e-03	0.915	2.0e-04	3.849

Table 5.3: Absolute errors and the computing time using the FST method for a swing option under the constant volatility with 400 time steps

	$N = 100$		$N = 200$		$N = 400$		$N = 800$	
Rights	Error	Time	Error	Time	Error	Time	Error	Time
$p = 1$	0.0852	0.05	0.0132	0.06	0.0057	0.15	0.0102	0.22
$p = 2$	0.1835	0.26	0.0427	0.35	0.0084	0.57	e.0e-04	0.95
$p = 3$	0.3261	0.46	0.1003	0.61	0.0451	0.92	0.0308	1.36

We show the absolute errors and the computing time for the FEM. Notice that the computing time in Table 5.2 is for calculating the swing option prices at all 400 mesh points, but we only show the price behavior when the spot price is at the money.

In Table 5.3, we show the behavior of the FST method. The computing time in this table is the time needed to calculate the price only at a single spot price. From this view point, the FEM is much faster than the FST method.

The convergence analysis for the Monte Carlo method is showed in Table 3.5. From these three tables, we can see that the accuracies of the FEM are very high and the computing time is much less than the other two approaches. Furthermore, although the FST method is relatively easy to apply, its application is quite limited, since in most cases it can be only used when the coefficients of the partial differential equation are constants. The Monte Carlo simulation is easy to set up, but it takes more time to calculate and the simulation results are not accurate enough compared to the FEM.

Figure 5.7 plots the prices for swing options with a constant volatility using the FEM when exercise rights up to 3. Figure 5.8 plots the exercise boundary values $S^*(t)$ of swing options.

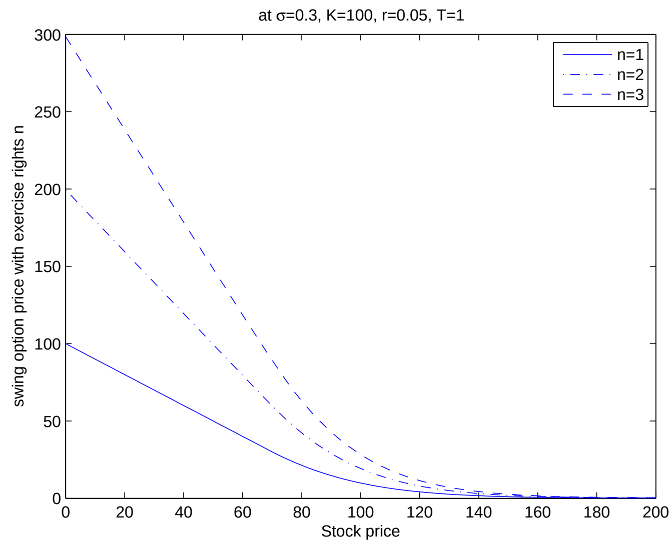

Figure 5.7: The prices of standard swing put option for up to 3 exercise rights

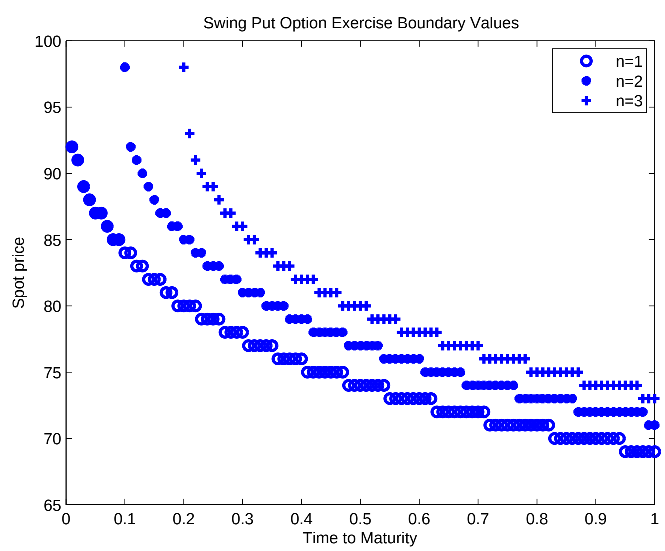

Figure 5.8: Exercise boundaries of standard swing put option for up to 3 exercise rights

From Figure 5.7, we can find that the swing and American options have the similar pattern. In Figure 5.8, we notice that when $n_1 \geq n_2$,

$$S^*_{n_1}(t) \geq S^*_{n_2}(t) \quad \forall t \in [0, T]$$

Since the value of a swing option with n exercise rights should less than the value of n times the price of the corresponding American option, in Figure 5.9, we compare the value swing option with 3 exercise rights and the value of 3 American options.

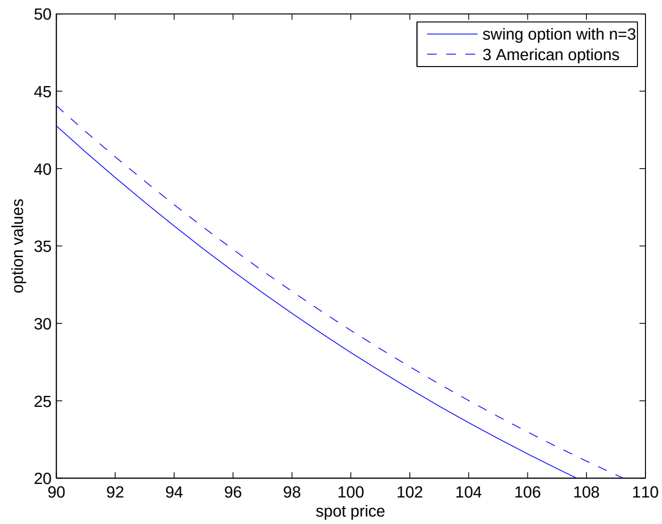

Figure 5.9: Prices comparison for swing and American options

Figure 5.9 shows that the value of n times the price of an American option is a upper bound for the corresponding swing option.

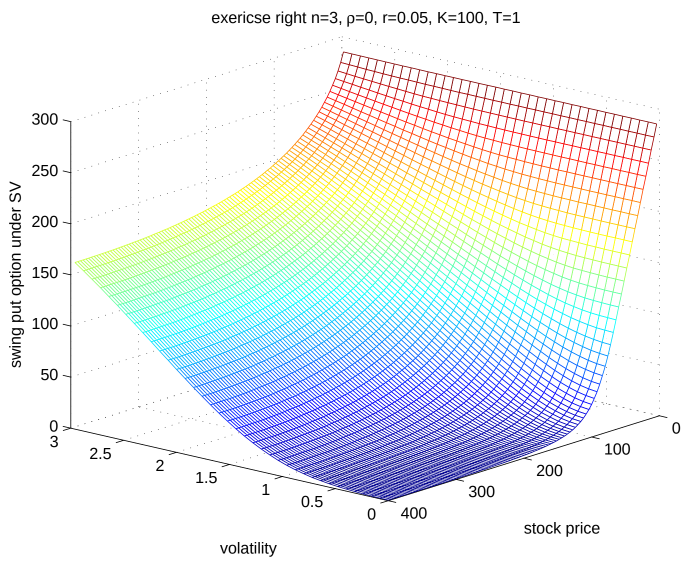

Figure 5.10: Swing put option under stochastic volatility

5.3 Swing Options under Stochastic Volatility

Now we consider the 'full-fledged' (by this, we mean the case when the number of swing rights can be greater than one) swing put option under stochastic volatility. We use the Stein-Stein's model, where the two Brownian Motions are uncorrelated. We set the parameters as follows: $\alpha = 1$, $m = 0.16$, $\beta = \frac{\sqrt{2}}{2}$, and $r = 0.05,\ T = 1,\ K = 100$.

Let N be the partition number for the S-plane, M be the partition number for the Y-plane, and N_t be the number of time steps. In our experiment, $N_t = 70, N = M = 101$. Figure 5.10 plots the prices for a swing put option under stochastic volatility with exercise rights $n = 3$.

We compare the results from the FEM with those from the Monte Carlo simulations. The settings for the parameters are as above.

Table 5.4: Prices of swing option under stochastic volatility when number of exercise rights is 3

Stock Price	Volatility	the FEM	Monte Carlo [stand.dev]
80	0.16	67.2005	67.6835 [0.44]
90	0.16	48.4735	49.3582 [0.70]
100	0.16	34.799	35.9164 [0.40]
110	0.16	25.3902	26.6868 [0.82]
120	0.16	19.0664	20.0486 [0.77]
80	0.40	74.5725	74.3090 [0.92]
90	0.40	57.3988	57.2476 [0.51]
100	0.40	44.306	44.1738 [0.85]
110	0.40	34.6676	34.7105 [0.57]
120	0.40	27.6907	27.6324 [0.88]

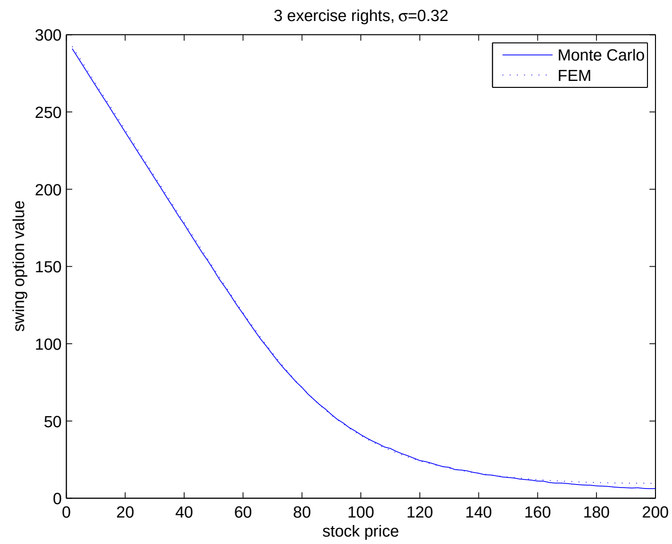

Figure 5.11: Swing option price at $\sigma = 0.32$ with 3 exercise rights

In Table 5.4, from the comparison, we can see that these two methods agree well. Unlike the FEM, which can provide a fixed numerical solution for the fixed parameter settings, the expected value calculated by the Monte Carlo method is itself a random variable. Each time when we simulate, we will get a different number. We know the value will be convergent to the true value when the sample size approaches to ∞.

In Figure 5.11, we plot the behaviors of the Monte Carlo Method and the FEM to evaluate a swing put option for the whole stock price domain.

In Figure 5.11, the behaviors of these two methods agree well for the whole stock price domain.

In Figure 5.12 and Figure 5.13, we choose two specific σ values and compare the price of a swing option under stochastic volatility and that of a swing option under the constant volatility case respectively.

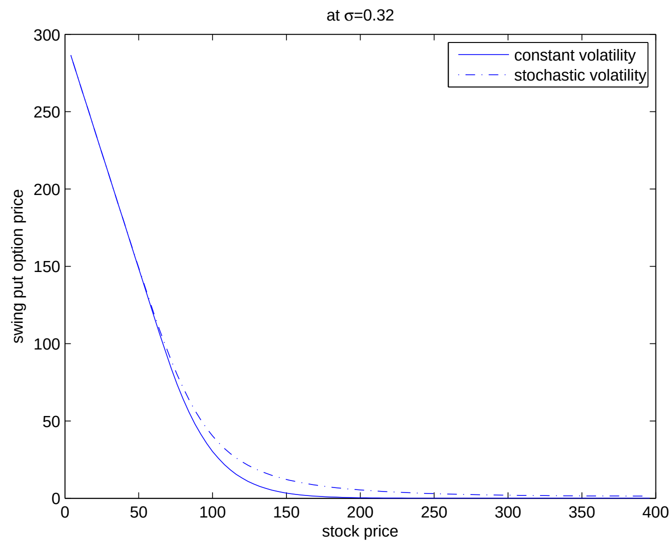

Figure 5.12: The prices of swing option when $\sigma = 0.32$

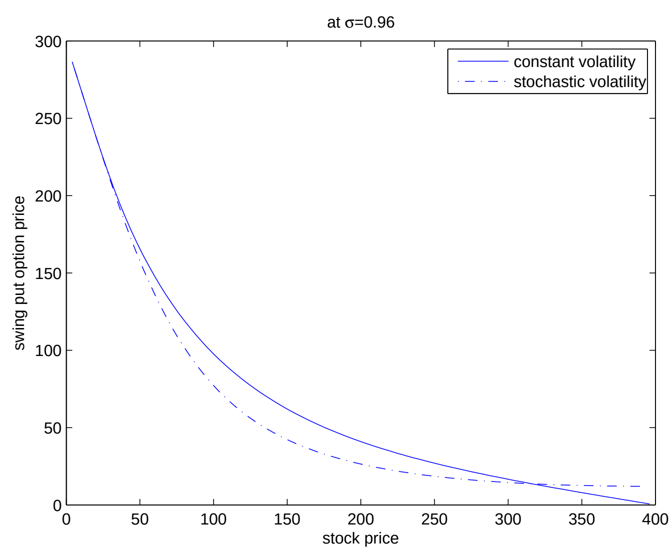

Figure 5.13: The prices of swing option when $\sigma = 0.96$

In the case of $\sigma = 0.32$, the prices of the two models exhibit the similar behavior. There is some difference around the strike price. When $S > 2K$, as the stock price increases, the difference between these two approaches becomes negligible. For the case of $\sigma = 0.96$, the asymptotic behaviors of these two models are different, so constant volatility models would cause errors for the pricing.

Chapter 6

Conclusion

The notion of the stochastic volatility was first included the study of European options and then later extended to that of American options. This enhancement captures the financial market behavior more closely than that under the simplifying assumption of a constant volatility. In this dissertation, we include the stochastic volatility in the swing option in order to make it more reflective of the real-world price movement. By transforming the solution process for the swing option to a sequence of single stopping time problems, we reduce the problem to a series of problems involving the valuations of European/American options under the stochastic volatility model. We develop an algorithm for pricing the swing option under the Stein-Stein's stochastic volatility model. The algorithm is flexible with respect to different payoff functions. We explore the behavior of the swing option under stochastic volatility, and compare the results with Monte Carlo simulations. The numerical results show that the finite element method is fast and accurate.

Although we develop the algorithm based on fact that the underlying asset follows the Geometric Brownian Motion process, we can extend this algorithm to one factor or two factor mean-reverting processes, which are many times used in the energy market, especially in the power sector. Other future work could be the study of the Greeks for the swing option under stochastic volatility, or a model including Lévy process.

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