UNIVERSAL OPERATOR SYSTEM STRUCTURES ON ORDERED SPACES AND THEIR APPLICATIONS

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Abstract

We are interested in the construction of some universal operator systems that have a given Archimedean ordered $*$-vector space as their ground level. Operator system theory was formulated in the late 1970's. Even though there hasn't been much development in this direction, operator systems play a great role in the study of operator space theory and quantum information theory.

The structure of operator systems and the positive cones they possess make the study of completely positive maps even more accessible and more interesting. In [23], we encounter the so-called minimal and maximal operator systems constructed over a given Archimedean ordered $*$-vector space. Based on these structures, we develop a technique to build our universal minimal and maximal operator systems. The investigation of the properties of completely positive maps from our universal minimal operator system into the universal maximal operator system, leads us to a new characterization of the "partially entanglement breaking maps" encountered in Quantum Information Theory.

Our methods come from the realm of completely positive maps, some duality techniques, and matrix theory.

Contents

Chapter 1

Introduction

1.1 Prologue

Operator system theory was initiated with Arveson's version of the Hahn-Banach theorem for completely positive operator-valued mappings [1]. Around the 1920-1930's, functional analysis had been successfully applied to the investigation of function algebras. The fact that these function algebras may be regarded as non-self adjoint subalgebras of commutative C*-algebras, lead to the exploration of their non-commutative analogues. Kadison and Singer [16] were among the first to explore this direction.

A key method in function algebra theory is to extend linear functionals from a function algebra to functionals on the surrounding C*-algebras. This is what made the Hahn-Banach theorem a central tool in classical functional analysis, and the same is true in the non-commutative context. Replacing scalar functionals with operator-valued mappings was the very necessary step that Arveson figured out for the non-commutative case. But this was not sufficient, since the usual Hahn-Banach theorem fails in this context. Therefore, Arveson realized that one could extend such mappings provided one used "matrix orders" and the "completely positive mappings" of Stinespring [28]. He was able to prove an influential extension theorem in this context, and study some of its applications [2].

This remarkable result of Arveson made other mathematicians aware of the natural "hidden structure" of operator linear spaces, encoded in their matrix orderings and norms. The easiness to approach this ordered theory gave rise to the corresponding theory of "operator systems", formulated by Choi and Effros [4] in the late 1970's. These newly formulated systems are the operator analogues of Kadison's function systems [15], a natural category of unital ordered Banach spaces. Wittstock [31] gave the general formulation of Arveson-Hahn-Banach theorem and Paulsen [21] was the first to prove such a result using the theory of operator systems. Operator system theory provides an abstract description of the order structure of self-adjoint unital subspaces of C*-algebras. Even though there has been relatively little development of the abstract theory of operator systems, many deep results of operator spaces come from regarding them as "corners" of operator systems. So new developments in the theory of operator systems could lead to new insights in the theory of operator spaces.

Many applications of the theory of completely bounded maps to operator theory are presented in Paulsen's book [20]. The extension and representation theorems for completely bounded maps show that the subspaces of C*-algebras carry naturally induced metric structure which is preserved by complete isometries. This structure has been characterized by Ruan's theorem [26] in terms of the axioms of an operator space.

1.2 Operator Spaces

Operator spaces can be looked at from two points of view, rather like one can approach C*-algebras either concretely as being $*$-closed subalgebras of $B(\mathcal{H})$, the bounded operator on some Hilbert space $\mathcal{H}$ or, abstractly, as a Banach algebra satisfying certain properties.

Very briefly, concrete operator spaces are closed linear subspaces of $B(\mathcal{H})$, for some Hilbert space $\mathcal{H}$. To get the right idea, one should really consider operator spaces as a category where the appropriate mappings are the completely bounded linear mappings between the operator spaces. Associated with an operator space $X \subseteq B(\mathcal{H})$, we have embedding of the space $M_n(X)$ of $n \times n$ matrices with entries in X into $M_n(B(\mathcal{H}))$ for all $n \in \mathbb{N}$. We regard $M_n(B(\mathcal{H}))$ as being the bounded operators $B(\mathcal{H}^n)$ on the Hilbert space $\mathcal{H}^n = \mathcal{H} \oplus \mathcal{H} \oplus \cdots \oplus \mathcal{H}$ (n times) using the standard idea of how matrices of operators act:

$$\begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^n x_{1i} v_i \\ \sum_{i=1}^n x_{2i} v_i \\ \vdots \\ \sum_{i=1}^n x_{ni} v_i \end{pmatrix}.$$

If $T : X \to Y$ is a linear operator, then we define $T^{(n)} : M_n(X) \to M_n(Y)$ by applying T to each entry of matrices with entries in X:

$$T^{(n)} \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix} = \begin{pmatrix} T(x_{11}) & T(x_{12}) & \cdots & T(x_{1n}) \\ T(x_{21}) & T(x_{22}) & \cdots & T(x_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ T(x_{n1}) & T(x_{n2}) & \cdots & T(x_{nn}) \end{pmatrix}.$$

In the case of operator space $X \subseteq B(\mathcal{H})$ and $Y \subseteq B(\mathcal{K})$, we have Hilbert space norms on the spaces $M_n(X)$ and $M_n(Y)$. The completely bounded norm of a linear operator $T : X \to Y$ between operator spaces X and Y is

$$\|T\|_{cb} = \sup_{n \in \mathbb{N}} \|T^{(n)}\|.$$

If $T : X \to Y$ is a completely bounded operator between operator spaces X and Y with $\|T\|_{cb} = \|T^{-1}\|_{cb} = 1$, then T is called a completely isometric isomorphism.

Abstractly, an operator space consists of a Banach space X together with a family of **matrix norms** $\{\|\cdot\|_n : M_n(X) \to \mathbb{R}\}_{n=1}^{\infty}$ on each $M_n(X)$ which satisfy Ruan's axioms:

(R1) $\|\alpha x \beta\|_m \leq \|\alpha\|\|x\|_n\|\beta\|,$ for all $x \in M_n(X), \alpha \in M_{m,n}, \beta \in M_{n,m}$

(R2) $\|x \oplus y\|_{n+m} = max\{\|x\|_n, \|y\|_m\},$ for all $x \in M_n(X), y \in M_m(X).$

Note that the norm of the scalar matrices $\alpha \in M_{m,n}$ and $\beta \in M_{n,m}$ are their norms as operators on a finite dimensional Hilbert space.

Every concrete operator space $X \subseteq B(\mathcal{H})$ gets norms on the matrix spaces $M_n(X) \subseteq B(\mathcal{H}^n)$. One can easily verify that these norms satisfies the two properties $(R1)$ and $(R2)$. Thus every concrete operator space is an operator space. Conversely, every abstract operator space X can be viewed as a concrete operator space via a **completely isometric embedding** $T : X \to B(\mathcal{H})$ which preserves the matrix norms, i.e. $T^{(n)} : M_n(X) \to M_n(B(\mathcal{H})) \cong B(\mathcal{H}^n)$ is an isometric embedding for each $n \in \mathbb{N}$. This result is due to Ruan [26], [7].

Every C*-algebra $\mathcal{A}$ is an operator space, since $\mathcal{A}$ can be embedded as a norm-closed $*$-subalgebra of some $B(\mathcal{H})$:

Theorem 1.2.1 (Gelfand-Naimark Theorem, [10])**.** *If $\mathcal{A}$ is a C*-algebra, then there exists a Hilbert space $\mathcal{H}$ and an isometric $*$-isomorphism of $\mathcal{A}$ onto a closed $*$-subalgebra of $B(\mathcal{H})$.*

Also, every Banach space X can be regarded as an operator space, since X can be embedded in a $B(\mathcal{H})$. There is an isometric embedding of X in the commutative C*-algebra $C(B_{X^*})$ of continuous functions on the unit ball $B_{X^*} = \{x^* : \|x^*\| \leq 1\}$ of the dual of X given by $x(x^*) = x^*(x)$. By using the matrix norms on this C*-algebra, we obtain matrix norms on $M_n(X)$ for all $n \in \mathbb{N}$. This is the smallest possible operator space that can be constructed and is denoted by MIN(X) [22].

There is also a canonical largest operator space constructed on a given Banach space X. It is denoted by MAX(X) in [22], and the MAX(X) matrix norms are clearly the largest possible, given by

$$\|x\|_n = \sup\{\|T^{(n)}(x)\| : T : X \to B(\mathcal{H}) \text{ an isometric embedding}\}.$$

These operator spaces are characterized by the universal mapping property: For any normed space X, and any operator space Y, $CB(\mathrm{MAX}(X), Y) = B(X, Y)$ and $CB(Y, \mathrm{MIN}(X)) = B(Y, X)$. We have the duality of MIN and MAX as follows: $\mathrm{MIN}(X)^* = \mathrm{MAX}(X^*)$ and $\mathrm{MAX}(X)^* = \mathrm{MIN}(X^*)$ completely isometrically. MIN and MAX can be viewed as functors from the category whose objects are normed spaces and whose morphisms are contractive linear maps into the category whose objects are operator spaces and whose morphisms are completely contractive maps. In this categorical sense, "first level" is really the functor from the category of operator spaces to normed spaces that "forgets" the additional structure. A lot of work has been done explaining the difference between the MIN and MAX functors, constructing other natural operator space structures on normed spaces, and exploring the behavior of these functors with respect to various natural tensor norms on each category. In [23], a parallel development has been done of the analogues in the operator system setting of the MIN and MAX functors.

1.3 Operator Systems and Completely Positive Maps

Given a Hilbert space $\mathcal{H}$ and $n \in \mathbb{N}$, we have the natural identification $M_n(B(\mathcal{H})) \cong B(\mathcal{H}^n)$, determined by matrix multiplication. $B(\mathcal{H})$ is matrix normed and matrix ordered, in the sense that each matrix space $M_n(B(\mathcal{H}))$ has a corresponding norm and ordering. The norms and orderings on these spaces are linked by some fundamental relations. Let $a \in M_n(B(\mathcal{H}))$, $b \in M_m(B(\mathcal{H}))$ and $A \in M_{n,m}$. Then, we have

(1) If $a \geq 0$ and $b \geq 0$, then $a \oplus b = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \geq 0$.

(2) $M_n(B(\mathcal{H}))^+ \cap (-M_n(B(\mathcal{H}))^+) = \{0\}$ for all $n \in \mathbb{N}$.

(3) Let $b = (b_{kl}) \in M_m(B(\mathcal{H}))$, $A = (a_{ik}) \in M_{n,m}$ and $C = (c_{lj}) \in M_{m,n}$. Then we define an $n \times n$ matrix over $B(\mathcal{H})$ by

$$AbC = \left[\sum_{k,l=1}^{m} a_{ik} b_{kl} c_{lj} \right]_{i,j=1}^{n}.$$

If $b \in M_m(B(\mathcal{H}))^+$ and $A \in M_{n,m}$ any scalar matrix, then $AbA^* \in M_n(B(\mathcal{H}))^+$.

Let $\mathcal{A}$ be a C*-algebra, and let $\mathcal{S}$ be a subset of $\mathcal{A}$. We set

$$\mathcal{S}^* = \{a : a^* \in \mathcal{S}\},$$

and we call $\mathcal{S}$ **self-adjoint** when $\mathcal{S} = \mathcal{S}^*$. If $\mathcal{A}$ has a unit $\mathbb{I}$ and $\mathcal{S}$ is a self-adjoint subspace of $\mathcal{A}$ containing $\mathbb{I}$, then we call $\mathcal{S}$ a (concrete) **operator system**. Consider $M_n(\mathcal{A})$ and denote the matrix unit $\mathbb{I}_n = I_n \otimes \mathbb{I}$, where I_n is the identity matrix of M_n. We endow $M_n(\mathcal{S})$ with the canonical $*$-operation, the norm and the relative order structure it inherits as a subspace of $M_n(\mathcal{A})$. Hence, the positive elements of $M_n(\mathcal{S})$ can be described as

$$M_n(\mathcal{S})^+ = M_n(\mathcal{S}) \cap M_n(\mathcal{A})^+, \text{ for all } n \in \mathbb{N}.$$

Using the Gelfand-Naimark Theorem, it follows that the properties (1), (2) and (3) given above, hold for matrices over $\mathcal{S}$. In particular, for any self-adjoint element $h \in M_n(\mathcal{S})$, we have

$$h = (h + \|h\| \cdot \mathbb{I}_n) - \|h\| \cdot \mathbb{I}_n,$$

where $h+\|h\|\cdot\mathbb{I}_n \geq 0$ and $\|h\|\cdot\mathbb{I}_n \geq 0$. Thus, all the self-adjoint elements of $M_n(\mathcal{S})$, denoted by $M_n(\mathcal{S})_{sa}$, can be given by

$$M_n(\mathcal{S})_{sa} = M_n(\mathcal{S})^+ - M_n(\mathcal{S})^+, \text{ for all } n \in \mathbb{N}.$$

A moment's reflection will convince the reader that for each $n \geq 1$, $M_n(\mathcal{S})$ is an operator system.

As one might expect, there is an abstract characterization (up to complete order isomorphism) for the operator systems involving properties (1), (2) and (3) above, which will be explained in detail in Section 2.1 of Chapter 2. To this end, let V be a complex vector space and assume there exists an involutive linear map $v \mapsto v^*$ on V with $(v^*)^* = v$ for all $v \in V$. Such a space is called **$*$-vector space**. Let $V_{sa} = \{v \in V : v = v^*\}$ be the set of all self-adjoint elements in V and note that every element x in V can be written as $v = h + ik$ with $h = (v + v^*)/2$ and $k = (v - v^*)/2i$ both in S_{sa}. We call $e \in V$ an **order unit** for V provided that for every $v \in V_{sa}$ there exists some positive real number $r > 0$ such that $re \geq v$. We call an order unit e **Archimedean** if $re \geq v$ for all $r > 0$. For $(v_{ij}) \in M_n(V)$ we set $(v_{ij})^* = (v_{ji}^*)$, so that $M_n(V)$ is also a $*$-vector space. We call e a **matrix order unit** provided that $e_n = I_n \otimes e$, where I_n is the identity matrix for M_n, is an order unit for $M_n(V)$ for all n, and an **Archimedean matrix order unit** provided each e_n is Archimedean. To make V an operator system, we equip V with **a matrix ordering** $\{C_n\}_{n=1}^{\infty}$ and an **Archimedean matrix order unit** e, where each C_n is a distinguished cone in $M_n(V)_{sa}$ that plays the role of the "positive" operators.

Every operator system is an ordered $*$-vector space V with an Archimedean order unit at the first level and conversely, given any Archimedean order unit space, there are possibly many different operator systems that all have the given Archimedean order unit space as their first level [4, 20].

If $\mathcal{S}$ and $\mathcal{T}$ are two given (concrete) operator systems, then a linear map $\phi : \mathcal{S} \to \mathcal{T}$ is called **n-positive** if its natural extension $\phi^{(n)} : M_n(\mathcal{S}) \to M_n(\mathcal{T})$ given by

$$\phi^{(n)}((a_{ij})) = (\phi(a_{ij})),$$

is positive. ϕ is called **completely positive** if ϕ is n-positive for all $n \in \mathbb{N}$. A completely positive map is **unital** if it maps units to units in each matrix space. Moreover, we call ϕ a **complete order isomorphism** provided that ϕ is invertible with both ϕ and ϕ^{-1} completely positive.

Completely positive maps have been applied to many applicable research areas. One of the most important characterizations that makes completely positive maps such a wonderful tool in many areas, is the existence of an operator-sum representation, as described in the following theorem:

Theorem 1.3.1 (Choi-Kraus, [3], [17])**.** *Let $\mathcal{H}, \mathcal{K}$ be finite dimensional Hilbert spaces, and let $\phi : B(\mathcal{H}) \to B(\mathcal{K})$ be a completely positive linear map. Then, there exists a (non-unique) family of operators $\{A_1, A_2, \ldots, A_n\} \subseteq B(\mathcal{H}, \mathcal{K})$ which determine the map ϕ through the equation*

$$\phi(T) = \sum_{i=1}^{n} A_i T A_i^*, \text{ for all } T \in B(\mathcal{H}).$$

Note that such operators described in the theorem above, are the so called **Kraus operators**. In linear algebra, the **trace** of an $n \times n$ square matrix $A = (a_{ij})$ is defined to be the sum of the elements on the main diagonal (the diagonal from the upper left to the lower right) of A, i.e. $\mathrm{tr}(A) = a_{11} + a_{22} + \cdots + a_{nn} = \sum_{i=1}^{n} a_{ii}$.

A completely positive map $\phi : B(\mathcal{H}) \to B(\mathcal{K})$, whose operator-sum representation is given by $\phi(\cdot) = \sum_{i=1}^{n} A_i(\cdot)A_i^*$ for some family of Kraus operators $\{A_1, A_2, \dots, A_n\} \subseteq B(\mathcal{H}, \mathcal{K})$, is called **trace-preserving** provided that $\mathrm{tr}(\phi(X)) = \mathrm{tr}(X)$ for each $X \in B(\mathcal{H})$. This is equivalent to

$$\sum_{i=1}^{n} A_i^* A_i = I_{\mathcal{H}},$$

where $I_{\mathcal{H}}$ is the identity operator on $\mathcal{H}$. This is also equivalent to the "dual map" of ϕ being unital. We say ϕ is unital if also

$$\phi(I_{\mathcal{H}}) = \sum_{i=1}^{n} A_i A_i^* = I_{\mathcal{K}}.$$

Because of the properties they possess, trace-preserving completely positive maps are one of the key natural ingredients in Quantum Information Theory. They are known as quantum channels. We will give a mathematical formulation of fundamental notions in quantum information theory, and a motivation for our work.

1.4 Basics in Quantum Information Theory

Quantum information theory has been studied by researchers from various backgrounds. Their approach can be broadly divided into two categories: one based on information theory, and the other based on quantum mechanics. An investigation of the quantum information and quantum computing literature reveals that many techniques from operator theory and operator algebras have been, or could be, used to build mathematical groundwork for the physical theories in these areas. Quantum mechanics is the mathematical framework for the development of the physical theories. Motivated by the postulates of quantum mechanics, an assumption typically made in quantum information theory is that every quantum operation on a closed quantum system is reversible. In order to treat information processing in quantum systems, it is necessary to mathematically formulate fundamental postulates of quantum mechanics: quantum systems and its states, quantum evolutions and operations, quantum measurements, and composite systems [19].

1.4.1 Formulation of Postulates in Language of State Vectors

First, we consider the quantum systems. A **quantum system** is described by a complex vector space with inner product, i.e. a Hilbert space $\mathcal{H}$, known as the **state space** of the system. The physical state of the system is completely described by its **state vector**, which is a unit vector in the system's state space. More precisely, it can be explained by a one-dimensional subspace spanned by this vector. The simplest quantum mechanical system and the fundamental one in quantum mechanics is the **qubit**. A qubit has a two-dimensional state space. With an orthonormal basis $\{e_0, e_1\}$ the most general state vector of a qubit can be represented as

$$u = \alpha e_0 + \beta e_1,$$

where $\alpha, \beta \in \mathbb{C}$ and $\langle u, u\rangle = u^* u = |\alpha|^2 + |\beta|^2 = 1$. A good example that can represent a qubit is photon polarizations. Photons are massless spin-1 particles; they can have two independent polarizations, transverse to the direction of propagation. Under a rotation about the axis of propagation, the two linear polarization state vectors $\vec{h}$ and $\vec{v}$ (representing horizontal and vertical polarizations respectively) transform as

$$\vec{h} \to (\cos\theta)\vec{h} + (\sin\theta)\vec{v}$$
$$\vec{v} \to (-\sin\theta)\vec{h} + (\cos\theta)\vec{v}.$$

The matrix representation of this transform is $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$. The state space of "photons of light" are all the points in $\mathbb{C}^2$.

The **evolution of a closed quantum system** is described by a unitary transformation.

As time passes, the state vector u_1 of a quantum system at time t_1 is related to its state vector u_2 of the system at time t_2 by a unitary operator U which depends only on the given times t_1, t_2 of change

$$u_2 = Uu_1.$$

So closed quantum systems evolve according to unitary evolution. The evolution of closed systems is all very well, but to explain what happens when the quantum system is no longer closed, and not necessarily subject to unitary evolution, there is need to describe the measurements on quantum systems.

Quantum measurements are described by a collection $\{M_m\}$ of measurement operators, also known as Kraus operators. These are operators acting on the state space of the system being measured. If the state vector of a quantum system is u before the measurement, then the outcome m occurs with probability

$$p(m) = \langle M_m u, M_m u\rangle = \langle M_m^* M_m u, u\rangle,$$

and the new state vector of the system after the measurement is

$$u_m = \frac{M_m u}{\sqrt{\langle M_m^* M_m u, u\rangle}}.$$

The probabilities of all outcomes add up to one

$$1 = \sum_m p(m) = \sum_m \langle M_m^* M_m u, u\rangle, \text{ for all states } u,$$

so that the measurement operators satisfy the **completeness equation**

$$\sum_m M_m^* M_m = I.$$

For example, measuring a qubit to be e_0 or e_1 (think of e_0 and e_1 as the basis vectors of $\mathbb{C}^2$) is thus a measurement with measurement operators

$$M_0 = e_0 e_0^* = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \; M_1 = e_1 e_1^* = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

The state vector $u = \alpha e_0 + \beta e_1$ then has probability

$$p(0) = \langle M_0^* M_0 u, u \rangle = \langle M_0 u, u \rangle = |\alpha|^2$$

of yielding e_0, and probability

$$p(1) = \langle M_1^* M_1 u, u \rangle = \langle M_1 u, u \rangle = |\beta|^2$$

of yielding e_1 after the measurement, with new state vectors

$$u_0 = \frac{M_0 u}{|\alpha|} = \frac{\alpha}{|\alpha|} e_0 \text{ and } u_1 = \frac{M_1 u}{|\beta|} = \frac{\beta}{|\beta|} e_1 \;, \text{ respectively.}$$

Let M be a self-adjoint operator on the state space of the system being observed. Such an operator is called an **observable**. Look at the spectral decomposition of M

$$M = \sum_m \lambda_m P_m,$$

where P_m is the projection onto the eigenspace of M with eigenvalue λ_m. The possible outcomes of the measurement correspond to eigenvalues λ_m of the observable. By measuring the state vector u, the probability of getting result λ_m is given by

$$p(m) = \langle P_m u, u \rangle.$$

Given that, the measured state of the quantum system is

$$u_m = \frac{P_m u}{\sqrt{p(m)}}.$$

The family of these projections forms the so called **projective measurement** of the quantum system, which together with unitary dynamics are sufficient to implement a general measurement of quantum system. Projective measurement have very nice properties related to probability. The average value of the observable M, i.e. the expected value of it, can be given by

$$\begin{aligned} E(M) &= \sum_m \lambda_m p(m) \\ &= \sum_m \lambda_m \langle P_m u, u\rangle \\ &= \Big\langle \left(\sum_m \lambda_m P_m\right) u, u\Big\rangle \\ &= \big\langle M u, u\big\rangle. \end{aligned}$$

Projective measurements are repeatable in the sense that if we perform a projective measurement once, and obtain the outcome m, repeating the measurement gives the outcome m again and does not change the state. To see this, suppose u was the initial state. After the first measurement the state is $u_m = \dfrac{P_m u}{\sqrt{\langle P_m u, u\rangle >}}$. Applying P_m to u_m does not change it, so we have $\langle P_m u_m, u_m\rangle = 1$, and therefore repeated measurement gives the result m each time, without changing the state. Many other quantum measurements are not repeatable in the same sense as a projective measurement. This property makes them less preferable than other important quantum measurements.

There is another interesting family of quantum measurements called POVM (abbrev. for Positive Operator-Valued Measurement). POVMs are best viewed as a special case of the general measurement formalism, providing the simplest means by which one can study general measurement statistics, without the necessity for knowing the post-measurement state. A POVM is a set of positive operators $\{E_m\}$, that satisfy the completeness equation $\sum_m E_m = I$, and the probability of each outcome m is given by $p(m) = \langle E_m u, u\rangle$, for any state vector u. An important difference between projective measurement and POVM is that the elements of a POVM are not necessarily orthogonal, with the consequence that the number of elements in the POVM, can be larger than the dimension of the Hilbert space they act in. Every POVM can be associated to some unitarily equivalent general measurement.

A **composite quantum system** is a combined system composed of two or more distinct physical systems. The state space of such systems is described by the tensor product of the state spaces of the component physical systems, i.e. if $\mathcal{H}_1$ and $\mathcal{H}_2$ are two quantum systems whose state spaces are given by the orthonormal basis $\{u_i\}_{i\in I}$ and $\{v_j\}_{j\in J}$ respectively, then the composite quantum system $\mathcal{H}_1 \otimes \mathcal{H}_2$ is given by $\{u_i \otimes v_j\}_{i\in I, j\in J}$.

If a state vector of a composite system can be be written as a tensor product of state vectors of its component systems, then it is called **separable**. Otherwise, it is called **entangled**. The entangled states play a crucial role in quantum computation and quantum information.

1.4.2 Alternate Formulation in Language of Density Matrices

Up to now, we have formulated the postulates of quantum mechanics using the language of state vectors. But in certain circumstances, the description of the current condition of a quantum system, called **the quantum state**, is difficult to be represented by means of state vectors. For this reason, an alternate formulation is developed using a tool known as the **density operator** or **density matrix**. This alternate formulation is mathematically equivalent to the state vector approach, but it provides a more convenient language for thinking and operating on quantum systems whose state vectors are not commonly known.

More precisely, suppose a quantum system is in one of a number of state vectors u_i with respective probabilities p_i, where i is an index. Such a pair $\{p_i, u_i\}$ is called an **ensemble of pure states**. The density operator for the system is given by the equation

$$\rho = \sum_i p_i P_i,$$

where $P_i = u_i u_i^*$ is the projection operator onto $\mathrm{span}\{u_i\}$.

Suppose that the evolution of a closed system is described by the unitary operator U that carries an initial state vector to some other state vector. If the system is initially in the state u_i with probability p_i, then after the evolution occurs, the system will be in the state Uu_i with same probability p_i (probability doesn't change under unitary evolutions of states). Thus, the evolution of density operator is described as follows

$$\rho = \sum_i p_i(u_i u_i^*) \overset{U}{\mapsto} \sum_i p_i(Uu_i)(Uu_i)^* = \sum_i p_i U(u_i u_i^*)U^* = U\rho U^*.$$

Given quantum measurement operators $\{M_m\}$ and an initial state u_i in which the quantum system is before the measurement, the probability of getting result m is

$$p(m|i) = \langle M_m u_i, M_m u_i\rangle = \mathrm{tr}(M_m^* M_m(u_i u_i^*)) = \mathrm{tr}(M_m^* M_m P_i),$$

where 'tr' stands for the trace of the operator. The total probability getting result m is

$$p(m) = \sum_i p(m|i)p_i = \sum_i p_i \mathrm{tr}(M_m^* M_m P_i) = \mathrm{tr}(M_m^* M_m \rho),$$

and all possible probabilities add up to one, i.e. $\sum_m p(m) = 1$. After a measurement which yields the result m, we obtain a new ensemble of state vectors

$$u_i^m = \frac{M_m u_i}{\sqrt{\mathrm{tr}(M_m^* M_m P_i)}}$$

with respective probabilities $p(i|m)$. Using elementary probability theory, one can easily show that the corresponding density operator ρ_m of such ensemble $\{u_i^m, p(i|m)\}$ is

$$\begin{aligned}
\rho_m &= \sum_i p(i|m)(u_i^m)(u_i^m)^* \\
&= \sum_i \frac{p(m|i)p(i)}{p(m)} \frac{(M_m u_i)(M_m u_i)^*}{\mathrm{tr}(M_m^* M_m P_i)} \\
&= \sum_i p_i \frac{M_m P_i M_m^*}{\mathrm{tr}(M_m^* M_m \rho)} \\
&= \frac{M_m \rho M_m^*}{\mathrm{tr}(M_m^* M_m \rho)}.
\end{aligned}$$

The fact that the quantum measurement operators on a quantum system $\mathcal{H}$ satisfy the completeness equation implies that the trace of the density operator ρ is equal to one

$$\mathrm{tr}(\rho) = \mathrm{tr}(I_{\mathcal{H}}\rho) = \mathrm{tr}(\sum_m M_m^* M_m \rho) = \sum_m \mathrm{tr}(M_m^* M_m \rho) = \sum_m p(m) = 1.$$

This result combined together with $\mathrm{tr}(P_i) = \mathrm{tr}(u_i u_i^*) = \langle u_i, u_i \rangle = 1$ shows that all probabilities p_i given by ensemble pairs add up to one

$$1 = \mathrm{tr}(\rho) = \mathrm{tr}(\sum_i p_i P_i) = \sum_i p_i \mathrm{tr}(P_i) = \sum_i p_i = 1.$$

Moreover, if h is an arbitrary vector in the existing operator system $\mathcal{H}$ described by the ensembles $\{p_i, u_i\}$ for some index i, then

$$\langle \rho h, h \rangle = \sum_i p_i \langle P_i h, h \rangle$$

$$\begin{aligned} &= \sum_i p_i \langle u_i, h\rangle \langle h, u_i\rangle \\ &= \sum_i p_i |\langle u_i, h\rangle|^2 \geq 0, \end{aligned}$$

which shows the positivity of the density operator ρ. To summarize, an operator ρ is a density operator for some ensemble $\{p_i, u_i\}$ of the quantum system if and only if ρ is a positive-definite operator with trace one.

A quantum system whose state vector u is known, is said to be in a **pure state**, and the density operator describing the state of the quantum system in this case, has the form $\rho = uu^*$. In contrast, if the density operator ρ is not expressible in this form, then the quantum system is said to be in a **mixed state**. In general, a density operator can be formed from an ensemble of density operators $\{p_i, \rho_i\}$, each of which arises from some ensemble $\{p_{ij}, u_{ij}\}$ of pure states, so that each state vector u_{ij} has probability $p_i p_{ij}$, i.e.

$$\rho = \sum_{i,j} p_i p_{ij} u_{ij} u_{ij}^* = \sum_i p_i \left(\underbrace{\sum_{i,j} p_{ij} u_{ij} u_{ij}^*}_{\rho_i} \right) = \sum_i p_i \rho_i.$$

We can say that the density operator ρ is a **mixture** of density operators ρ_i, each of which is a mixture of state vectors u_{ij}. In the case of a composite quantum system, say $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n$ where each $\mathcal{H}_i$ is prepared in the state ρ_i, the composite density operator ρ is given by the tensor product of each of them $\rho = \rho_1 \otimes \rho_2 \otimes \cdots \otimes \rho_n$.

Let $\mathcal{E} : B(\mathcal{H}) \to B(\mathcal{H})$ be a completely positive map that transforms a given quantum state ρ, in which the quantum system $\mathcal{H}$ is prepared, into another quantum state ρ', i.e. $\mathcal{E}(\rho) = \rho'$. This map is called a **quantum operation**. All quantum operations $\mathcal{E} : B(\mathcal{H}) \to B(\mathcal{H})$ are convex linear in the sense that: if ρ is obtained by a randomly selected ensemble

$\{p_i, \rho_i\}$ such that $\rho = \sum_i p_i\rho_i$, then

$$\mathcal{E}(\rho) = \mathcal{E}(\sum_i p_i\rho_i) = \sum_i p_i\mathcal{E}(\rho_i).$$

The physical quantum operation is a general tool for describing the evolution of quantum systems in a wide variety of circumstances. Evolution of closed quantum systems occurs via unitary transformations, and it corresponds to a map $\rho \overset{U}{\mapsto} U\rho U^*$ for some unitary operator U.

In the background of quantum computing, the quantum systems of interest are "open" as they are exposed to external environments during computations. In such cases, the open system is regarded as part of a larger closed system given by the composite of the system and the environment. If $\mathcal{H}_s$ and $\mathcal{H}_e$ are the system and environment Hilbert spaces, then the closed system is represented on $\mathcal{H} = \mathcal{H}_e \otimes \mathcal{H}_s$. The characterization of evolutions $\mathcal{E} : B(\mathcal{H}_s) \to B(\mathcal{H}_s)$ in open quantum systems $\mathcal{H}_s$, requires first that density operators are mapped to density operators $\mathcal{E}(\rho) = \rho'$; i.e. the probability densities are mapped to probability densities

$$\mathrm{tr}(\rho) = 1 = \mathrm{tr}(\mathcal{E}(\rho)).$$

Thus, such a map should be positive and **trace-preserving**. However, this property must be preserved even when the system is exposed to all possible environments. In terms of the map, if $\mathcal{E} : B(\mathcal{H}_s) \to B(\mathcal{H}_s)$ describes an evolution of the system, then the extension map

$$I_{\mathcal{H}_e} \otimes \mathcal{E} : B(\mathcal{H}_e \otimes \mathcal{H}_s) \to B(\mathcal{H}_e \otimes \mathcal{H}_s)$$

must be also positive and trace-preserving for all possible environment Hilbert spaces $\mathcal{H}_e$. Hence, the widely accepted working definition of a **quantum operation**(or evolution) on a Hilbert space $\mathcal{H}$, is a completely positive, trace-preserving map $\mathcal{E}$ on $B(\mathcal{H})$.

Deriving from Theorem 1.3.1, every quantum operation (i.e. every completely positive trace-preserving map) $\mathcal{E} : B(\mathcal{H}) \to B(\mathcal{K})$ has an "operator-sum representation" of the form

$$\mathcal{E}(\rho) = \sum_i E_i \rho E_i^*,$$

for some set of (non-unique) Kraus operators $\{E_i\}$ which map input Hilbert space $\mathcal{H}$ to the output Hilbert space $\mathcal{K}$, and $\sum_i E_i^* E_i \leq I_{\mathcal{H}}$. In the literature, trace-preserving quantum operations are also known as **quantum channels.**

A linear functional $s : B(\mathcal{H}) \to \mathbb{C}$ is called positive provided $s(T) \geq 0$ for all positive operators $T \in B(\mathcal{H})$. Moreover, s is called a state provided $s(I_{\mathcal{H}}) = 1$. If $\mathcal{H}$ is a quantum system, then s is a quantum state. A quantum state $s : B(\mathcal{H} \otimes \mathcal{K}) \to \mathbb{C}$ is called **separable** if it is a convex combination of tensor states

$$s = \sum_i r_i s_i \otimes t_i,$$

where $s_i : B(\mathcal{H}) \to \mathbb{C}$ and $t_i : B(\mathcal{K}) \to \mathbb{C}$ are states on the component systems, and $r_i \geq 0$ with $\sum_i r_i = 1$. States that are not separable are said to be **entangled**.

Let $\phi : B(\mathcal{L}) \to B(\mathcal{K})$ be a linear map, where $\mathcal{L}$ and $\mathcal{K}$ are finite dimensional Hilbert spaces with $\dim(\mathcal{L}) = n$ and $\dim(\mathcal{K}) = m$. Then we define the **Choi matrix of** ϕ to be

$$C_\phi = (\phi(e_i e_j^*)) \in B(\mathcal{H} \otimes \mathcal{K}),$$

where $\{e_i\}_{i=1}^n$ is an orthonormal basis of $\mathcal{L}$. This association between ϕ and C_ϕ turns out to be an isomorphism, which is known as Choi-Jamiolkowski isomorphism [3]. Because much is already known about linear operators, the Choi-Jamiolkowski isomorphism provides a simple way of studying linear maps on operators – just study the associated linear operators instead.

Given a quantum state $s : B(\mathcal{H} \otimes \mathcal{K}) \to \mathbb{C}$, one can show that the Choi matrix of s is its density matrix, basically $C_s = (s(e_i e_j^* \otimes f_k f_l^*))$ where $\{e_i\}$ and $\{f_k\}$ are orthonormal basis for $\mathcal{H}$ and $\mathcal{K}$ respectively. Let $\mathcal{E} : B(\mathcal{L}) \to B(\mathcal{K})$ be a quantum channel and let $I_{\mathcal{H}} \otimes \mathcal{E} : B(\mathcal{H} \otimes \mathcal{L}) \to B(\mathcal{H} \otimes \mathcal{K})$ be its canonical extension map for all possible $\mathcal{H}$. Then $\mathcal{E}$ is called an **entanglement breaking channel** provided that $s \circ (I_{\mathcal{H}} \otimes \mathcal{E}) : B(\mathcal{H} \otimes \mathcal{L}) \to \mathbb{C}$ is a separable state for all quantum states $s : B(\mathcal{H} \otimes \mathcal{K}) \to \mathbb{C}$.

The structure of entanglement breaking channels and their properties have contributed a number of results related to the hard problem of additivity of capacity in quantum information theory. Motivated by these results, there was a big need to generalize the concept of entanglement breaking channels. This lead to the definition of the classes of **partially entanglement breaking** quantum channels [5], [12]. To define such classes, the notion of **partial separability** of quantum states was introduced using Schmidt number of the density matrices.

1.4.3 The Schmidt Number of a Density Matrix

Theorem 1.4.1 (Schmidt Decomposition Theorem, [19])**.** *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces of dimensions n and m respectively. For any vector U in the tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$, there exist orthonormal sets $\{u_1, u_2, \ldots, u_k\} \subseteq \mathcal{H}_1$ and $\{v_1, v_2, \ldots, v_k\} \subseteq \mathcal{H}_2$ for $k = \min(n, m)$, such that*

$$v = \sum_{i=1}^{k} \alpha_i u_i \otimes v_i, \textit{ for some nonnegative real numbers } \alpha_i \geq 0.$$

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces of dimensions n and m respectively, and let ρ be the density matrix that describes the current condition of the composite quantum system $\mathcal{H}_1 \otimes \mathcal{H}_2$. For simplicity of notations, we will consider the quantum systems $\mathcal{H}_1$ and $\mathcal{H}_2$

with the equivalent representations $\mathbb{C}^n$ and $\mathbb{C}^m$ respectively, and the density matrix as a square matrix in $M_n \otimes M_m$.

The Schmidt Decomposition Theorem is a basic tool in quantum information theory. It is essentially the restatement of the Singular Value Decomposition in a different context. The formal proof of this theorem works by noticing that there is a linear isomorphism between $\mathbb{C}^n \otimes \mathbb{C}^m$ and $M_{n,m}$ given by associating a vector $u_e \otimes v_e \in \mathbb{C}^n \otimes \mathbb{C}^m$ with the matrix $u_e v_e^* \in M_{n,m}$ and extending linearly. We will denote the matrix associated to the vector U by A_u. In this context, applying the Singular Value Decomposition to A_u gives the Schmidt Decomposition of U. The least number of terms required in the summation is known as the **Schmidt Rank** of U. One can realize that, the Schmidt rank of U is equal to the number of nonzero singular values of the matrix A_u associated to U, i.e. the rank of A_u. In a similar way, the nonnegative real constants α_e's are exactly the singular values of A_u, and they are often called the **Schmidt coefficients**.

Further, since each $u_e v_e^* \in M_{n,m}$ has rank 1, we see that even if we remove the requirement that the sets above be orthonormal, it is impossible to write U as the sum of fewer elementary tensors. To summarize, any vector $U \in \mathbb{C}^n \otimes \mathbb{C}^m$ can be written as

$$U = \sum_{e=1}^{k} (u_e \otimes v_e),$$

for some sets of vectors $\{u_1, u_2, \ldots, u_k\} \subseteq \mathbb{C}^n$ and $\{v_1, v_2, \ldots, v_k\} \subseteq \mathbb{C}^m$ for $k \leq \min(n, m)$. And any rank one positive semi-definite matrix UU^* can therefore be written as

$$UU^* = \sum_{e=1}^{k} (u_e \otimes v_e) \sum_{f=1}^{k} (u_f \otimes v_f)^* = \sum_{e,f=1}^{k} (u_e u_f^* \otimes v_e v_f^*).$$

In other words, given a vector U of Schmidt rank at most k, we have

$$UU^* \in \left\{ \sum_{e,f=1}^{k} (u_e u_f^* \otimes v_e v_f^*) : \{u_1, u_2, \dots, u_k\} \subseteq \mathbb{C}^n,\, \{v_1, v_2, \dots, v_k\} \subseteq \mathbb{C}^m \right\}.$$

Let $s : M_n \otimes M_n (\cong B(\mathcal{H} \otimes \mathcal{K})) \to \mathbb{C}$ be a quantum state. Then s can be represented by its density matrix ρ_s given by the Choi matrix of s, i.e. $\rho_s = (s(E_{ij} \otimes E_{kl}))$, where $\{E_{ij}\}$ and $\{E_{kl}\}$ are the canonical matrix units for M_n and M_m, respectively. Being a positive definite matrix, ρ_s can be written as a sum of rank one positive semi-definite matrices $\rho_s = \sum_{l=1}^{p} U_l U_l^*$ with each $U_l \in \mathbb{C}^n \otimes \mathbb{C}^m$.

If the density matrix ρ_s of a given state $s : M_n \otimes M_m \to \mathbb{C}$ is a finite sum of rank one positive semi-definite matrices UU^* with $U \in \mathbb{C}^n \otimes \mathbb{C}^m$ of Schmidt rank at most k with $k \leq \min(n, m)$, then the least such number k is called the **Schmidt number** [29] of ρ_s.

The Schmidt number of a density matrix tells us the "level of entanglement or separability" of the state. A state $s : M_n \otimes M_m \to \mathbb{C}$ is called **maximally entangled** if the Schmidt number of its density matrix is $\min(n, m)$. Also, note that separable states are represented by density matrices of the form $\rho = \sum_j \sigma_j \otimes \tau_j$, where each $\sigma_j = \sum_e u_e^j (u_e^j)^* \geq 0$, $\tau_j = \sum_f v_f^j (v_f^j)^* \geq 0$. These are exactly the density matrices, whose Schmidt numbers are equal to 1.

A state $s : M_n \otimes M_m \to \mathbb{C}$ is called **k-separable** [29], [12] if the Schmidt number of its density matrix ρ_s is at most k with $k \leq \min(n, m)$. The quantum channels that carry any quantum states into k-separable states, are called **k-partially entanglement breaking channels**.

1.5 Motivation and Work

The study of trace-preserving completely positive maps has recently made a big impact in quantum information theory. But to talk about the notion of complete positivity, one should be able to define the operator systems on which these maps are applied. In [23], two operator systems have been constructed over a given Archimedean order unit space V, denoted as $\mathrm{OMIN}(V)$ and $\mathrm{OMAX}(V)$, as the analogues of MIN and MAX functors, and their properties were developed accordingly. Moreover, it was shown that the entanglement breaking maps between matrix algebras M_n and M_m are exactly the completely positive maps between these two new operator systems $\mathrm{OMIN}(M_n)$ and $\mathrm{OMAX}(M_m)$.

Motivated by this work, in this thesis we introduce other general operator systems that all have the given Archimedean order unit space as their first level.
Chapter 2 is the main work of this thesis that describes these important steps. In Section 2.1, we give the necessary preliminaries for the construction of any operator system on a given Archimedean order unit $*$-vector space.
Given an Archimedean order unit $*$-vector space, there are possibly many different operator systems that all have the given Archimedean order unit space as their ground level. This fact motivates us into finding a way to describe such operator systems. Hence, in Section 2.2 we introduce a key technique to construct a general k-minimal operator system $\mathrm{OMIN}_k(V)$ and k-maximal operator system $\mathrm{OMAX}_k(V)$ for a fixed $k \in \mathbb{N}$. We develop their properties, characterizations and dualizations. We show that the special cases of these universal operator systems correspond to the minimal and the maximal operator systems introduced in [23].
Furthermore, given an operator system $\mathcal{S}$ and a fixed $k \in \mathbb{N}$, in Section 2.3 we construct two new important operator systems on $\mathcal{S}$, the super k-minimal operator system $\mathrm{OMIN}_k^s(\mathcal{S})$ and the super k-maximal operator system $\mathrm{OMAX}_k^s(\mathcal{S})$. The properties that characterize

these super operator system structures and the results from this developed theory, lead us to a new interpretation of k-partially entanglement breaking maps between matrix algebras encountered in quantum information theory [13], [5], [27], [14]. Hence, in Section 2.4 we characterize the k-partially entanglement breaking maps between two full matrix algebras in terms of super operator systems $\mathrm{OMIN}_k^s(\cdot)$ and $\mathrm{OMAX}_k^s(\cdot)$.

Speaking of completely positive maps and their operator-sum representations, there is a lot of interest in the family of Kraus operators representing the given completely positive map. Kraus operators are very important for constructions of many different quantum states, such as Matrix Product States (MPS). Briefly, MPS summarize many of the physical properties of quantum spin chains. Of particular interest in various physical contexts is the subset of translationally invariant MPS, also known as finitely correlated states [9], [25]. Their importance stems from the fact that with a simple tensor, A, one can fully describe relevant states of N spins, which at least in principle should require to deal with an exponential number of parameters when written in a basis in the corresponding tensor Hilbert space $\mathcal{H}^{\otimes N}$. Thus, all physical properties of such states are contained in A. For each such MPS, there is a canonical form given as follows:

Let us consider a system with periodic boundary of N(large but finite) sites, each of them with an associated d-dimensional Hilbert space. Then a translationally invariant MPS ξ on some n-dimensional virtual Hilbert spaces that are connected to the real physical d-dimensional spaces through a map, is defined by a family of Kraus operators $\mathcal{P}_1 = \{A_i \in M_n, i = 1, \ldots, d\}$

$$\xi = \sum_{i_1,\ldots,i_N} \mathrm{tr}(A_{i_1} \cdots A_{i_N}) e_{i_1} \otimes \cdots \otimes e_{i_N},$$

where e_{i_j} are the orthonormal basis for each $\mathbb{C}^n$. There exists a close relation between such an MPS and the completely positive map given by $\phi(X) = \sum_i A_i X A_i^*$. A lot of applications of these states in quantum information theory problems [25] are related to the injectivity of the following map:

$$X \overset{\Gamma_L}{\longmapsto} \sum_{i_1,\ldots,i_L} \operatorname{tr}(XA_{i_1}\cdots A_{i_L}) e_{i_1}\otimes\cdots\otimes e_{i_L},$$

where $A_{i_j} \in \mathcal{P}_1$. Note that Γ_L is injective if and only if the set of L-fold products $\{A_{i_1}\cdots A_{i_L} : 1 \le i_1,\ldots,i_L \le d\}$ spans the entire space of $n\times n$ matrices, M_n. Moreover, if the map ϕ associated to this family of Kraus operators is trace-preserving, i.e. $\sum_i A_i A_i^* = I_n$, then injectivity of Γ_L implies the injectivity of $\Gamma_{L'}$ for all $L' \ge L$. In [25], Michael Wolf has conjectured that this spanning happens at some level $L \le n^2$, provided a nice set of initial matrices is given.

Motivated by these results, in Chapter 3, we investigate the necessary and sufficient conditions for an arbitrary finite set of matrices $P_1 = \{A_1, A_2, \ldots, A_l\}$ in the full matrix algebra M_n of square matrices such that their set of m-fold products

$$P_m = \underbrace{P_1 \cdot P_1 \cdot \cdots \cdot P_1}_{\text{m times}} = \left\{ \prod_{i=1}^{m} A_{k_i} : \text{ where } k_i \in \{1,2,\ldots,l\} \right\}$$

span the whole matrix algebra M_n. We show some special cases of this result. In Section 3.1, we provide background and necessary definitions and in the rest of the chapter, we show our work. In the last section, we relate this result to completely positive maps and quantum channels.

Chapter 2

Universal Operator System Structures On Ordered Spaces

2.1 Preliminaries

Let V be a complex vector space. An **involution** on V is a conjugate linear map $* : V \to V$ given by $v \mapsto v^*$, such that $v^{**} = v$ and $(\lambda v + w)^* = \bar{\lambda} v^* + w^*$ for all $\lambda \in \mathbb{C}$ and $v, w \in V$. The complex vector space V together with the involution map is called a **$*$-vector space**. If V is a $*$-vector space, then we let $V_{sa} = \{v \in V | v = v^*\}$ be the real vector space of self-adjoint elements of V.

A **cone** $W \subseteq V$ is a nonempty subset of a real vector space V, such that $W + W \subseteq W$ and $\mathbb{R}^+ W \subseteq W$ where $\mathbb{R}^+ = [0, \infty)$. Moreover, W is called a **proper cone** if $W \cap (-W) = \{0\}$. An **ordered $*$-vector space** (V, V^+) is a pair consisting of a $*$-vector space V and a proper cone $V^+ \subseteq V_{sa}$. The elements of V^+ are called positive and there is a partial order $\geq$(respectively, $\leq$) on V_{sa} defined by $v \geq w$ (respectively, $w \leq v$) if and only if $v - w \in V^+$ for $v, w \in V_{sa}$.

An element $e \in V_{sa}$ is called an **order unit** for V if for all $v \in V_{sa}$, there exists a real number $t > 0$ such that $te \geq v$. This order unit e is called **Archimedean order unit** if whenever $v \in V$ and $te + v \in V^+$ for all real $t > 0$, we have that $v \in V^+$. In this case, we call the triple (V, V^+, e) an **Archimedean ordered unital $*$-vector space** or an **AOU space** for short.

Let (V, V^+), (W, W^+) be two ordered *-vector spaces with order units e, e' respectively. A linear map $\phi : V \to W$ is called **positive** if $\phi(V^+) \subseteq W^+$, and **unital** if it is positive and $\phi(e) = e'$. Moreover, ϕ is an **order isomorphism** if ϕ is bijective, and both ϕ, ϕ^{-1} are positive. Note that, if $\phi : V \to W$ is positive, then $\phi(v^*) = \phi(v)^*$ for all $v \in V$.

Let V be a $*$-vector space and let $M_{n,m}(V)$ denote the set of all $n \times m$ matrices with entries in V . The natural addition and scalar multiplication turn $M_{n,m}(V)$ into a complex vector space. We often write $M_{n,m} = M_{n,m}(\mathbb{C})$, and let $\{E_{i,j}\}_{i,j=1}^{n,m}$ denote its canonical matrix unit system. For a given matrix $A \in M_{n,m}$, we write $\bar{A}, A^t$ and A^* for the complex conjugate, transpose and complex adjoint of A, respectively. If $n = m$, we write $M_{n,n} = M_n$ and I_n for the identity matrix. The matrix units determine the linear identifications $M_{n,m}(V) \cong M_{n,m} \otimes V \cong V \otimes M_{n,m}$, where

$$v = (v_{ij}) \mapsto \sum_{i,j=1}^{n,m} E_{i,j} \otimes v_{ij} \text{ and } v = (v_{ij}) \mapsto \sum_{i,j=1}^{n,m} v_{i,j} \otimes E_{ij}, \text{ respectively.}$$

More often than not, we will use the first linear identification with the matrix coefficients on the right. There are two basic natural operations which link the finite matrix linear spaces $M_{n,m}(V)$: the direct sum and the matrix product. Given $v \in M_{n,m}(V)$ and $w \in M_{p,q}(V)$, then we define the direct sum $v \oplus w \in M_{n+p,m+q}(V)$ by

$$v \oplus w = \begin{bmatrix} v & 0 \\ 0 & w \end{bmatrix} \in M_{n+p,m+q}(V).$$

On the other hand, given $A = (a_{ki}) \in M_{p,n}$, $B = (b_{jl}) \in M_{m,q}$ and $v = (v_{ij}) \in M_{n,m}(V)$, we define the matrix product $AvB \in M_{p,q}(V)$ by

$$AvB = \left[\sum_{i,j=1}^{n,m} a_{ki} v_{ij} b_{jl}\right]_{k,l=1}^{p,q} \in M_{p,q}(V).$$

Note that, if $V = M_r$ and we use the identification $M_{n,m}(M_r) \cong M_{n,m} \otimes M_r$, then we have for any $X \in M_{p,n}$, $a \in M_{n,m}(M_r)$ and $Y \in M_{m,q}$

$$XaY = (X \otimes I_r)a(Y \otimes I_r) \in M_{p,q}(M_r).$$

Let V, W be two $*$-vector spaces. Given a linear map $\phi : V \to W$ and $n, m \in \mathbb{N}$, we have a corresponding map $\phi^{(n,m)} : M_{n,m}(V) \to M_{n,m}(W)$ defined by $\phi^{(n,m)}(v) = (\phi(v_{ij}))$. We let $\phi^{(n)} = \phi^{(n,n)} : M_n(V) \to M_n(W)$.

If we are given v, w, A and B as above, then one can easily verify that

$$\phi^{(n+p,m+q)}(v \oplus w) = \phi^{(n,m)}(v) \oplus \phi^{(p,q)}(w)$$

and

$$\phi^{(p,q)}(AvB) = A\phi^{(n,m)}(v)B.$$

Moreover, if $\phi : V \to W$ is a linear map and $W = M_k$, then we have for any $X \in M_{p,n}$, $a \in M_{n,m}(V)$ and $Y \in M_{m,q}$

$$\phi^{(p,q)}(XaY) = X\phi^{(n,m)}(a)Y = (X \otimes I_k)\phi^{(n,m)}(a)(Y \otimes I_k).$$

Let V be a $*$-vector space. We define a $*$-operation on $M_n(V)$ by letting $[v_{ij}]^* = [v_{ji}^*]$. With respect to this operation, $M_n(V)$ is a $*$-vector space. We let $M_n(V)_{sa}$ be the set of all self-adjoint elements of $M_n(V)$. Let $\{C_n\}_{n=1}^{\infty}$ be a family of proper cones $C_n \subset M_n(V)_{sa}$ for all $n \in \mathbb{N}$, such that they are **compatible**, i.e $X^* C_n X \subseteq C_m$ for all $X \in M_{n,m}$, $m \in \mathbb{N}$. We call each such C_n a **matrix cone**, the family of these matrix cones a **matrix ordering on V**, and the pair $(V, \{C_n\}_{n=1}^{\infty})$ a **matrix ordered $*$-vector space**.

Let $(V, \{C_n\}_{n=1}^{\infty})$ and $(W, \{C_n'\}_{n=1}^{\infty})$ be matrix ordered $*$-vector spaces. Then a linear map $\phi : V \to W$ is called **completely positive** if $\phi^{(n)}(C_n) \subseteq C_n'$ for all $n \in \mathbb{N}$. Moreover, ϕ is called a **complete order isomorphism** if ϕ is invertible and both ϕ, ϕ^{-1} are completely positive.

Let $(V, \{C_n\}_{n=1}^{\infty})$ be a matrix ordered $*$-vector space. Let $e \in V_{sa}$ be the distinguished order unit for V. Consider the corresponding diagonal matrix $e_n = e \otimes I_n \in M_n(V)_{sa}$ for all $n \in \mathbb{N}$, where I_n is the unit of M_n. We say that e is a **matrix order unit** for V if e_n is an order unit for the ordered $*$-vector space $(M_n(V), C_n)$ for each n. We say e is an **Archimedean matrix order unit** if e_n is an Archimedean order unit for the ordered $*$-vector space $(M_n(V), C_n)$ for each n. Finally, we say that the triple $(V, \{C_n\}_{n=1}^{\infty}, e)$ is an **(abstract) operator system**, if V is a $*$-vector space, $\{C_n\}_{n=1}^{\infty}$ is a matrix ordering on V, and e is an Archimedean matrix order unit.

We denote by $B(\mathcal{H})$ the space of all bounded linear operators acting on a Hilbert space $\mathcal{H}$. A **concrete operator system** $\mathcal{S}$ is a subspace of $B(\mathcal{H})$ such that $\mathcal{S} = \mathcal{S}^*$ and $I_{\mathcal{H}} \in \mathcal{S}$. As is the case for many classes of subspaces (and subalgebras) of $B(\mathcal{H})$, there is an abstract characterization of concrete operator systems, as was shown in [23]. If $\mathcal{S} \subseteq B(\mathcal{H})$ is a concrete operator system, then we observe that $\mathcal{S}$ is a $*$-vector space, $\mathcal{S}$ inherits an order structure from $B(\mathcal{H})$, and has $I_{\mathcal{H}}$ as an Archimedean order unit. Moreover, since $\mathcal{S} \subseteq B(\mathcal{H})$, we have that $M_n(\mathcal{S}) \subseteq M_n(B(\mathcal{H})) \cong B(\mathcal{H}^n)$ and hence $M_n(\mathcal{S})$ inherits a natural order structure from $B(\mathcal{H}^n)$ and the $n \times n$ diagonal matrix

$$I_n \otimes I_{\mathcal{H}} = \begin{bmatrix} I_{\mathcal{H}} & & & 0 \\ & I_{\mathcal{H}} & & \\ & & \ddots & \\ 0 & & & I_{\mathcal{H}} \end{bmatrix}$$

is an Archimedean order unit for $M_n(\mathcal{S})$. In other words, $\mathcal{S}$ is an abstract operator system $(\mathcal{S}, \{M_n(\mathcal{S})^+\}_{n=1}^{\infty}, I_{\mathcal{H}})$, where each matrix cone $M_n(\mathcal{S})^+$ contains $n \times n$ positive matrices in $M_n(B(\mathcal{H}))$ for all $n \in \mathbb{N}$. We will call this matrix ordering $\{M_n(\mathcal{S})^+\}_{n=1}^{\infty}$ as **the natural operator system structure** of $\mathcal{S}$ inherited by the order structure of $B(\mathcal{H})$. The following result of Choi and Effros [4], [20] shows that the converse is also true.

Theorem 2.1.1 (Choi-Effros)**.** *Every concrete operator system $\mathcal{S}$ is an (abstract) operator system. Conversely, if $(V, \{C_n\}_{n=1}^{\infty}, e)$ is an (abstract) operator system, then there exists a Hilbert space $\mathcal{H}$, a concrete operator system $\mathcal{S} \subseteq B(\mathcal{H})$, and a complete order isomorphism $\phi : V \to \mathcal{S}$ with $\phi(e) = I_{\mathcal{H}}$.*

Definition 2.1.2. *Let (V, V^+) be an ordered $*$-vector space with order unit e. A linear function $\phi : V \to M_k$ is called* **unital** *if $\phi(e) = I_k$, and ϕ is called* **positive** *if $\phi(V^+) \subseteq M_k^+$. A positive linear function ϕ is called* **diagonal** *if*

$$\phi(e) = D_k = \begin{bmatrix} d_1 & & & 0 \\ & d_2 & & \\ & & \ddots & \\ 0 & & & d_k \end{bmatrix}, \; d_i \in \mathbb{R}^+, \; 1 \leq i \leq k.$$

Furthermore, ϕ is called **a rank r diagonal map** *if*

$$\phi(e) = D_r \oplus 0 = \begin{bmatrix} D_r & 0 \\ 0 & 0 \end{bmatrix}, \, 1 \leq r \leq k,$$

where D_r is an $r \times r$ strictly entry-wise positive diagonal matrix.

All positive unital maps ϕ are rank k diagonal maps with $\phi(e) = I_k$. Set

$$S_k(V) = \{\phi : V \to M_k \,|\, \phi \text{ unital positive maps } \}.$$

Remark 2.1.3. *Let (V, V^+) be an ordered $*$-vector space with order unit e.*

(i) Let $\phi : V \to M_k$ be a linear map. We can think of ϕ as a $k \times k$ matrix of linear functionals $\phi_{ij} : V \to \mathbb{C}$, i.e. $\phi = [\phi_{ij}]_{i,j=1}^k \in M_k(V')$.

(ii) If $\phi : V \to M_k$ is a positive linear function, then $\phi(v^*) = \overline{\phi(v)}^t$ for all $v \in V$, where t stands for the transpose.

(iii) If ϕ is a positive linear map, then all the diagonal entries of ϕ are positive linear functionals. Moreover, if ϕ is unital, then the diagonal entries are states.

(iv) If ϕ is a rank r diagonal map, $\phi(e) = D_r \oplus 0$, $1 \leq r \leq k$, then $\phi_{ii}(e) = 0$ for all $i \geq r+1$. This implies $\phi_{ii} = 0$ on V, since it is a positive linear functional. The positivity of ϕ implies that all other ϕ_{ij}, except those of indices $(1,1) \leq (i,j) \leq (r,r)$, are the zero functionals. Denote

$$\tilde{\phi} = [\phi_{ij}]_{i,j=1}^r : V \to M_r.$$

One can easily verify that $\tilde{\phi}$ is positive and $\tilde{\phi}(e) = D_r$. Hence, any rank r diagonal map can be written as $\phi = \tilde{\phi} \oplus 0$, where $\tilde{\phi} : V \to M_r$ is a diagonal map, $1 \leq r \leq k$.

The following proposition is a generalization of Proposition 3.12 and Proposition 3.13 encountered in [24]:

Proposition 2.1.4. *Let (V, V^+, e) be an AOU space. If $v \in V$ and $\phi(v) \geq 0$ for each $\phi \in S_k(V)$, then $v \in V^+$. Furthermore, if $\phi(v) = 0$ for all such ϕ, then $v = 0$.*

Proof. Let $s : V \to \mathbb{C}$ be a state on V, i.e. $s \in S(V)$. Define

$$\phi = I_k \otimes s = \begin{bmatrix} s & & & 0 \\ & s & & \\ & & \ddots & \\ 0 & & & s \end{bmatrix}_{k \times k} : V \to M_k.$$

Then $\phi \in S_k(V)$. Let $v \in V$. Then $\phi(v) \geq 0$ if and only if $s(v) \geq 0$. This implies $v \in V^+$. Moreover, $\phi(v) = 0$ if only if $s(v) = 0$, which implies $v = 0$.(see [24], Proposition 3.12 and Proposition 3.13) ☐

Lemma 2.1.5. *Let (V, V^+, e) be an AOU space. Let $\phi : V \to M_k$ be a non-zero positive map. Then ϕ is unitarily equivalent to a diagonal map $\psi : V \to M_k$ of rank $r \leq k$.*

Proof. Let $\phi : V \to M_k$ be a positive map such that $\phi(e) = P \in M_k^+$. The rank of the matrix P is at least 1 and at most k. Without loss of generality, assume $\text{rank}(P) = r$, for some $1 \leq r \leq k$. There exists a unitary U, such that $U^*PU = D_r \oplus 0$, where D_r is an $r \times r$ diagonal matrix with positive diagonal entries. Define $\psi : V \to M_k$ by

$$\psi(\cdot) = U^*\phi(\cdot)U.$$

It is straightforward to check that ψ is a positive linear map with $\psi(e) = D_r \oplus 0$, i.e. ψ is a rank r diagonal map, $1 \leq r \leq k$. Hence, ϕ is unitarily equivalent to such a map. ☐

Let (V, V^+) be an ordered $*$-vector space with order unit e. We endow the real subspace V_{sa} with the so-called **order seminorm** $\|v\| = \inf\{r | -re \leq v \leq re\}$. We extend this order seminorm on V_{sa} to a **$*$-seminorm** on V that preserves the $*$-operation, i.e. $\|v^*\| = \|v\|$ for all $v \in V$. We define **the order seminorm on V** to be a $*$-seminorm $|||\cdot|||$ on V with the property that $|||v||| = \|v\|$ for all $v \in V_{sa}$. If e is an Archimedean order unit, then all these order seminorms become norms because $|||v||| = 0$ implies $v \leq 0$ and $v \geq 0$. Every order seminorm $\|\cdot\|$ on V induces an **order topology** on V, the topology with a basis consisting of balls $B_\epsilon(v) = \{w \in V : \|v - w\| < \epsilon\}$ for $v \in V$ and $\epsilon > 0$. Note that since $\|\cdot\|$ is not necessarily a norm, this topology is not necessarily Hausdorff.

Remark 2.1.6. *Let $A \in M_k$ be a $k \times k$ matrix. Recall the usual matrix norm*

$$\|A\| = \sup\{\|Ax\| : x \in \mathbb{C}^k \text{ with } \|x\| \leq 1\}.$$

From matrix theory, we know that if we divide A into block matrices as follows

$$A = \begin{bmatrix} A_r & * \\ * & * \end{bmatrix} \in M_k,$$

where $A_r \in M_r$, $1 \leq r \leq k$, then $\|A\| \geq \|A_r\|$. Moreover, if $A = A_r \oplus 0$, then the norms are the same, i.e. $\|A\| = \|A_r\|$.

Definition 2.1.7. *Let (V, V^+) be an ordered $*$-vector space with order unit e. We define the* **k-minimal order seminorm** $\|\cdot\|_{k-min} : V \to [0, +\infty)$ *by*

$$\|v\|_{k-min} = \sup\big\{\|\phi(v)\| \ : \phi \in S_k(V)\big\}.$$

Note: When k=1, the k-minimal order seminorm becomes the usual minimal order seminorm defined in [24] by $\|v\|_m = \sup\{|s(v)| : s \text{ is a state}\}$. And $\|\cdot\|_m \leq |||\cdot|||$ for every other $*$-seminorm $|||\cdot|||$ on V.

By definition, we have $\|e\|_{k-min} = \|e\|_m = |||e||| = 1$. If (V, V^+, e) is an AOU space, and $\phi : V \to M_k$ is positive such that the norm of $\|\phi(e)\| \leq 1$ with respect to k-minimal norm, then ϕ is called a **contraction**.

Proposition 2.1.8. *Let (V, V^+) be an ordered $*$-vector space with an order unit e and let $k \in \mathbb{N}$. Then*

$$\|v\|_{k-min} = \sup\left\{ \|\phi(v)\| \, : \phi \in \bigcup_{r=1}^{k} S_r(V) \right\}.$$

Proof. For fixed $k \in \mathbb{N}$, let $r \leq k$. If $r = k$, then it is clear that $\|v\|_{k-min} = \|v\|_{r-min}$. Assume $r < k$ and let $\phi \in S_k(V)$. Write

$$\phi = [\phi_{ij}]_{i,j=1}^{k} = \begin{bmatrix} [\phi_{ij}]_{i,j=1}^{r} & * \\ * & * \end{bmatrix}.$$

Denote $[\phi_{ij}]_{i,j=1}^{r} = \phi_r$. Then, one can easily verify that $\phi_r \in S_r(V)$. Hence, we have

$$\|\phi(v)\| = \left\| \begin{bmatrix} \phi_r(v) & * \\ * & * \end{bmatrix} \right\| \geq \|\phi_r(v)\|.$$

By taking supremum over all $\phi \in S_k(V)$, we obtain

$$\|v\|_{k-min} \geq \|\phi_r(v)\|, \text{ for all } \phi_r \in S_r(V).$$

This implies

$$\|v\|_{k-min} \geq \|v\|_{r-min}.$$

As a result, we conclude that

$$\|v\|_{k-min} = \sup\left\{ \|\phi(v)\| \, : \phi \in \bigcup_{r=1}^{k} S_r(V) \right\}.$$

$\square$

Theorem 2.1.9. *Let (V, V^+, e) be an AOU space. Let $|||\cdot|||$ be any order norm on V such* *that* $\|\cdot\|_{k-min} \leq |||\cdot|||$ *and let* $\phi : V \to M_k$ *be a positive map. If* $\|\phi\|$ *denotes the norm of the positive map* ϕ *with respect to the order norm* $|||\cdot|||$, *then* $\|\phi\| = \|\phi(e)\|_{M_k}$. *Moreover, if* ϕ *is unital, then* $\|\phi\| = 1$.

Proof. By Lemma 2.1.5 above, we have that any positive map $\phi : V \to M_k$ is unitarily equivalent to a rank $r \leq k$ diagonal map $\psi : V \to M_k$ such that $\psi = (\tilde{\psi}) \oplus 0$, with $\tilde{\psi}(e) = D_r$, for all $1 \leq r \leq k$. Therefore, $\|\phi\| = \|\psi\|$. Note that $\|\psi\| = \|\tilde{\psi}\|$. Hence, it's enough to show that $\|\phi\| = \|\phi(e)\|$ for any diagonal map ϕ of rank k. Since $\phi(e) = D_k \geq 0$ invertible, then $\psi = \phi(e)^{-1/2}\phi\phi(e)^{-1/2}$ is a unital positive map, and for any $v \in V$, we have

$$\begin{aligned}
\|\phi(v)\| &= \|\phi(e)^{1/2}\psi(v)\phi(e)^{1/2}\| \leq \|\phi(e)\|^{1/2} \cdot \|\psi(v)\| \cdot \|\phi(e)\|^{1/2} \\
&\leq \|\phi(e)\| \cdot \sup\{\|\varphi(v)\| : \varphi \in S_k(V)\} \\
&= \|\phi(e)\| \cdot \|v\|_{k-min} \leq \|\phi(e)\| \cdot |||v|||.
\end{aligned}$$

So, we have $\|\phi\| \leq \|\phi(e)\|$. In addition, since $\|e\|_{k-min} = |||e||| = 1$, it follows that $\|\phi\| = \|\phi(e)\|$. Moreover, if ϕ is unital, then $\|\phi\| = 1$. □

2.2 The k-Minimal and the k-Maximal Operator System Structures on Ordered Spaces

Let (V, V^+, e) be an AOU space. A matrix ordering on (V, V^+, e) is a matrix ordering $\mathcal{C} = \{C_n\}_{n=1}^{\infty}$ on V such that $C_1 = V^+$. An **operator system structure** on (V, V^+, e) is a matrix ordering $\{C_n\}_{n=1}^{\infty}$ such that $(V, \{C_n\}_{n=1}^{\infty}, e)$ is an operator system with $C_1 = V^+$. Given an operator system $(\mathcal{S}, \{P_n\}_{n=1}^{\infty}, e)$ and a unital positive map $\varphi : V \to \mathcal{S}$ such that $V^+ = \varphi^{-1}(P_1)$, one obtains an operator system structure on V by setting $C_n = \varphi_n^{-1}(P_n)$. We shall call this **the operator system structure induced by** φ. Conversely, given an operator system structure on V, by letting $\mathcal{S} = V$ and letting φ be the identity map, then we see that the given operator system structure is the one induced by φ.

If $\mathcal{P} = \{P_n\}_{n=1}^{\infty}$ and $\mathcal{Q} = \{Q_n\}_{n=1}^{\infty}$ are two matrix orders on V , we say that $\mathcal{P}$ is **stronger** than $\mathcal{Q}$ (respectively, $\mathcal{Q}$ is **weaker** than $\mathcal{P}$) if $P_n \subseteq Q_n$ for all $n \in \mathbb{N}$. Note that $\mathcal{P}$ is stronger than $\mathcal{Q}$ if and only if for every n, and every $A, B \in M_n(V)_{sa}$, the inequality $A \leq_{\mathcal{P}} B$ implies that $A \leq_{\mathcal{Q}} B$, where the subscripts are used to denote the partial orders induced by $\mathcal{P}$ and $\mathcal{Q}$, respectively. Equivalently, $\mathcal{P}$ is stronger than $\mathcal{Q}$ if and only if the identity map on V is completely positive from $(V, \{P_n\}_{n=1}^{\infty})$ to $(V, \{Q_n\}_{n=1}^{\infty})$.

2.2.1 The Definition of the k-Minimal Operator System Structure

Before setting up the k-minimal operator system structure on an AOU space (V, V^+, e), recall the weakest operator system structure, introduced in [23] and denoted by $\mathcal{C}^{min}(V) = \{C_n^{min}(V)\}_{n=1}^{\infty}$, where

$$\begin{aligned} C_n^{min}(V) &= \big\{(v_{ij}) \in M_n(V) : (s(v_{ij})) \in M_n^+, \text{ for all } s \in S(V)\big\} \\ &= \big\{(v_{ij}) \in M_n(V) : (f(v_{ij})) \in M_n^+, \ f \text{ positive linear functional}\big\} \\ &= \big\{(v_{ij}) \in M_n(V) : \alpha^*(v_{ij})\alpha \in V^+, \text{ for all } \alpha \in \mathbb{C}^n\big\}. \end{aligned}$$

$\mathcal{C}^{min}(V)$ is the operator system structure on V, induced by the inclusion of V into $C(S(V))$, the C^*-algebra of continuous funtions on $S(V)$, set of states on V. And OMIN(V) is the operator system $(V, \mathcal{C}^{min}(V), e)$, which can be identified as a subspace of $C(S(V))$, up to complete order isomorphism.

In the next result, we generalize the complex version of Kadison's characterization of function systems [24], [20]:

Theorem 2.2.1. *Let (V, V^+, e) be an AOU space and fix $k \in \mathbb{Z}^+$. Give V the order topology generated by the k-minimal order norm, denoted as V_{k-min}, and endow the space of unital positive linear maps $S_k(V) = \{\phi : V \to M_k \,|\phi(V^+) \subseteq \mathbb{R}^+ \cup \{0\}$ with $\phi(e) = I_k\}$ with the* *corresponding weak***-topology. Then* $S_k(V)$ *is a compact space, and the map*

$$\Gamma : V \to M_k(C(S_k(V))) \text{ given by } \Gamma(v)(\phi) = \phi(v)$$

is an injective map that is an order isomorphism onto its range with the property that $\Gamma(e) = I_k$. *Furthermore,* Γ *is an isometry with respect to the k-minimal order norm on* V *and the sup norm on* $M_k(C(S_k(V)))$.

Proof. If (V, V^+, e) is an AOU space, then its dual V^* is a normed $*$-vector space, not necessarily an AOU space. For fixed $k \in \mathbb{N}$, one can show that $M_k(V^*) = \{\phi = (\phi_{ij}) : V \to M_k \,|\, \phi_{ij} \in V^* \text{ for all } 1 \leq i, j \leq k\}$ is a normed $*$-vector space, too. Then the unit ball of $M_k(V^*)$ is defined as

$$(M_k(V^*))_1 = \{\phi \in M_k(V^*) : \|\phi\| \leq 1\}.$$

Endowing V with any order norm $|||\cdot|||$ makes $S_k(V)$ a subset of the unit ball of $M_k(V^*)$. In addition, suppose that $\{\phi_\lambda\}_{\lambda\in\Lambda} \subseteq S_k(V)$ is a net of these maps, and $\lim \phi_\lambda = \phi$ in the weak*-topology for some $\phi \in M_k(V^*)$. Then for any $v \in V^+$ we have that $\lim \phi_\lambda(v) = \phi(v)$, and since $\phi_\lambda(v)$ is non-negative for all λ, it follows that $\phi(v) \geq 0$ for all $v \in V^+$. Hence ϕ is a positive linear map. Moreover, $\phi(e) = \lim \phi_\lambda(e) = \lim I_k = I_k$ so that ϕ is unital. Thus $S_k(V)$ is closed in the weak*-topology.

In the case of the k-minimal order norm, we have $S_k(V) \subseteq (M_k(V^*_{k-min}))_1$, and the latter $(M_k(V^*_{k-min}))_1 \cong (M_k(V_{k-min}))^*_1$. It follows from Alaoglu's Theorem([6], Theorem 3.1), that $(M_k(V_{k-min}))^*_1$ is compact in the weak*-topology, which implies that $(M_k(V^*_{k-min}))_1$ is compact, too. Since $S_k(V)$ is a closed subset of this compact ball, we have that $S_k(V)$ is compact in the weak*-topology.

Consider the continuous matrix-valued functions $\hat{v} : S_k(V) \to M_k$ given by $\hat{v}(\phi) = \phi(v) \in M_k$. The collection of such continuous functions $\{\hat{v} : S_k(V) \to M_k\}$ together with

$\|\cdot\|_{k-min}$ norm, form the unital C^*-algebra $M_k(C(S_k(V)))$, i.e.

$$M_k(C(S_k(V))) \equiv \big\{\hat{v} : S_k(V) \to M_k | \hat{v} \text{ continuous matrix-valued function } \big\}.$$

Let $\Gamma : V \to M_k(C(S_k(V)))$ be the map given by $\Gamma(v)(\phi) = \phi(v)$. If $\Gamma(v) = 0$ for some $v \in V$, then $\phi(v) = 0$ for all $\phi \in S_k(V)$. It follows from Proposition 2.3 that $v = 0$. Therefore, Γ is one-to-one.

In addition, if $v \in V^+$, then for any $\phi \in S_k(V)$ we have that $\Gamma(v)(\phi) = \phi(v) \in M_k^+$ by the positivity of ϕ. Hence the function $\Gamma(v)$ takes on nonnegative values and $\Gamma(v) \in M_k(C(S_k(V)))^+$. Conversely, if $\Gamma(v) \in M_k(C(S_k(V)))^+$, then for all $\phi \in S_k(V)$ we have that $\phi(v) = \Gamma(v)(\phi) \geq 0$. This implies $v \in V^+$ by Proposition 2.3. Therefore, Γ is an order isomorphism onto its range. Finally, if $v \in V$, then

$$\begin{aligned} \|v\|_{k-min} &= \sup\{\|\phi(v)\| \mid \phi \in S_k(V)\} \\ &= \sup\{\|\Gamma(v)(\phi)\| : \phi \in S_k(V)\} \\ &= \|\Gamma(v)\|_\infty. \end{aligned}$$

so that Γ is an isometry with respect to the k-minimal order norm on V and the sup norm on $M_k(C(S_k(V)))$. □

Remark 2.2.2. *Since unital C^*-algebras are operator systems, the order isomorphism map Γ of Kadison's Representation Theorem induces an operator system structure $\{C_n\}_{n=1}^\infty$ on V. We have $C_1 = V^+ = \Gamma^{-1}(P_1)$, where P_1 denotes the set of nonnegative matrix-valued continuous functions on $S_k(V)$. In addition, we say $(v_{ij}) \in C_n$ if and only if $(\Gamma(v_{ij})) \in M_n(M_k(C(S_k(V))))^+$, if and only if $(\phi(v_{ij})) \in M_{nk}^+$ for every $\phi \in S_k(V)$.*

Definition 2.2.3. *Let (V, V^+, e) be an AOU space. For each $n \in \mathbb{N}$ set*

$$C_n^{k-min}(V) = \Big\{(v_{ij}) \in M_n(V) : (\phi(v_{ij})) \geq 0, \text{ for all } \phi \in S_k(V)\Big\},$$

$\mathcal{C}^{k-min}(V) = \{C_n^{k-min}(V)\}_{n=1}^{\infty}$ *and define* $\text{OMIN}_k(V) = (V, \mathcal{C}^{k-min}(V), e)$.

By the definition and the remark above, $\mathcal{C}^{k-min}(V)$ is the operator system structure on V induced by the inclusion of V into $M_k(C(S_k(V)))$. We call $\mathcal{C}^{k-min}(V)$ **the k-minimal operator system structure** on V, and we call $\text{OMIN}_k(V)$ **the k-minimal operator system**.

Properties of the k-minimal operator system structure:

(1) When $k = 1$, $C_n^{1-min}(V) = C_n^{min}(V)$ for all $n \in \mathbb{N}$:

$$\begin{aligned} (v_{ij}) \in C_n^{1-min}(V) &\iff (s(v_{ij})) \in M_n^+, \text{ for all } s \in S_1(V) = S(V), \\ &\iff \alpha^*(v_{ij})\alpha \in V^+, \text{ for all } \alpha \in \mathbb{C}^n. \end{aligned}$$

(2) $C_n^{k-min}(V) \subseteq C_n^{min}(V)$, for all $n \in \mathbb{Z}^+$:

Let $(v_{ij}) \in C_n^{k-min}(V)$ for some fixed $k \in \mathbb{Z}^+$, and let $\alpha \in \mathbb{C}^n$. Then

$$0 \leq (\alpha^* \otimes I_k)(\phi(v_{ij}))(\alpha \otimes I_k) = \phi(\alpha^*[v_{ij}]\alpha), \text{ for all } \phi \in S_k(V).$$

This implies $\alpha^*(v_{ij})\alpha \in V^+$, for all $\alpha \in \mathbb{C}^n$, i.e. $(v_{ij}) \in C_n^{min}(V)$.

(3) $C_n^{h-min}(V) \subseteq C_n^{k-min}(V)$ for all $h \geq k$:

Let $(v_{ij}) \in C_n^{h-min}(V)$. The equality holds when $h = k$. Suppose $h > k$ and let $\phi \in S_k(V)$ and $s \in S(V)$. Define $\Phi : V \to M_h$ by

$$\Phi = \phi \oplus \underbrace{s \oplus \cdots \oplus s}_{\text{(h-k) times}} = \begin{bmatrix} \phi & & 0 \\ & s & & \\ & & \ddots & \\ 0 & & & s \end{bmatrix}.$$

One can easily verify that Φ is a well-defined positive linear function with $\Phi(e) = I_h$, i.e. $\Phi \in S_h(V)$. This implies $(\Phi(v_{ij})) \geq 0$. Thus, we have:

$$0 \leq (\Phi(v_{ij})) = \begin{bmatrix} \phi(v_{ij}) & & & 0 \\ & s(v_{ij}) & & \\ & & \ddots & \\ 0 & & & s(v_{ij}) \end{bmatrix}_{i,j}.$$

By the canonical reshuffling, we obtain:

$$\begin{aligned} 0 \leq (\Phi(v_{ij})) \quad &\iff \quad \begin{bmatrix} (\phi(v_{ij})) & & & 0 \\ & (s(v_{ij})) & & \\ & & \ddots & \\ 0 & & & (s(v_{ij})) \end{bmatrix} \geq 0 \\ &\iff \quad (\phi(v_{ij})) \geq 0, \text{ for all } \phi \in S_k(V) \\ &\iff \quad (v_{ij}) \in C_n^{k-min}(V). \end{aligned}$$

(4) The identity map $\imath : \mathrm{OMIN}_h(V) \to \mathrm{OMIN}_k(V)$ is completely positive, whenever $h \geq k$.

Proposition 2.2.4. *Let $(W, \{C_n\}_{n=1}^{\infty})$ be a matrix ordered $*$-vector space. For a fixed $k \in \mathbb{N}$, let $\phi : W \to M_k$ be a linear map. Then ϕ is completely positive if and only if ϕ is k-positive.*

Proof. If ϕ is completely positive, then ϕ is k-positive for each $k \in \mathbb{N}$. Now assume ϕ is k-positive. Before showing $\phi^{(n)}(w) \geq 0$ for all $w \in C_n, n \geq k$, we will prove the following result:

Given any vector $x \in \mathbb{C}^n \otimes \mathbb{C}^k$, there exists an isometry $\beta : \mathbb{C}^k \to \mathbb{C}^n$ and a vector $\tilde{x} \in \mathbb{C}^k \otimes \mathbb{C}^k$ such that $(\beta \otimes I_k)(\tilde{x}) = x$ for all $n \geq k$ in $\mathbb{N}$.

- Let $e_i = e_i^{(k)} = (0, \dots, 0, 1_i, 0, \dots, 0)$ be the usual basis vectors for $\mathbb{C}^k$, and let $x \in \mathbb{C}^n \otimes \mathbb{C}^k$. Then there exist unique vectors $x_i \in \mathbb{C}^n$, $i = 1, 2, \dots, k$ with $x = \sum_{i=1}^k x_i \otimes e_i^{(k)}$. Let $\mathcal{F} \subseteq \mathbb{C}^n$ be the subspace spanned by the vectors x_i. Then we have $\dim\mathcal{F} \leq k \leq n$. Thus, we may find an isometry $\beta : \mathbb{C}^k \to \mathbb{C}^n$ whose range contains $\mathcal{F}$. For each i, we have a unique vector $\tilde{x}_i \in \mathbb{C}^k$ such that $\beta(\tilde{x}_i) = x_i$. Thus, if $\tilde{x} = \sum_{i=1}^k \tilde{x}_i \otimes e_i^{(k)}$, then $(\beta \otimes I_k)(\tilde{x}) = x$.

- Now, let $w \in C_n$ and $n \geq k$. Then we have

$$\begin{aligned} \langle \phi^{(n)}(w)x, x\rangle &= \langle \phi^{(n)}(w)(\beta \otimes I_k)(\tilde{x}), (\beta \otimes I_k)(\tilde{x})\rangle \\ &= \langle (\beta^* \otimes I_k)\phi^{(n)}(w)(\beta \otimes I_k)(\tilde{x}), (\tilde{x})\rangle \\ &= \langle \phi^{(k)}(\beta^* w \beta)\tilde{x}, \tilde{x}\rangle \geq 0. \end{aligned}$$

Thus, ϕ is n-positive for all $n \in \mathbb{N}$, i.e. completely positive. □

Lemma 2.2.5. *Let $(W, \{C_n\}_{n=1}^\infty)$ be a matrix ordered $*$-vector space, let X be a compact space. If $\psi : W \to M_k(C(X))$ is k-positive, then ψ is completely positive.*

Proof. Define $\pi_x : M_k(C(X)) \to M_k$ to be the point-evaluation matrix function, i.e. $\pi_x((f_{ij})) = (f_{ij}(x))$. It is clear that π_x is a well-defined $*$-homomorphism. Moreover, π_x is completely positive. Consider $\pi_x \circ \psi : W \to M_k$. Let $(w_{ij}) \in C_k$, then $(\psi(w_{ij})) \in M_k(M_k(C(X)))^+$, which implies $(\pi_x(\psi(w_{ij}))) \in M_{k^2}^+$, since π_x is a completely positive map. The k-positivity of ψ implies $\pi_x \circ \psi : W \to M_k$ is a k-positive map, and therefore completely positive by Proposition 2.2.4. As a result, ψ is completely positive. □

Theorem 2.2.6. *Let (V, V^+, e) be an AOU space. If $(W, \{C_n\}_{n=1}^\infty)$ is a matrix ordered $*$-vector space and $\phi : W \to \mathrm{OMIN}_k(V)$ is k-positive, then ϕ is completely positive. Moreover, if $\tilde{V} = (V, \{\tilde{C}_n\}_{n=1}^\infty, e)$ is an operator system with $\tilde{C}_1 = V^+$ such that for every operator system W, any k-positive map $\psi : W \to \tilde{V}$ is completely positive and any positive* *map $f : \tilde{V} \to M_k$ is completely positive, then the identity map on V is a complete order isomorphism from $\tilde{V}$ onto* $\mathrm{OMIN}_k(V)$*.*

Proof. It is clear that, up to complete order isomorphism, $\mathrm{OMIN}_k(V)$ can be identified with a subspace of $M_k(C(S_k(V)))$. We know that $S_k(V)$ is a compact space. Substituting $X = S_k(V)$ in Lemma 2.2.5, we get $\phi : W \to M_k(C(S_k(V)))$ is completely positive, i.e. $\phi : W \to \mathrm{OMIN}_k(V)$ is completely positive.

Now, let $\tilde{V} = (V, \{\tilde{C}_n\}_{n=1}^{\infty}, e)$ be another operator system with $\tilde{C}_1 = V^+$ such that for every operator system W, any k-positive map $\psi : W \to \tilde{V}$ is completely positive and any positive map $f : \tilde{V} \to M_k$ is completely positive. By composing with evaluation maps, it follows that any positive map $f : \tilde{V} \to M_k(C(X))$ where X is a compact space, is completely positive. Since $\mathrm{OMIN}_k(V) \subseteq M_k(C(S_k(V)))$ and the identity map on V, $\imath : \tilde{V} \to \mathrm{OMIN}_k(V)$ is positive, we have that the identity map is completely positive. Hence, $M_k(\tilde{V})^+ \subseteq M_k(\mathrm{OMIN}_k(V))^+$.

Suppose there exists $(v_{ij}) \in M_k(\mathrm{OMIN}_k(V))^+ \backslash M_k(\tilde{V})^+$. Then there exists $f : M_k(V) \to \mathbb{C}$ such that $f((v_{ij})) < 0$, but $f(M_k(\tilde{V})^+) \subseteq \mathbb{R}^+$. Let $f_{ij} = f(v \otimes E_{ij})$ where E_{ij} is a matrix unit of M_k, and let $\phi : V \to M_k$ be given by $\phi(v) = (f_{ij}(v))$.

Then $\phi : V \to M_k$ is positive since for any $v \in V^+$ and $\lambda = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_k \end{pmatrix} \in \mathbb{C}^k$,

$$\langle \phi(v)\lambda, \lambda \rangle = \sum_{i,j} \bar{\lambda}_i f_{ij}(v)\lambda_j = \sum_{i,j=1}^{k} f(\bar{\lambda}_i (v \otimes E_{ij})\lambda_j) = f((v\bar{\lambda}_i\lambda_j)) \geq 0.$$

It follows that $\phi : \mathrm{OMIN}_k(V) \to M_k$ is completely positive by definition of the structure of $\mathrm{OMIN}_k(V)$. But $(\phi(v_{ij})) \ngeq 0$, a contradiction. Therefore, we conclude that $M_k(\tilde{V})^+ = M_k(\mathrm{OMIN}_k(V))^+$. Now, consider the identity map $\imath : \mathrm{OMIN}_k(V) \to \tilde{V}$. It follows that this identity map is k-positive since $\tilde{V}^+ = C_1^{k-min}(V) = V^+$ and $M_k(\tilde{V})^+ = M_k(\mathrm{OMIN}_k(V))^+$, and therefore completely positive by the characterization of $\tilde{V}$.

Hence, the identity map on V is a complete order isomorphism from $\tilde{V}$ onto $\mathrm{OMIN}_k(V)$.

□

Proposition 2.2.7. *Let (V, V^+, e) be an AOU space, and let $f : V \to M_k$ be any positive linear matrix-valued map, and let*

$$C_n^k(V) = \big\{(v_{ij}) \in M_n(V) | \, (f(v_{ij})) \in M_{nk}^+, \, f : V \to M_k \text{ p.l.m.}\big\}.$$

Then $\{C_n^k(V)\}_{n=1}^{\infty}$ is a matrix ordering on V, and $C_n^k(V) = M_n(\mathrm{OMIN}_k(V))^+$.

Proof. In order to show that $C_n^k(V)$ is a matrix ordering, it suffices to show that $C_n^k(V) = C_n^{k-min}(V)$ for all n. One can see that $C_n^k(V) \subseteq C_n^{k-min}(V)$ is trivial, since $S_k(V)$ is just a subset of all positive linear matrix-valued functions from V to M_k. On the other hand, let $(v_{ij}) \in C_n^{k-min}(V)$. We have shown that any positive linear map $f : V \to M_k$ is congruent to some $g \oplus 0 : V \to M_k$, where $g \in S_r(V)$ for some $1 \leq r \leq k$. Since $C_n^{k-min}(V) \subseteq C_n^{r-min}(V)$ for all $1 \leq r \leq k$, then $(g(v_{ij})) \geq 0$, which implies $(f(v_{ij})) \geq 0$. As a result, $C_n^{k-min}(V) \subseteq C_n^k(V)$. Hence, $C_n^k(V) = C_n^{k-min}(V)$ is a matrix ordering. □

Remark 2.2.8. *The above result shows that we can define the universal k-minimal operator system structure in a more general way, as*

$$C_n^{k-min}(V) = \big\{(v_{ij}) \in M_n(V) | \, (f(v_{ij})) \geq 0, \, f : V \to M_k \textit{ positive linear map } \big\}.$$

2.2.2 The Definition of the k-Maximal Operator System Structure

Given a $*$-vector space V, we identify the vector space $M_n(V)$ of all $n \times n$ matrices with entries in V with the (algebraic) tensor product $M_n \otimes V$ in the natural way. Then $M_n(V)$ equipped with the involution map as $[v_{ij}]^* = [v_{ji}^*]$, is a $*$-vector space. We have that $M_n(V)_{sa} = (M_n)_{sa} \otimes V_{sa}$, where the right-hand side is the algebraic tensor of real vector spaces.

Let (V, V^+, e) be an AOU space. Recall the strongest matrix ordering $\mathcal{D}^{max}(V) = \{D_n^{max}(V)\}_{n=1}^\infty$, where each matrix cone $D_n^{max}(V)$ is given by

$$\begin{aligned} D_n^{max}(V) &= \left\{ \sum_{i=1}^{k} a_i \otimes v_i : v_i \in V^+, a_i \in M_n^+, 1 \le i \le k, k \in \mathbb{N} \right\} \\ &= \left\{ A \operatorname{diag}(v_1, \dots, v_m)\, A^* : A \in M_{n,m}, v_i \in V^+, m \in \mathbb{N} \right\}, \end{aligned}$$

and e is just a matrix order unit for this ordering, as was shown in [23].

Definition 2.2.9. *Let (V, V^+, e) be an AOU space and let $\{C_n^{min}(V)\}_{n=1}^\infty$ be the minimal operator system structure on V. For some fixed $k \in \mathbb{N}$, set*

$$D_n^{k-max}(V) = \big\{ ADA^* \mid A \in M_{n,mk},\, D = diag(D_1, \dots, D_m), \text{ where } D_l \in C_k^{min}(V),\, 1 \le l \le m,\, m \in \mathbb{N} \big\}$$

and $\mathcal{D}^{k-max}(V) = \{D_n^{k-max}(V)\}_{n=1}^\infty$.

Proposition 2.2.10. *Let (V, V^+, e) be an AOU space. Then $\mathcal{D}^{k-max}(V)$ is a matrix ordering on V and e is a matrix order unit for this ordering. In particular, $\mathcal{D}^{1-max}(V)$ is the strongest matrix ordering on V.*

Proof. Need to check the three conditions of being a matrix ordering on V:

(1) $D_n^{k-max}(V)$ is a cone in $M_n(V)_{sa}$ for each $n \in \mathbb{N}$, and in particular, $D_1^{k-max}(V) = V^+$: For each $n \in \mathbb{N}$, D_n^{k-max} is a non-empty subset of $M_n(V)_{sa}$ as one can see in $D_n^{max}(V) \subseteq D_n^{k-max}(V) \subseteq C_n^{min}(V)$, with the following two poperties:

   (i) $D_n^{k-max}(V)$ is closed under positive scalar multiplication:

Let $\lambda \in \mathbb{R}^+$ and $ADA^* \in D_n^{k-max}(V)$, then

$$\begin{aligned}\lambda(ADA^*) &= (\sqrt{\lambda}A)D(\sqrt{\lambda}A)^* \\ &= A(\lambda)A^* \in D_n^{k-max}(V).\end{aligned}$$

(ii) $D_n^{k-max}(V)$ is closed under addition:

Let $ADA^*, B\tilde{D}B^* \in D_n^{k-max}(V)$, where $A \in M_{n,mk}$, $B \in M_{n,pk}$, $D = \mathrm{diag}(D_1, \ldots, D_m)$, $\tilde{D} = \mathrm{diag}(\tilde{D}_1, \ldots, \tilde{D}_p)$, then we have:

$$ADA^* + B\tilde{D}B^* = \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} D & 0 \\ 0 & \tilde{D} \end{bmatrix} \begin{bmatrix} A^* \\ B^* \end{bmatrix} \in D_n^{k-max}(V).$$

In particular, for $n = 1$, we have $V^+ = D_1^{max}(V) \subseteq D_1^{k-max}(V) \subseteq C_1^{min}(V) = V^+$, i.e. $D_1^{k-max}(V) = V^+$.

(2) $D_n^{k-max}(V) \cap -D_n^{k-max}(V) = \{0\}$ for all $n \in \mathbb{N}$:

Note that $D_n^{k-max}(V) \cap -D_n^{k-max}(V) \subseteq C_n^{min}(V) \cap -C_n^{min}(V) = \{0\}$.

(3) $X\, D_n^{k-max}(V)\, X^* \subseteq D_m^{k-max}(V)$ for all $X \in M_{m,n}$, for all $m, n \in \mathbb{N}$:

Let $ADA^* \in D_n^{k-max}(V)$, $X \in M_{m,n}$ for any $m, n \in \mathbb{N}$,

then $X(ADA^*)X^* = (XA)D(XA)^* \in D_m^{k-max}(V)$,

i.e. $XD_n^{k-max}(V)X^* \subseteq D_m^{k-max}(V)$ for all m, n.

Hence, (1), (2) and (3) show that $\mathcal{D}^{k-max}(V)$ is a matrix ordering on V. It remains to show that e is a matrix order unit for this ordering. It is clear that e is an (Archimedean) order unit for $D_1^{k-max}(V)$ since $D_1^{k-max}(V) = V^+$. Since $\mathcal{D}^{max}(V)$ is the strongest matrix ordering on V, then we have $D_n^{max}(V) \subseteq D_n^{k-max}(V)$ for all $n \in \mathbb{N}$. We know e_n is an order unit for $(M_n(V), D_n^{max}(V))$. It follows that e_n is an order unit for $(M_n(V), D_n^{k-max}(V))$, i.e. e is a matrix order unit for $\mathcal{D}^{k-max}(V)$. As a result, $\mathcal{D}^{k-max}(V)$ is a matrix ordering on V. In particular, for $k = 1$ we have $C_1^{min}(V) = V^+$, therefore $\mathcal{D}^{1-max}(V) = \mathcal{D}^{max}(V)$ is the strongest matrix ordering on V. $\square$

For a given AOU space (V, V^+, e) and a fixed $k \in \mathbb{N}$, the triple $(V, \mathcal{D}^{k-max}(V), e)$ is just a matrix ordered $*$-vector space, not yet an operator system. If e was an Archimedean matrix order unit for the matrix ordering $\mathcal{D}^{max}(V)$, then e would be an Archimedean matrix order unit for $\mathcal{D}^{k-max}(V)$ too, and henceforth $\mathcal{D}^{k-max}(V)$ would be an operator system structure on V. But, as shown in [23], $(V, \mathcal{D}^{max}(V), e)$ might not be an operator system in all cases. There are AOU spaces (V, V^+, e) such as $(C([0,1]), C([0,1])^+, 1)$, for which the matrix order unit e is not Archimedean. Specifically, let $V = C([0,1])$ be the vector space of complex-valued functions on the unit interval, with V^+ the usual cone of positive functions and e the constant function taking value 1, and let $P(t) = \begin{bmatrix} 1 & e^{2\pi it} \\ e^{-2\pi it} & 1 \end{bmatrix} \in M_2(C([0,1]))_{sa}$. Then, in [23] it has been shown that $re_2 + P(t) = \begin{bmatrix} 1+r & e^{2\pi it} \\ e^{-2\pi it} & 1+r \end{bmatrix} \in D_2^{max}(C([0,1]))$ for every $r > 0$, but $P(t) \notin D_2^{max}(C([0,1]))$. This shows that $e = 1$ can not be an Archimedean matrix order unit. As a result, $(C([0,1]), \mathcal{D}^{max}(C([0,1])), 1)$ can not be an operator system.

In order to transform $(V, \mathcal{D}^{max}(V), e)$ and consequently $(V, \mathcal{D}^{k-max}(V), e)$ into operator systems, we need to discuss the **Archimedeanization process** for matrix ordered spaces. This theory was developed in detail for ordered $*$-vector spaces in [24], and generalized to matrix ordered spaces with a matrix order unit e, in [23]. Here, we will just give a review of the basic steps and results that make this "matrix Archimedeanization" happen:

(i) For any ordered $*$-vector space (V, V^+) with order unit e, there is a functorial way to produce an AOU space, called the **Archimedeanization** of V, which is the largest quotient of V containing the class of e as an Archimedean order unit. Specifically, if (V, V^+, e) is an ordered $*$-vector space with order unit e, we define

$$D = \{v \in V_{sa} : re + v \in V^+ \text{ for all } r > 0\} \text{ and } N = \bigcap_{f \in S(V)} \ker f.$$

Then N is a complex subspace of V closed under the $*$-operation, so that the quotient

V/N is a $*$-vector space in the natural way and

$$(V/N)_{sa} = \{v + N : v \in V_{sa}\}.$$

We define an order on V/N by

$$(V/N)^+ = \{v + N : v \in D\}.$$

The Archimedeanization of V is defined to be

$$V_{Arch} = (V/N, (V/N)^+, e).$$

In [24], it is shown that V_{Arch} is an AOU space and it is characterized by the following universal property:

The quotient $q : V \to V_{Arch}$ is a positive linear map and whenever (W, W^+, e') is an AOU space and $\phi : V \to W$ is a unital positive linear map, there exists a unique positive linear map $\tilde{\phi} : V_{Arch} \to W$ with $\phi = \tilde{\phi} \circ q$.

(ii) If $(V, \{C_n\}_{n=1}^{\infty})$ is a matrix ordered $*$-vector space with matrix order unit e, then we **"matrix Archimedeanize"** as follows:

Let $(V, \{C_n\}_{n=1}^{\infty})$ be a matrix ordered $*$-vector space with matrix order unit e, then for each $n \in N$ define

$$N_n = \bigcap_{f \in S(M_n(V))} \ker f.$$

Note that using the notation of the previous paragraph, we have $N = N_1$. Moreover, $N_n = M_n(N)$ for each $n \in \mathbb{N}$. We may identify $M_n(V/N)$ with $M_n(V)/M_n(N)$. We see that

$$(M_n(V)/M_n(N))_{sa} = \{A + M_n(N) : A^* = A\},$$

and $(e + N)_n = e_n + M_n(N)$. Moreover, for any $X \in M_{n,m}(\mathbb{C})$ we have that $X^* M_n(N) X \subseteq M_m(N)$. Set

$$C_n^{Arch} = \big\{A + M_n(N) \in M_n(V)/M_n(N) : \\ (re_n + A) + M_n(N) \in C_n + M_n(N) \, for \, all \, r > 0\big\},$$

and let

$$V_{Arch} = (V/N, \{C_n^{Arch}\}_{n=1}^{\infty}, e + N).$$

Then $V_{Arch} = (V/N, \{C_n^{Arch}\}_{n=1}^{\infty}, e + N)$ is a matrix ordered $*$-vector space, and $e + N$ is an Archimedean matrix order unit for this space. One can realize that the Archimedeanization of a matrix ordered space $(V, \{C_n\}_{n=1}^{\infty}, e)$ is obtained by forming the Archimedeanization of $(M_n(V), C_n, e_n)$ at each matrix level. This matrix Archimedeanization is characterized by the following universal property:

Let $(V, \{C_n\}_{n=1}^{\infty}, e)$ be a matrix ordered $*$-vector space with matrix order unit e, and let V_{Arch} be the Archimedeanization of V with Archimedean matrix order unit $e + N$. Then there exists a unital surjective completely positive linear map $q : V \to V_{Arch}$ with the property that whenever $(W, \{C'_n\}_{n=1}^{\infty}, e')$ is an operator system with Archimedean order unit e', and $\phi : V \to W$ is a unital completely positive linear map, then there exists a unique completely positive linear map $\tilde{\phi} : V_{Arch} \to W$ with $\phi = \tilde{\phi} \circ q$. Moreover, if V' is any ordered $*$-vector space with an Archimedean order unit and $q' : V \to V'$ is a unital surjective positive linear map with the above property, then V' is isomorphic to V_{Arch} via a unital complete order isomorphism.

(iii) Let $(V, \{C_n\}_{n=1}^{\infty}, e)$ be a matrix ordered $*$-vector space with matrix order unit e, such that (V, C_1, e) is an AOU space. In this case, we obtain $N = \{0\}$ and $V/N = V$.

Then, the operator system V_{Arch} obtained by the matrix Archimedianization of V, consists of the enlarged matrix ordering $\{C_n^{Arch}\}_{n=1}^{\infty}$, given by $C_1^{Arch} = C_1$ and

$$C_n^{Arch} = \{A \in M_n(V) : re_n + A \in C_n \text{ for all } r > 0\} \text{ for all } n \geq 2,$$

together with the Archimedean matrix order unit e. Each C_n^{Arch} is equal to the closure of C_n in the order topology of $M_n(V)$.

Thus, the Archimedeanized matrix ordered $*$-vector space $(V, \mathcal{D}^{max}(V), e)$ with underlying space V, matrix ordering $\mathcal{C}^{max}(V) = \{C_n^{max}(V)\}_{n=1}^{\infty}$, given by $C_1^{max}(V) = D_1^{max} = V^+$ and

$$C_n^{max}(V) = \left\{ A \in M_n(V) : re_n + A \in D_n^{max}(V) \text{ for all } r > 0 \right\},$$

is the so called **maximal operator system** $\mathrm{OMAX}(V) = (V, \mathcal{C}^{max}(V), e)$.

Definition 2.2.11. *Let (V, V^+, e) be an AOU space. We set*

$$C_n^{k-max}(V) = \left\{ A \in M_n(V) :\ re_n + A \in D_n^{k-max}(V) \ \text{ for all } r > 0 \right\},$$

$\mathcal{C}^{k-max}(V) = \{C_n^{k-max}(V)\}_{n=1}^{\infty}$ *and define* $\mathrm{OMAX}_k(V) = (V, \mathcal{C}^{k-max}(V), e)$.

By the definition and the results above, we have that the enlarged matrix ordering $\mathcal{C}^{k-max}(V)$ is an operator system structure on V, which we shall call the **k-maximal operator system structure** on V and $\mathrm{OMAX}_k(V)$ the **k-maximal operator system** on V.

Properties of the k-maximal operator system structure:

(1) When $k = 1$, $C_n^{1-max}(V) = C_n^{max}(V)$ for all $n \in \mathbb{N}$:

This is obvious just because of the fact that $C_1^{min}(V) = V^+$.

Hence, the definitions of both cones coincide for all $n \in \mathbb{N}$.

(2) $C_n^{max}(V) \subseteq C_n^{k-max}(V)$, for all $k \in \mathbb{N}$ and for all $n \in \mathbb{N}$.

(3) $C_n^{k-max}(V) \subseteq C_n^{h-max}(V)$ for all $k \leq h$ in $\mathbb{N}$:

Let $(v_{ij}) \in C_n^{k-max}(V)$. The equality holds for $h = k$.

Suppose $k < h$, then:

(i) If $(v_{ij}) \in D_n^{k-max}(V)$, then $(v_{ij}) = ADA^*$ for some $A \in M_{n,mk}$ and $D = \operatorname{diag}(D_1, D_2, \ldots, D_m)$, where each $D_i \in C_k^{min}(V)$ for all $1 \leq i \leq m$, $m \in \mathbb{N}$. Write $A = \begin{bmatrix} A_1 & A_2 & \cdots & A_m \end{bmatrix}$, where each $A_i \in M_{n,k}$, $1 \leq i \leq m$. Transform the matrix A into:

$$\tilde{A} = \begin{bmatrix} A_1\, 0 & A_2\, 0 & \cdots & A_m\, 0 \end{bmatrix} \in M_{n,hk},$$

by adding $(h-k)$ columns of 0 after each block A_i. Using the same trick, transform the block diagonal matrix D into a bigger block diagonal matrix $\tilde{D} = \operatorname{diag}(\tilde{D}_1, \tilde{D}_2, \ldots, \tilde{D}_m)$, where each diagonal block $\tilde{D}_i$ is maximized by adding a $(h-k) \times (h-k)$ diagonal block of 0, i.e.

$$\tilde{D}_i = \begin{bmatrix} D_i & 0 \\ 0 & 0 \end{bmatrix} \in C_h^{min}(V).$$

Then $(v_{ij}) = ADA^* = \tilde{A}\tilde{D}\tilde{A}^* \in D_n^{h-max}(V)$ and $D_n^{k-max}(V) \subseteq D_n^{h-max}(V)$ for all $k \leq h$.

(ii) Let $(v_{ij}) \in C_n^{k-max}(V)$. Then $re_n + (v_{ij}) \in D_n^{k-max}(V)$ for all $r > 0$. Then by case (i), $re_n + (v_{ij}) \in D_n^{h-max}(V)$ too. Therefore, $(v_{ij}) \in C_n^{h-max}(V)$ and $C_n^{k-max}(V) \subseteq C_n^{h-max}(V)$.

(4) The identity map $\imath : OMAX_k(V) \to OMAX_h(V)$ is completely positive, whenever $k \leq h$.

Lemma 2.2.12. *Let (W, W^+, e) be an AOU space, and let $\{P_n\}_{n=1}^{\infty}$ be an operator system structure on W with $P_1 = W^+$. If $p \in W^+$, $(w_{ij}) \in M_n(W)$ are such that $r(p \otimes I_n) + (w_{ij}) \in P_n$ for all $r > 0$, then $(w_{ij}) \in P_n$.*

Proof. Let $\|\cdot\|$ be an order seminorm for this structure. If $p = 0$, then $(w_{ij}) \in P_n$ is obvious. Let $0 \neq p \in W^+$, and replace p by $\frac{p}{\|p\|} \in W^+$. This implies $r(\frac{p}{\|p\|} \otimes I_n) + (w_{ij}) \in P_n$, too, for all $r > 0$. Since $\|\frac{p}{\|p\|}\| = 1$ and $e - \frac{p}{\|p\|} \in W^+$, then we have

$$re_n + (w_{ij}) = r((e - \frac{p}{\|p\|}) \otimes I_n) + r(\frac{p}{\|p\|} \otimes I_n) + (w_{ij}) \in P_n,$$

for all $r > 0$. Therefore, $(w_{ij}) \in P_n$. □

Theorem 2.2.13. *Let (V, V^+, e) be an AOU space and let $(W, \{P_n\}_{n=1}^{\infty}, e')$ be an (abstract) operator system. If $\phi : \mathrm{OMAX}_k(V) \to W$ is a k-positive map for some given $k \in \mathbb{N}$, then $\phi : \mathrm{OMAX}_k(V) \to W$ is completely positive.*
Moreover, if $\tilde{V} = (V, \{C_n\}_{n=1}^{\infty}, e)$ is an operator system on V with $C_1 = V^+$ such that for every operator system W, any k-positive map $\psi : \tilde{V} \to W$ is completely positive and any positive map $f : M_k \to \tilde{V}$ is completely positive, then the identity map on V is a complete order isomorphism from $\tilde{V}$ onto $\mathrm{OMAX}_k(V)$.

Proof. Assume $\phi : \mathrm{OMAX}_k(V) \to W$ is a k-positive map which is equivalent to ϕ being k-positive on $\mathrm{OMIN}(V)$ since $C_i^{k-max}(V) = C_i^{min}(V)$ for all $i = 1, 2, \ldots, k$. Let $(v_{ij}) \in M_n(\mathrm{OMAX}_k(V))^+ = C_n^{k-max}(V)$. Then:

(1) If $(v_{ij}) \in D_n^{k-max}(V)$, then $(v_{ij}) = ADA^*$ for some $A \in M_{n,mk}$,
$D = \mathrm{diag}(D_1, D_2, \ldots, D_m)$ where $D_i \in C_k^{min}(V)$ for all $1 \leq i \leq m$, $m \in \mathbb{N}$. Then, we have

$$\begin{aligned}\phi^{(n)}\big((v_{ij})\big) &= \phi^{(n)}\big(ADA^*\big) = A\phi^{(mk)}(D)A^* \\ &= A\ \mathrm{diag}\big(\phi^{(k)}(D_1),\, \phi^{(k)}(D_2), \ldots, \phi^{(k)}(D_m)\big)A^* \in M_n(W)^+,\end{aligned}$$

since each $\phi^{(k)}(D_i) \in M_k(W)^+$ because $\phi : \text{OMIN}(V) \to W$ k-positive.

(2) If $(v_{ij}) \in C_n^{k-max}(V)$, then $re_n + (v_{ij}) \in D_n^{k-max}(V)$ for all $r > 0$. It follows that

$$\phi(re_n + (v_{ij})) = r(I_n \otimes \phi(e)) + \phi^{(n)}\big((v_{ij})\big) \in M_n(W)^+ \text{ for all } r > 0.$$

Therefore by Lemma 2.2.12, $\phi^{(n)}\big((v_{ij})\big) \in M_n(W)^+$. As a result, $\phi : \text{OMAX}_k(V) \to W$ is completely positive.

Now, let $\tilde{V} = (V, \{C_n\}_{n=1}^{\infty}, e)$ be another operator system on V with $C_1 = V^+$ such that for every operator system W, any k-positive map $\psi : \tilde{V} \to W$ is completely positive and any positive map $f : M_k \to \tilde{V}$ is completely positive. We have that the identity map $\imath : \tilde{V} \to \text{OMIN}(V)$ is positive and hence completely positive. Thus, $M_k(\tilde{V})^+ \subseteq M_k(\text{OMIN}(V))^+ = M_k(\text{OMAX}_k(V))^+$. It follows that $\imath : \tilde{V} \to \text{OMAX}_k(V)$ is k-positive and therefore completely positive by the characterization of $\tilde{V}$.

Conversely, assume $(v_{ij}) \in M_k(\text{OMIN}(V))^+ = M_k(\text{OMAX}_k(V))^+$ and consider the map $f : M_k \to V$, given by $f((a_{ij})) = \sum_{i,j=1}^{k} a_{ij}v_{ij}$. We will show that f is positive, $f((a_{ij})) \geq 0$ for all $(a_{ij}) \in M_k^+$. Since any positive definite matrix in M_k can be written as a sum of rank one positive matrices, it is enough to consider the case of rank one matrices. Thus, let $(a_{ij}) = \big(\bar{\beta}_i\beta_j\big)$ be a rank one positive matrix and $s : V \to \mathbb{C}$ be a state on V. Then

$$s(f((\bar{\beta}_i\beta_j))) = \sum_{i,j=1}^{k} \bar{\beta}_i\beta_j s(v_{ij}) = \left\langle (s(v_{ij})) \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_k \end{pmatrix}, \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_k \end{pmatrix} \right\rangle \geq 0.$$

Since this is true for all states, we have that $f((\bar{\beta}_i\beta_j)) \in V^+$. This implies that $f : M_k \to V$ is positive and hence $f : M_k \to \tilde{V}$ is completely positive. It follows that $(v_{ij}) = (f(E_{ij})) \in M_k(\tilde{V})^+$. Thus $M_k(\text{OMAX}_k(V))^+ \subseteq M_k(\tilde{V})^+$. This implies that the identity map $\imath : \text{OMAX}_k(V) \to \tilde{V}$ is k-positive and therefore completely positive.

Hence, the identity map on V is a complete order isomorphism from $\tilde{V}$ onto $\text{OMAX}_k(V)$. $\square$

Lemma 2.2.14. *Let (V, V^+, e) be an AOU space and let $(W, \{P_n\}_{n=1}^{\infty}, e')$ be an (abstract) operator system. Given $k \in \mathbb{N}$, if $\phi : \text{OMAX}_k(V) \to W$ is completely positive, then $\phi : \text{OMIN}(V) \to W$ is k-positive.*

Proof. Let $\phi : \text{OMAX}_k(V) \to W$ be a completely positive map. Then $\phi^{(n)}(C_n^{k-max}(V)) \subseteq P_n$ for all $n \in \mathbb{N}$. For any $1 \leq h \leq k$, we have

$$C_n^{h-max}(V) \subseteq C_n^{k-max}(V),$$

which implies

$$(*) \quad \phi^{(n)}(C_n^{h-max}(V)) \subseteq P_n, \text{ for all } n \in \mathbb{N}.$$

Also, note that, given any $k \in \mathbb{N}$, there is no order relation between cones of k-minimal matrix ordering $\mathcal{C}^{k-min}(V)$ and k-maximal one $\mathcal{C}^{k-max}(V)$, except for the one single k^{th} matrix level, i.e.

$$(**) \quad C_k^{k-max}(V) = C_k^{min}(V).$$

Combining $(*)$ and $(**)$, we obtain

$$\phi^{(h)}(C_h^{min}(V)) = \phi^{(h)}(C_h^{h-max}(V)) \subseteq P_h,$$

for all $h = 1, 2, \ldots, k$, i.e. $\phi : \text{OMIN(V)} \to W$ is k-positive. $\square$

Corollary 2.2.15. *Let (V, V^+, e) be an AOU space and let $(W, \{P_n\}_{n=1}^{\infty}, e')$ be an operator system. Then $\phi : \text{OMIN}(V) \to W$ is k-positive if and only if $\phi : \text{OMAX}_k(V) \to W$ is completely positive.*

The following result gives an alternative way to describe the k-maximal operator system structure $\mathcal{C}^{k-max}(V)$:

Proposition 2.2.16. *Let (V, V^+, e) be an AOU space and fix $k \in \mathbb{N}$. Then $(v_{ij}) \in C_n^{k-max}(V)$ if and only if $(\phi(v_{ij})) \in M_n(B(\mathcal{H}))^+$ for all unital k-positive maps $\phi : \mathrm{OMIN}(V) \to B(\mathcal{H})$ and for all Hilbert spaces $\mathcal{H}$.*

Proof. Suppose $\phi : \mathrm{OMIN}(V) \to B(\mathcal{H})$ is k-positive with $\mathcal{H}$ an arbitrary Hilbert space. Then $\phi : \mathrm{OMAX}_k(V) \to B(\mathcal{H})$ is completely positive by Theorem 2.2.13. For each $n \in \mathbb{N}$ set

$$\begin{aligned} P_n^k(V) \quad = \quad & \big\{(v_{ij}) \in M_n(V) : (\phi(v_{ij})) \in M_n(B(\mathcal{H}))^+ \text{ for all} \\ & \phi : OMIN(V) \to B(\mathcal{H}) \text{ unital k-positive }, \mathcal{H} \text{ Hilbert space } \big\}. \end{aligned}$$

It is clear that $C_n^{k-max}(V) \subseteq P_n^k(V)$ for all n. On the other hand, using Theorem 2.1.1, given the abstract operator system $\mathrm{OMAX}_k(V)$, there exists a Hilbert space $\mathcal{H}_0$, a concrete operator system $\mathcal{S} \subseteq B(\mathcal{H}_0)$ and a complete order isomorphism $\phi_0 : \mathrm{OMAX}_k(V) \to \mathcal{S} \subseteq B(\mathcal{H}_0)$ with $\phi_0(e) = I_{\mathcal{H}_0}$. Then by Lemma 2.2.14, we have $\phi_0 : \mathrm{OMIN}(V) \to B(\mathcal{H}_0)$ is unital k-positive. Let $(v_{ij}) \in P_n^k(V)$. Then $(\phi_0(v_{ij})) \in M_n(\mathcal{S})^+ \subseteq M_n(B(\mathcal{H}_0))^+$. It follows that $(v_{ij}) \in \phi_0^{-1}(M_n(\mathcal{S})^+) \subseteq C_n^{k-max}(V)$ since ϕ_0 is a complete order isomorphism. Hence, $P_n^k(V) \subseteq C_n^{k-max}(V)$ for all n. As a result,

$$\begin{aligned} C_n^{k-max}(V) \quad = \quad & \big\{(v_{ij}) \in M_n(V) : (\phi(v_{ij})) \in M_n(B(\mathcal{H}))^+ \text{ for all} \\ & \phi : \mathrm{OMIN}(V) \to B(\mathcal{H}) \text{ unital k-positive, } \mathcal{H} \text{ Hilbert space}\big\}. \end{aligned}$$

□

2.2.3 The Matricial State Spaces of $\mathcal{C}^{k-max}(V)$ and $\mathcal{C}^{k-min}(V)$

A matricial order on a $*$-vector space induces a natural matrix order on its dual space. In this section, we describe the correspondence between the various operator system structures that an AOU space can be endowed with and the corresponding matricial state spaces.

Unfortunately, duals of AOU spaces are not in general AOU spaces, but they are normed $*$-vector spaces. As was shown in [24], the order norm on the self-adjoint part V_{sa} of an AOU space (V, V^+, e) has many possible extensions to a norm on V, but all these norms are equivalent and hence the set of continuous linear functionals on V with respect to any of these norms coincides with the same space which we shall denote by V' and call **the dual space of** V. For a functional $f \in V'$ we let $f^* \in V'$ be the functional given by $f^*(v) = \overline{f(v^*)}$; the mapping $f \to f^*$ turns V' into a $*$-vector space.

Given an AOU space (V, V^+, e) and its dual V', then let $M_{n,m}(V')$ denote the set of all $n \times m$ matrices with entries in V', $n, m \in \mathbb{N}$. Then $M_{n,m}(V')$ together with natural addition and scalar multiplication is a complex vector space, which can be linearly identified as $M_{n,m}(V') \cong M_{n,m} \otimes V' \cong V' \otimes M_{n,m}$ by using the canonical matrix unit system $\{E_{i,j}\}_{i,j=1}^{n,m}$ of $M_{n,m}$. The direct sum and the matrix product operations that link these matrix linear spaces are defined in the same way as described in Section 2.1.

Let $f : M_{n,m}(V) \to \mathbb{C}$ be a linear map on the complex vector space $M_{n,m}(V)$. We define $f_{ij} : V \to C$ by $f_{ij}(a) = f(E_{ij} \otimes a)$, $a \in V$. Then for any $v = (v_{ij}) \in M_{n,m}(V)$, we have $f(v) = \sum_{i,j} f_{ij}(v_{ij})$. We denote the vector space of such linear maps by $\mathcal{L}(M_{n,m}(V), \mathbb{C})$. Given $X = (x_{ki}) \in M_{p,n}$ and $Y = (y_{jl}) \in M_{m,q}$, we define $Xf : M_{p,m}(V) \to \mathbb{C}$ and $fY : M_{n,q}(V) \to \mathbb{C}$ by

$$Xf = \left(\sum_{i=1}^{n} x_{ki} f_{ij}\right)_{k,j=1}^{p,m} \text{ and } fY = \left(\sum_{j=1}^{m} f_{ij} y_{jl}\right)_{i,l=1}^{n,q}, \text{ respectively.}$$

Lemma 2.2.17. *Let (V, V^+, e) be an AOU space and $f : M_{n,m}(V) \to \mathbb{C}$ be a linear map, $n, m \in \mathbb{N}$. If $X \in M_{p,n}$ and $Y \in M_{m,q}$, $p, q \in \mathbb{N}$, then $Xf : M_{p,m}(V) \to \mathbb{C}$ and fY :*

$M_{n,q}(V) \to \mathbb{C}$ *are linear and*

$$(Xf)(v) = f(X^t v) \text{ and } (fY)(w) = f(wY^t),$$

where $v \in M_{p,m}(V)$ *and* $w \in M_{n,q}(V)$.

Proof. Let $f = (f_{ij}) : M_{n,m}(V) \to \mathbb{C}$ be a linear map and let $X = (x_{ki}) \in M_{p,n}$ and $Y = (y_{jl}) \in M_{m,q}$ be two arbitrary scalar matrices. It is trivial that both Xf and fY are linear functions on $M_{p,m}(V)$ and $M_{n,q}(V)$, respectively. Let $v = (v_{kj}) \in M_{p,m}(V)$ and $w = (w_{il}) \in M_{n,q}(V)$. Then we have

$$\begin{aligned}
(Xf)(v) &= \sum_{k,j=1}^{p,m} (Xf)_{kj}(v_{kj}) = \sum_{k,j=1}^{p,m} (\sum_{i=1}^{n} x_{ki} f_{ij})(v_{kj}) \\
&= \sum_{k,j=1}^{p,m} \sum_{i=1}^{n} x_{ki} f_{ij}(v_{kj}) = \sum_{i,j=1}^{n,m} (\sum_{k=1}^{p} f_{ij}(x_{ki} v_{kj})) \\
&= \sum_{i,j=1}^{n,m} f_{ij}(\sum_{k=1}^{p} x_{ki} v_{kj}) = \sum_{i,j=1}^{n,m} f_{ij}((X^t v)_{ij}) = f(X^t v)
\end{aligned}$$

and

$$\begin{aligned}
(fY)(w) &= \sum_{i,l=1}^{n,q} (fY)_{il}(w_{il}) = \sum_{i,l=1}^{n,q} (\sum_{j=1}^{m} (f_{ij} y_{jl})(w_{il}) \\
&= \sum_{i,l=1}^{n,q} \sum_{j=1}^{m} f_{ij}(w_{il}) y_{jl} = \sum_{i,j=1}^{n,m} (\sum_{l=1}^{q} f_{ij}(w_{il} y_{jl})) \\
&= \sum_{i,j=1}^{n,m} f_{ij}(\sum_{l=1}^{q} w_{il} y_{jl}) = \sum_{i,j=1}^{n,m} f_{ij}((wY^t)_{ij}) = f(wY^t).
\end{aligned}$$

□

Let (V, V^+, e) be an AOU space and let $f = (f_{ij}) : M_{n,m}(V) \to \mathbb{C}$ be a linear map. There exists a linear map from the vector space of linear maps from $M_{n,m}(V)$, $\mathcal{L}(M_{n,m}(V), \mathbb{C})$, into the vector space of linear maps from V into $M_{n,m}$, denoted by $\mathcal{L}(V, M_{n,m})$, and vice versa.

Hence, given $f \in \mathcal{L}(M_{n,m}(V), \mathbb{C})$, we associate to f a linear map $\phi_f = (f_{ij}) : V \to M_{n,m}$ by the following formula:

$$\phi_f(a) = (f_{ij}(a)) \in M_{n,m},\ a \in V.$$

On the other hand, given $\phi = (\phi_{ij}) \in \mathcal{L}(V, M_{n,m})$, we associate to ϕ a linear map $f_\phi = (\phi_{ij}) : M_{n,m}(V) \to \mathbb{C}$ by the following formula:

$$f_\phi(v) = \sum_{i,j=1}^{n,m} \phi_{ij}(v_{ij}) \in \mathbb{C},\ v = (v_{ij}) \in M_{n,m}(V).$$

Based on this correspondence between these vector spaces of linear maps, if $f = (f_{ij}) \in \mathcal{L}(M_{n,m}(V), \mathbb{C})$ with each $f_{ij} \in V'$, then $\phi_f = (f_{ij})$ can be regarded as sitting inside $M_{n,m}(V')$. Conversely, we identify $\phi = (\phi_{ij}) \in M_{n,m}(V')$ with the linear map $f_\phi : M_{n,m}(V) \to \mathbb{C}$ defined as above.

Let $\phi = (\phi_{ij}) : V \to M_{n,m}$ be a linear map with $\phi_{ij} \in V'$. Given $A \in M_{n,p}$ and $B \in M_{m,q}$, $p, q \in \mathbb{N}$, then $A^*\phi B \in M_{p,q}(V')$ since both $M_{n,m}(V')$ and $M_{p,q}(V')$ are complex vector spaces and matrix product is a well-defined operation on them (see Section 2.1 for more details).
Write $A = \begin{bmatrix} a_1 & a_2 & \cdots & a_p \end{bmatrix}$ and $B = \begin{bmatrix} b_1 & b_2 & \cdots & b_q \end{bmatrix}$ where $a_k \in \mathbb{C}^n$ and $b_l \in \mathbb{C}^m$, $1 \leq k \leq p$, $1 \leq l \leq q$. Then we can write $A^*\phi B = (a_k^*\phi b_l) \in M_{p,q}(V')$, where each $a_k^*\phi b_l \in V'$. We identify $A^*\phi B \in M_{p,q}(V')$ with the linear map $F_{A^*\phi B} = (a_k^*\phi b_l) : M_{p,q}(V) \to \mathbb{C}$ given by

$$F_{A^*\phi B}((v_{kl})) = \sum_{k,l=1}^{p,q} (A^*\phi B)_{kl}(v_{kl}) = \sum_{k,l=1}^{p,q} (a_k^*\phi b_l)(v_{kl}),\ (v_{kl}) \in M_{p,q}(V).$$

Note that when $p = q = 1$, we have $A \in \mathbb{C}^n$, $B \in \mathbb{C}^m$ and $A^*\phi B \in V'$. Moreover, $F_{A^*\phi B} : V \to \mathbb{C}$ is given by $A^*\phi B$ itself. One can straightforwardly show that

$$F_{A^*\phi B}(a) = (A^*\phi B)(a) = A^*\phi(a)B, \text{ for all } a \in V.$$

The next lemma shows how to evaluate such maps when $p \neq 1$, $q \neq 1$. Before showing this result, we will discuss the matrix-vector correspondence and introduce a new notation which we will be using widely in the next results.

The Matrix – Vector Correspondence

Let $X \in M_{n,m}$ be a scalar matrix, $n, m \in \mathbb{N}$. Write X in terms of its columns $X = \begin{bmatrix} x_1 & x_2 & \cdots & x_m \end{bmatrix}$ with $x_j \in \mathbb{C}^n$, $1 \leq j \leq m$. We set

$\mathrm{vec}(X) = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \in \mathbb{C}^m \otimes \mathbb{C}^n$ and call $\mathrm{vec}(X)$ **the vectorization** of the matrix X.

One can think of this process as a linear map

$$\mathrm{vec} : M_{n,m} \to \mathbb{C}^m \otimes \mathbb{C}^n \text{ given by } \mathrm{vec}(E_{ij}) = e_j \otimes e_i,$$

where $\{e_i\}_{i=1}^n \subseteq \mathbb{C}^n$ and $\{e_j\}_{j=1}^m \subseteq \mathbb{C}^m$ are the canonical orthonormal bases.

Lemma 2.2.18. *Let (V, V^+, e) be an AOU space and V' be its dual. If $\phi = (\phi_{ij}) \in M_{n,m}(V')$, $A \in M_{n,p}$ and $B \in M_{m,q}$ are given, $n, m, p, q \in \mathbb{N}$, then the linear map $F_{A^*\phi B} : M_{p,q}(V) \to \mathbb{C}$ is given by*

$$F_{A^*\phi B}(v) = \mathrm{vec}(A)^* \phi^{(p,q)}(v) \mathrm{vec}(B), \textit{ for all } v \in M_{p,q}(V).$$

Proof. Let $\phi = (\phi_{ij}) \in M_{n,m}(V)$, $A = \begin{bmatrix} a_1 & a_2 & \cdots & a_p \end{bmatrix} \in M_{n,p}$ and $B = \begin{bmatrix} b_1 & b_2 & \cdots & b_q \end{bmatrix}$ where $a_k \in \mathbb{C}^n$ and $b_l \in \mathbb{C}^m$, $1 \leq k \leq p$, $1 \leq l \leq q$, be given. Then $A^*\phi B \in M_{p,q}(V')$ and $F_{A^*\phi B} \in \mathcal{L}(M_{p,q}(V), \mathbb{C})$. Let $v = (v_{kl}) \in M_{p,q}(V)$, then we have

$$F_{A^*\phi B}(v) = \sum_{k,l=1}^{p,q} (A^*\phi B)_{kl}(v_{k,l}) = \sum_{k,l=1}^{p,q} (a_k^* \phi b_l)(v_{kl})$$

$$
\begin{aligned}
&= \sum_{k,l=1}^{p,q} a_k^* \phi(v_{kl}) b_l = \begin{bmatrix} a_1^* & a_2^* & \cdots & a_p^* \end{bmatrix} (\phi(v_{kl})) \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_q \end{bmatrix} \\
&= \mathrm{vec}(A)^* \phi^{(p,q)}(v) \mathrm{vec}(B).
\end{aligned}
$$

□

Definition 2.2.19. *Given an operator system structure $\{P_n\}_{n=1}^{\infty}$ on an AOU space (V, V^+, e), then* **the dual of each cone** P_n *is given by*

$$P_n^d = \{f : M_n(V) \to \mathbb{C} | \, f \text{ linear and } f(P_n) \subseteq \mathbb{R}^+\}.$$

Given $f \in P_n^d$, we define $f_{ij} : V \to \mathbb{C}$ by $f_{ij}(v) = f(v \otimes E_{ij})$, where E_{ij}'s are the canonical matrix units for M_n.

Given an operator system structure $\{P_n\}_{n=1}^{\infty}$ on an AOU space (V, V^+, e) and $f \in P_n^d$, then the functionals f_{ij} belong to V', as was shown in [23]. Identifying each $f \in P_n^d$ with $(f_{ij}) \in M_n(V')$, we shall regard P_n^d as sitting inside $M_n(V')$.

The dual cones of a given operator system structure $\{P_n\}_{n=1}^{\infty}$ on an AOU space (V, V^+, e) form a matrix ordering on the dual normed space V'. Moreover, given a matrix ordering $\{Q_n\}_{n=1}^{\infty}$ on V', one can construct an operator system structure on V as the following result shows:

Theorem 2.2.20 ([23], Theorem 4.3)**.** *Let $\{P_n\}_{n=1}^{\infty}$ be an operator system structure on the AOU space (V, V^+, e). Then $\{P_n^d\}_{n=1}^{\infty}$ is a matrix ordering on the ordered $*$-vector space V' with $P_1^d = (V^+)^d$. Conversely, if $\{Q_n\}_{n=1}^{\infty}$ is any matrix ordering on the $*$-vector space V'*

with $Q_1 = (V^+)^d$ *and we set*

$$^dQ_n = \{v \in M_n(V) : f(v) \geq 0 \text{ for all } f \in Q_n\},$$

then $\{^dQ_n\}_{n=1}^{\infty}$ *is an operator system structure on* (V, V^+, e).

Note that the weak*-topology on V' endows $M_n(V')$ with a topology which coincides with the weak*-topology that comes from the identification of $M_n(V')$ with the dual of $M_n(V)$. Thus, we shall refer to this topology, unambiguously, as the weak*-topology on $M_n(V')$.

The mappings $P_n \to P_n^d$ and $Q_n \to {}^dQ_n$ establish a one-to-one inclusion-reversing correspondence between operator system structures $\{P_n\}_{n=1}^{\infty}$ on (V, V^+, e) and matrix orderings $\{Q_n\}_{n=1}^{\infty}$ on V' with $Q_1 = (V^+)^d$ for which each Q_n is weak*-closed (see [23] for more details.)

Hence, having the inclusion of the following matrix orderings on V,

$$C_n^{max}(V) \subseteq C_n^{k-max}(V)(\text{resp. } C_n^{k-min}(V)) \subseteq C_n^{min}(V)$$

implies the reverse inclusion of the corresponding matrix orderings on V'

$$(C_n^{min}(V))^d \subseteq (C_n^{k-max}(V))^d(\text{resp. } (C_n^{k-min}(V))^d) \subseteq (C_n^{max}(V))^d,$$

and vice-versa.

Definition 2.2.21. *Let* (V, V^+, e) *be an AOU space. For a fixed* $k \in \mathbb{N}$, *set*

$$Q_n^{k-min}(V') = \Big\{F_{X^*GX} : M_n(V) \to \mathbb{C} \,|\, X \in M_{mk,n}, G = diag(\phi_1, \ldots, \phi_m),$$
$$\text{with } \phi_i : V \to M_k \text{ positive linear map, } m \in \mathbb{N}\Big\},$$

and

$$Q_n^{k-max}(V') = \left\{ (f_{ij}) \in M_n(V') : \left(f_{ij}^{(k)}(a) \right) \in M_{nk}^+, \text{ for all } a \in C_k^{min}(V) \right\}.$$

Proposition 2.2.22. *Let (V, V^+, e) be an AOU space. Then $\{Q_n^{k-min}(V')\}_{n=1}^{\infty}$ and $\{Q_n^{k-max}(V')\}_{n=1}^{\infty}$ are matrix orderings on V' with $Q_1^{k-min}(V') = (V^+)^d$ and $Q_1^{k-max}(V') = (V^+)^d$.*

Proof. One can straightforwardly check that both these families of cones are matrix orderings on V'. Here, we will just show $Q_1^{k-min}(V') = (V^+)^d$ and $Q_1^{k-max}(V') = (V^+)^d$.

(1) $Q_1^{k-min}(V') = (V^+)^d$:

Let $F_{X^*GX} \in Q_1^{k-min}(V') \subseteq V'$ with $X = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} \in \mathbb{C}^{mk}$ where $x_i \in \mathbb{C}^k$, and $G = \mathrm{diag}(\phi_1, \ldots, \phi_m)$, where $\phi_i : V \to M_k$ is a positive linear map.

Let $v \in V^+$, then

$$F_{X^*GX}(v) = (X^*GX)(v) = \left(\sum_{i=1}^m x_i^* \phi_i x_i \right)(v) = \sum_{i=1}^m x_i^* \underbrace{(\phi_i(v))}_{\geq 0} x_i \geq 0.$$

This implies that $F_{X^*GX} \in (V^+)^d$, i.e. $Q_1^{k-min}(V') \subseteq (V^+)^d$.

Conversely, let $f \in (V^+)^d$. Then the map

$$\phi = I_k \otimes f = \begin{bmatrix} f & & & 0 \\ & f & & \\ & & \ddots & \\ 0 & & & f \end{bmatrix} : V \to M_k$$

is a well-defined positive linear map on V.

Let $\alpha = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, then $f = \alpha^* \phi \alpha \in Q_1^{k-min}(V')$. Hence, $(V^+)^d \subseteq Q_1^{k-min}(V')$. As a result, $Q_1^{k-min}(V') = (V^+)^d$.

(2) $Q_1^{k-max}(V') = (V^+)^d$:

Let $f \in Q_1^{k-max}(V') \subseteq V'$. Then, by the definition of $Q_n^{k-max}(V')$, we have $f^{(k)}(a) \geq 0$ for all $a \in C_k^{min}(V)$. Let $a = v \otimes E_{11} \in M_k(V)$ with $v \in V^+$, so $a \in C_k^{min}(V)$. Then

$$f^{(k)}(a) = f(v) \otimes E_{11} \geq 0 \text{ implies } f(v) \geq 0 \text{ for all } v \in V^+.$$

Therefore, $f \in (V^+)^d$. Hence, $Q_1^{k-max}(V') \subseteq (V^+)^d$.

Conversely, let $f \in (V^+)^d$. Then $f : OMIN(V) \to \mathbb{C}$ is completely positive, which implies $f^{(k)}(C_k^{min}(V)) \subseteq M_k^+$, i.e. $f \in Q_1^{k-max}(V')$ and $(V^+)^d \subseteq Q_1^{k-max}(V')$. As a result, $Q_1^{k-max}(V') = (V^+)^d$. □

Theorem 2.2.23. *Let (V, V^+, e) be an AOU space. Then ${}^dQ_n^{k-min}(V') = C_n^{k-min}(V)$ and $(C_n^{k-max}(V))^d = Q_n^{k-max}(V')$.*

Proof. We will show that $\{{}^dQ_n^{k-min}(V')\}_{n=1}^{\infty}$ is the k-minimal operator system structure on V, and $\{Q_n^{k-max}(V')\}_{n=1}^{\infty}$ is the dual of the k-maximal operator system structure on V.

(1) ${}^dQ_n^{k-min}(V') = C_n^{k-min}(V)$:

Let $v = (v_{ij}) \in C_n^{k-min}(V)$ and $F_{X^*GX} \in Q_n^{k-min}(V')$, where $X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_m \end{bmatrix} \in M_{mk,n}$ with each $X_i \in M_{k,n}$, $1 \leq i \leq m$, and $G = \text{diag}(\phi_1, \ldots, \phi_m)$ with $\phi_i : V \to M_k$ a positive linear map. One can easily check that $X^*GX = \sum_{i=1}^m X_i^*\phi_i X_i$ and $F_{X^*GX} = \sum_{i=1}^m F_{X_i^*\phi_i X_i}$. Then

$$\begin{aligned} F_{X^*GX}(v) &= \sum_{i=1}^{m} F_{X_i^*\phi_i X_i}(v) \\ &= \sum_{i=1}^{m} \mathrm{vec}(X_i)^* \phi_i^{(n)}(v) \mathrm{vec}(X_i) \geq 0, \end{aligned}$$

since $\phi_i^{(n)}(v) \geq 0$ for all i. This implies $v \in {}^dQ_n^{k-min}(V')$ and $C_n^{k-min}(V) \subseteq {}^dQ_n^{k-min}(V')$.

Conversely, let $v = (v_{ij}) \in {}^dQ_n^{k-min}(V)$ and let $\phi \in S_k(V)$. Let $\Lambda = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix} \in \mathbb{C}^{nk}$ with $\lambda_i \in \mathbb{C}^k$, for all $1 \leq i \leq n$. Then, we have

$$\begin{aligned} \Lambda^*\phi^{(n)}(v)\Lambda &= \sum_{i,j=1}^{n} \lambda_i^* \phi(v_{ij})\lambda_j = \sum_{i,j=1}^{n} (\lambda_i^*\phi\lambda_j)(v_{ij}) \\ &= F_{X^*\phi X}(v) \geq 0, \end{aligned}$$

where $X = \begin{bmatrix} \lambda_1 & \lambda_2 & \cdots \lambda_n \end{bmatrix} \in M_{k,n}$, $X^*\phi X \in M_n(V')$ and $F_{X^*\phi X} \in Q_n^{k-min}(V')$. This implies $\phi^{(n)}(v) \geq 0$ for all unital positive maps ϕ on V, i.e. $v = (v_{ij}) \in C_n^{k-min}(V)$ and ${}^dQ_n^{k-min}(V') \subseteq C_n^{k-min}(V)$. Hence, we conclude that ${}^dQ_n^{k-min}(V') = C_n^{k-min}(V)$.

(2) $(C_n^{k-max}(V))^d = Q_n^{k-max}(V')$:

Let $F = (f_{ij}) \in Q_n^{k-max}(V')$ and let $A^*DA \in D_n^{k-max}(V)$. Write $A = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{bmatrix} \in M_{mk,n}$ where each $A_l = \begin{bmatrix} C_1^l & C_2^l & \cdots & C_n^l \end{bmatrix} \in M_{k,n}$ with C_i^l being the i^{th} column of A_l for all $1 \leq l \leq m$, $1 \leq i \leq n$, and $D = \mathrm{diag}(D_1, D_2, \cdots, D_m)$ with $D_l \in C_k^{min}(V)$. Then, we have

$$F(A^*DA) = \sum_{l=1}^{m} F(A_l^* D_l A_l) = \sum_{l=1}^{m} F([(C_i^l)^* D_l (C_j^l)])$$

$$\begin{aligned} &= \sum_{l=1}^{m}\sum_{i,j=1}^{n} f_{ij}((C_i^l)^* D_l (C_j^l)) = \sum_{l=1}^{m}\sum_{i,j=1}^{n} (C_i^l)^* f_{ij}^{(k)}(D_l)\,(C_j^l) \\ &= \sum_{l=1}^{m} \mathrm{vec}(A_l)^* \underbrace{\left[f_{ij}^{(k)}(D_l)\right]}_{\geq 0} \mathrm{vec}(A_l) \geq 0. \end{aligned}$$

This shows that $F(D_n^{k-max}(V)) \subseteq \mathbb{R}^+$. Now, let $v = (v_{ij}) \in C_n^{k-max}(V)$ such that $re_n + v \in D_n^{k-max}(V)$ for all $r > 0$. Then

$$rF(e_n) + F(v) = F(re_n + v) \geq 0, \text{ for all } r > 0.$$

Therefore, $F(v) \geq 0$ and $F(C_n^{k-max}(V)) \subseteq \mathbb{R}^+$. As a result, $F \in (C_n^{k-max}(V))^d$ and $Q_n^{k-max}(V') \subseteq (C_n^{k-max}(V))^d$.

Conversely, let $F = (f_{ij}) \in (C_n^{k-max}(V))^d$, $a \in C_k^{min}(V)$ and $\Lambda = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{bmatrix} \in \mathbb{C}^{nk}$ with each $\lambda_i \in \mathbb{C}^k$, $1 \leq i \leq n$. Then, we have

$$\begin{aligned} \Lambda^* \left[f_{ij}^{(k)}(a)\right] \Lambda &= \sum_{i,j=1}^{n} \lambda_i^* \, f_{ij}^{(k)}(a)\, \lambda_j = \sum_{i,j=1}^{n} f_{ij}(\lambda_i^* a \lambda_j) \\ &= F([\lambda_i^* a \lambda_j]) = F(X^* a X) \geq 0, \end{aligned}$$

where $X = \begin{bmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_n \end{bmatrix} \in M_{k,n}$ and $XaX^* \in C_n^{k-max}(V)$. Therefore $\left[f_{ij}^{(k)}(a)\right] \geq 0$ for all $a \in C_k^{min}(V)$, i.e. $F = (f_{ij}) \in Q_n^{k-max}(V')$ and $(C_n^{k-max}(V))^d \subseteq Q_n^{k-max}(V')$. It follows that $(C_n^{k-max}(V))^d = Q_n^{k-max}(V')$. □

Remark 2.2.24. *Although the cone $Q_n^{k-min}(V')$ defined above is not weak***-closed, Theorem 2.2.23 shows that $(C_n^{k-min}(V))^d$ is the weak***-closure of $Q_n^{k-min}(V')$. Also, note that the cone $Q_n^{k-max}(V')$ is weak***-closed(easy to show) and $\{{}^dQ_n^{k-max}(V')\}$ is an operator system structure on V. This implies that $({}^dQ_n^{k-max}(V'))^d = Q_n^{k-max}(V')$.*

2.2.4 Comparisons of Various Structures on a Given Operator System

Given a unital C*-algebra or, more generally, an operator system $\mathcal{S}$ such that at the first level it is an AOU space, then we may form new operator systems, $\mathrm{OMIN}_k(\mathcal{S})$ and $\mathrm{OMAX}_k(\mathcal{S})$ for a given $k \in \mathbb{N}$. Also, since it is a normed space, we may form the operator spaces $\mathrm{MIN}(\mathcal{S})$ and $\mathrm{MAX}(\mathcal{S})$. In this section, we compare these structures, describing when they are identical and, more generally, when the identity map between these various structures is a completely bounded isomorphism.

Proposition 2.2.25. *Let (V, V^+, e) be an AOU space and let $k \in \mathbb{N}$ be given. Then for $v \in V$, we have that*

$$\|v\|_{\mathrm{OMAX}_k(V)} = \sup\{\|\varphi(v)\| \,|\, \varphi : V \to B(\mathcal{H})\},$$

where the supremum is taken over all Hilbert spaces and over all unital k-positive maps φ on $\mathrm{OMIN}(V)$*.*

Proof. Suppose that $\varphi : V \to B(\mathcal{H})$ is a unital k-positive map on $\mathrm{OMIN}(V)$. By Theorem 2.2.13, $\varphi : \mathrm{OMAX}_k(V) \to B(\mathcal{H})$ is completely positive and hence it is completely contractive. It follows that φ is contractive and hence

$$\|\varphi(v)\| \leq \|v\|_{\mathrm{OMAX}_k(V)}, \text{ for all } v \in V.$$

On the other hand, if $\varphi : \mathrm{OMAX}_k(V) \to B(\mathcal{H})$ is a unital complete isometry, then φ is completely positive and $\|v\|_{\mathrm{OMAX}_k(V)} = \|\varphi(v)\|$, for all $v \in V$. Therefore, we conclude that

$$\|v\|_{\mathrm{OMAX}_k(V)} = \sup_{\mathcal{H},\varphi}\{\|\varphi(v)\| \,|\, \varphi : \mathrm{OMIN}(V) \to B(\mathcal{H}) \text{ unital k-positive }\}.$$

$\square$

Proposition 2.2.26. *Let $\mathcal{S}$ be an operator system and let $k \in \mathbb{N}$. Then*

(1) The identity map id *from* $\mathrm{OMIN}_k(\mathcal{S})$ *to* $\mathcal{S}$ *is completely bounded with* $\|\mathrm{id}\|_{cb} = C$ *if and only if for every operator system* $\mathcal{T}$, *every unital k-positive map* $\phi : \mathcal{T} \to \mathcal{S}$ *is completely bounded and the supremum of the completely bounded norms of all such maps is* C.

(2) The identity map id *from* $\mathcal{S}$ *to* $\mathrm{OMAX}_k(\mathcal{S})$ *is completely bounded with* $\|\mathrm{id}\|_{cb} = K$ *if and only if for every operator system* $\mathcal{T}$, *every unital k-positive map* $\phi : \mathcal{S} \to \mathcal{T}$ *is completely bounded and the supremum of the completely bounded norms of all such maps is* K.

Proof. Refer to [23], Proposition 5.3 and Proposition 5.4. □

Proposition 2.2.27. *Let (V, V^+, e) be an AOU space. Then the identity map from* $\mathrm{OMAX}_k(V)$ *to* $\mathrm{MAX}(V_{k-min})$ *is completely bounded if and only if for every Hilbert space $\mathcal{H}$, every bounded map $\phi : V_{k-min} \to B(\mathcal{H})$ decomposes as*

$$\phi = (\phi_1 - \phi_2) + i(\phi_3 - \phi_4),$$

where each $\phi_j : \mathrm{OMIN}(V) \to B(\mathcal{H})$ *is k-positive.*

Proof. Assume that the decompositions of all such bounded maps holds, and suppose that $\mathrm{MAX}(V_{k-min}) \subseteq B(\mathcal{H})$ completely isometrically for some Hilbert space $\mathcal{H}$. Let $\phi : V_{k-min} \to \mathrm{MAX}(V_{k-min})$ be the identity map and let ϕ_j, $j = 1, 2, 3, 4$ be a k-positive mapping on $\mathrm{OMIN}(V)$ such that $\phi = (\phi_1 - \phi_2) + i(\phi_3 - \phi_4)$. Since each ϕ_j is k-positive on $\mathrm{OMIN}(V)$, then $\phi_j : \mathrm{OMAX}_k(V) \to B(\mathcal{H})$ is completely positive, and hence completely bounded for $j = 1, 2, 3, 4$. Hence $\phi : \mathrm{OMAX}_k(V) \to B(\mathcal{H})$ is completely bounded, too.
Conversely, if the identity map from $\mathrm{OMAX}_k(V)$ to $\mathrm{MAX}(V_{k-min})$ is completely bounded and $\phi : V_{k-min} \to B(\mathcal{H})$ is bounded, then $\phi : \mathrm{MAX}(V_{k-min}) \to B(\mathcal{H})$ is completely bounded

and hence $\phi : \mathrm{OMAX}_k(V) \to B(\mathcal{H})$ is completely bounded. Applying Wittstock's decomposition theorem, we have that $\phi = (\phi_1 - \phi_2) + i(\phi_3 - \phi_4)$, where each $\phi_j : \mathrm{OMAX}_k(V) \to B(\mathcal{H})$ is completely positive, and hence k-positive as a map from $\mathrm{OMIN}(V)$ into $B(\mathcal{H})$. $\square$

Let $\mathcal{S} \subseteq B(\mathcal{H})$ be a concrete operator system. It is clear that $(\mathcal{S}, \mathcal{S}^+, I_{\mathcal{H}})$ is an Archimedean order unit space. Considering $(\mathcal{S}, \mathcal{S}^+, I_{\mathcal{H}})$ as an AOU space allows us to construct other operator system structures on $\mathcal{S}$ different from the natural one it already has.

Let's begin with the minimal and the maximal operator system structures $\mathcal{C}^{min}(\mathcal{S})$ and $\mathcal{C}^{max}(\mathcal{S})$ respectively. One can easily verify that

$$C_n^{max}(\mathcal{S}) \subseteq M_n(\mathcal{S})^+ \subseteq C_n^{min}(\mathcal{S}), \text{ for all } n \in \mathbb{N},$$

and equality holds for $n = 1$. Every property and result developed for these two operator system structures on abstract AOU spaces hold the same in this setting too, as was shown in [23].

Now, for a fixed $k \in \mathbb{N}$, let's consider the k-minimal and k-maximal operator sytem structures $\mathcal{C}^{k-min}(\mathcal{S})$ and $\mathcal{C}^{k-max}(\mathcal{S})$ respectively. As we have shown in the previous section, we have that for $n = 1$

$$C_1^{k-max}(\mathcal{S}) = C_1^{k-min}(\mathcal{S}) = \mathcal{S}^+,$$

and for $1 < i \leq k$

$$C_i^{k-max}(\mathcal{S}) = C_i^{min}(\mathcal{S}) \supseteq C_i^{k-min}(\mathcal{S}).$$

Using the definitions of these operator system structures, we can show that for all $n > k$ we have

$$C_n^{k-max}(\mathcal{S}) \nsubseteq C_n^{k-min}(\mathcal{S}) \text{ and } C_n^{k-max}(\mathcal{S}) \nsupseteq C_n^{k-min}(\mathcal{S}).$$

Moreover, neither $C_n^{k-max}(\mathcal{S})$, nor $C_n^{k-min}(\mathcal{S})$ contains or is contained in the cone $M_n^+(\mathcal{S})$

of natural positives, i.e.

$$C_n^{k-max}(\mathcal{S}) \nsubseteq M_n(\mathcal{S})^+ \quad \text{and} \quad M_n(\mathcal{S})^+ \nsubseteq C_n^{k-max}(\mathcal{S}),$$
$$C_n^{k-min}(\mathcal{S}) \nsubseteq M_n(\mathcal{S})^+ \quad \text{and} \quad M_n(\mathcal{S})^+ \nsubseteq C_n^{k-min}(\mathcal{S}).$$

This shows that there is no general relation between the natural operator system structure on $\mathcal{S}$ and the k-minimal or the k-maximal one.

Based on this fact, a natural question arises: Can we construct such general operator system structures on a given operator system $\mathcal{S}$, that generalize the weakest and the strongest operator system structures OMIN($\mathcal{S}$) and OMAX($\mathcal{S}$) respectively, and have an order relation with the natural operator system structure of $\mathcal{S}$? Yes, we can. We will show the details of what we call **super k-minimal/k-maximal operator system structures** in the next section.

2.3 The Super k-Minimal and the Super k-Maximal Operator System Structures on Given Operator Systems

Let $\mathcal{S} \subseteq B(\mathcal{H})$ be an operator system. Then $(\mathcal{S}, \mathcal{S}^+, I_{\mathcal{H}})$ is an Archimedean order unit space. Given a fixed $k \in \mathbb{N}$, let $\psi : \mathcal{S} \to M_k$ be a unital k-positive linear map.

Proposition 2.3.1 ([20], Exercise 6.2)**.** *Let $\mathcal{S}$ be an operator system, and $\phi : \mathcal{S} \to M_k$ be a k-positive map for some fixed $k \in \mathbb{N}$ with $\phi(I_{\mathcal{S}}) = P \geq 0$. Then there exists a unital k-positive map $\psi : \mathcal{S} \to M_k$ such that*

$$\phi(v) = P^{1/2}\psi(v)P^{1/2}, \ \textit{for all } v \in \mathcal{S}.$$

Proof. Without loss of generality, assume $S \subseteq B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Assume $\phi : \mathcal{S} \to M_k$ is a k-positive map for some fixed $k \in \mathbb{N}$ with $\phi(I_{\mathcal{S}}) = P$. Let Q be the projection onto the range of P, i.e. $PQ = QP = P$, and let R be positive with $(1-Q)R = 0$ and $RPR = Q$. Let $\xi \in \mathcal{H}$ be a unit vector. Set

$$\psi(v) = R\phi(v)R + \langle v\xi, \xi\rangle(I_k - Q).$$

One can straightforwardly check that ψ is a unital k-positive map.

Before we prove that $\phi(v) = P^{1/2}\psi(v)P^{1/2}$ for all $v \in \mathcal{S}$, we will state some important facts from matrix theory that are needed for the proof:

(1) For all $v \in \mathcal{S}$, range $\phi(v) \subseteq$ range $P \subseteq$ range $P^{1/2}$ and ker $\phi(v) \supseteq$ ker $P \supseteq$ ker $P^{1/2}$.

(2) Let $z_1, z_2 \in \mathbb{C}^k$. Then each z_i can be written as $z_i = x_i + y_i$ with $x_i = P^{1/2}\alpha_i \in$ range $P^{1/2}$ for some $\alpha_i \in \mathbb{C}^k$, and $y_i \in$ ker $P^{1/2}$. Having range $\phi(v) \subseteq$ range $P^{1/2}$ implies $\langle\phi(v)z_1, z_2\rangle = \langle\phi(v)x_1, x_2\rangle$.

(3) $P^{1/2}QP^{1/2} = P$ where $P \geq 0$ and Q projection onto the range of P.

(4) $RP = P^{1/2}$: We know $RPR = Q$ and $QP = P$. It follows that

$$(RP)^2 = (RP)(RP) = (RPR)P = QP = P \Rightarrow RP = P^{1/2}.$$

Now we will show $\phi(v) = P^{1/2}\psi(v)P^{1/2}$ for all $v \in \mathcal{S}$. Using the fact(3) above, one can easily show that $P^{1/2}\psi(v)P^{1/2} = P^{1/2}R\phi(v)RP^{1/2}$.

Finally, using facts (1), (2) and (4), we have

$$\begin{aligned}
\langle P^{1/2}R\phi(v)RP^{1/2}z_1, z_2\rangle &= \langle P^{1/2}R\phi(v)RP^{1/2}x_1, x_2\rangle \\
&= \langle\phi(v)RP^{1/2}x_1, RP^{1/2}x_2\rangle = \langle\phi(v)RP^{1/2}(P^{1/2}\alpha_1), RP^{1/2}(P^{1/2}\alpha_2)\rangle \\
&= \langle\phi(v)RP\alpha_1, RP\alpha_2\rangle = \langle\phi(v)P^{1/2}\alpha_1, P^{1/2}\alpha_2\rangle \\
&= \langle\phi(v)x_1, x_2\rangle = \langle\phi(v)z_1, z_2\rangle.
\end{aligned}$$

Hence, $\phi(v) = P^{1/2}\psi(v)P^{1/2}$ for all $v \in \mathcal{S}$. □

Let $\mathcal{S}$ be an operator system. Denote the set of all unital k-positive linear maps $\psi : \mathcal{S} \to M_k$ by $S_k^s(\mathcal{S})$, i.e.

$$S_k^s(\mathcal{S}) = \big\{\psi : \mathcal{S} \to M_k \,|\, \psi \text{ k-positive with } \psi(I_{\mathcal{S}}) = I_k\big\}.$$

By Proposition 2.2.4, every k-positive map $\phi : \mathcal{S} \to M_k$ is completely positive. Therefore one can describe $S_k^s(\mathcal{S})$ as the set of unital completely positive maps from $\mathcal{S}$ to M_k. It is obvious that $S_k^s(\mathcal{S}) \subseteq S_k(\mathcal{S})$, the set of all unital positive linear maps on $\mathcal{S}$. One can verify that $S_k^s(\mathcal{S})$ is a closed convex subset of the compact space $S_k(\mathcal{S})$, therefore $S_k^s(\mathcal{S})$ is compact.

Recall $M_k(C(S_k(\mathcal{S})))$, the unital C*-algebra of $k \times k$ continuous matrix-valued functions on $S_k(\mathcal{S})$. Every continuous map $\hat{v} : S_k(\mathcal{S}) \to M_k$ is continuous on $S_k^s(\mathcal{S}) \subseteq S_k(\mathcal{S})$, and every continuous map on $S_k^s(\mathcal{S})$ can be extended to a continuous map on $S_k(\mathcal{S})$, since both $S_k(\mathcal{S})$ and $S_k^s(\mathcal{S})$ are compact. Let $\tilde{\Gamma} : \mathcal{S} \to M_k(C(S_k^s(\mathcal{S})))$ be defined by $\tilde{\Gamma}(a)(\psi) = \psi(a)$, $a \in \mathcal{S}$. One can show that this map is a well-defined order isomorphism onto its range with $\tilde{\Gamma}(I_{\mathcal{S}}) = I_k$ (see Theorem 2.2.1 for more details). This order isomorphism induces an operator system structure $\{C_n\}_{n=1}^{\infty}$ on $\mathcal{S}$, with $C_1 = S^+ = \tilde{\Gamma}^{-1}(P_1)$ where P_1 denotes the set of non-negative continuous matrix valued functions on $S_k^s(V)$. And $(a_{ij}) \in C_n$ if and only if $(\Psi(a_{ij})) \in M_n(M_k(C(S_k^s(\mathcal{S}))))^+$ if and only if $(\psi(a_{ij})) \in M_{nk}^+$ for every $\psi \in S_k^s(\mathcal{S})$.

Definition 2.3.2. *Let $\mathcal{S}$ be an operator system. For each $n \in \mathbb{N}$, set*

$$ {}^sC_n^{k-min}(\mathcal{S}) = \left\{(a_{ij}) \in M_n(\mathcal{S}) : (\phi(a_{ij})) \geq 0 \textit{ for all } \phi \in S_k^s(\mathcal{S})\right\},$$

${}^s\mathcal{C}^{k-min}(\mathcal{S}) = \{{}^sC_n^{k-min}(\mathcal{S})\}_{n=1}^{\infty}$ *and define* $\mathrm{OMIN}_k^s(\mathcal{S}) = (\mathcal{S}, {}^s\mathcal{C}^{k-min}(\mathcal{S}), I_{\mathcal{S}})$.

Note that ${}^s\mathcal{C}^{k-min}(\mathcal{S})$ is the operator system structure on $\mathcal{S}$ induced by the inclusion of $\mathcal{S}$ into $M_k(C(S_k^s(\mathcal{S})))$, which we will call **the super k-minimal operator system structure** on $\mathcal{S}$. We call $\mathrm{OMIN}_k^s(\mathcal{S})$ **the super k-minimal operator system**.

Proposition 2.3.3. *Let $\mathcal{S}$ be an operator system, fix $k, h \in \mathbb{N}$ with $k \leq h$. Then*

(1) $\mathrm{id} : \mathcal{S} \to \mathrm{OMIN}^s_k(\mathcal{S})$ *is completely positive.*

(2) $\mathrm{id} : \mathrm{OMIN}^s_h(\mathcal{S}) \to \mathrm{OMIN}^s_k(\mathcal{S})$ *is completely positive.*

(3) Given an operator system $\mathcal{T}$, if the linear map $\phi : \mathcal{T} \to \mathcal{S}$ is k-positive, then $\phi : \mathcal{T} \to \mathrm{OMIN}^s_k(\mathcal{S})$ is completely positive.

Proof. Refer to Section 2.2.1. □

Lemma 2.3.4. *Let $\mathcal{S}$ be an operator system and let $k \in \mathbb{N}$. Set*

$$^sC^k_n(\mathcal{S}) = \left\{ (a_{ij}) \in M_n(\mathcal{S}) : (\phi(a_{ij})) \in M^+_{nk}, \phi : \mathcal{S} \to M_k \text{ k-positive} \right\}.$$

Then $\{^sC^k_n(\mathcal{S})\}_{n=1}^\infty$ is the super k-minimal operator system structure on $\mathcal{S}$.

Proof. It is enough to show that $^sC^k_n(\mathcal{S}) = {}^sC^{k-min}_n(\mathcal{S})$ for all $n \in \mathbb{N}$. The set of unital k-positive maps $S^s_k(\mathcal{S})$ is just a subset of all k-positive maps on $\mathcal{S}$. It follows that $^sC^k_n(\mathcal{S}) \subseteq {}^sC^{k-min}_n(\mathcal{S})$ for all n. On the other hand, let $(a_{ij}) \in {}^sC^{k-min}_n(\mathcal{S})$ and let $\phi : \mathcal{S} \to M_k$ be a k-positive map with $\phi(I_\mathcal{S}) = P \geq 0$. By Proposition 2.3.1, there exists a unital k-positive map $\psi \in S^s_k(\mathcal{S})$, such that $\phi(\cdot) = P^{1/2}\psi(\cdot)P^{1/2}$. Hence, we have

$$\big(\phi(a_{ij})\big) = \big(P^{1/2}\psi(a_{ij})P^{1/2}\big) = (I_k \otimes P^{1/2})\big(\psi(a_{ij})\big)(I_k \otimes P^{1/2}) \geq 0.$$

This shows $(a_{ij}) \in {}^sC^k_n(\mathcal{S})$ and $^sC^k_n(\mathcal{S}) \supseteq {}^sC^{k-min}_n(\mathcal{S})$. As a result, we conclude that $\{^sC^k_n(\mathcal{S})\}_{n=1}^\infty$ is, in fact, the super k-minimal operator system structure on $\mathcal{S}$. □

Proposition 2.3.5. *Let $\mathcal{S}$ be an operator system and $k \in \mathbb{N}$ be fixed. Set*

$$^sD^{k-max}_n(\mathcal{S}) = \Big\{ ADA^* \mid A \in M_{n,mk},\ D = diag(D_1, \ldots, D_m), \text{ where } D_l \in M_k(\mathcal{S})^+,\ 1 \leq l \leq m,\ m \in \mathbb{N} \Big\}.$$

Then ${}^s\mathcal{D}^{k-max}(\mathcal{S}) = \{{}^sD_n^{k-max}(\mathcal{S})\}_{n=1}^{\infty}$ *is a matrix ordering on* $\mathcal{S}$*, and* $I_{\mathcal{S}}$ *is a matrix order unit for this ordering.*

Proof. Refer to Proposition 2.2.10. □

Given an operator system $\mathcal{S}$, we know $\mathcal{S}$ is an AOU space $(\mathcal{S}, \mathcal{S}^+, I_{\mathcal{S}})$. Then by the proposition above, we have that $(\mathcal{S}, {}^s\mathcal{D}^{k-max}(\mathcal{S}), I_{\mathcal{S}})$ is a matrix ordered $*$-vector space for any fixed $k \in \mathbb{N}$. One can verify that this matrix ordered $*$-vector space $(\mathcal{S}, {}^s\mathcal{D}^{k-max}(\mathcal{S}), I_{\mathcal{S}})$ is an operator system, provided $\mathcal{S}$ is a finite-dimensional space. If $\mathcal{S}$ is infinite-dimensional space, then ${}^s\mathcal{D}^{k-max}(\mathcal{S})$ is just a matrix ordering, not an operator system structure. In order to transform this into an operator system, we should matrix Archimedeanize this matrix ordered space in the same way as it was done in Section 2.2.2.

Definition 2.3.6. *Let* $\mathcal{S}$ *be an operator system and* $k \in \mathbb{N}$*. Set*

$${}^sC_n^{k-max}(\mathcal{S}) = \left\{ A \in M_n(\mathcal{S}) : r(I_n \otimes I_{\mathcal{S}}) + A \in {}^sD_n^{k-max}(\mathcal{S}) \ \textit{for all } r > 0 \right\},$$

${}^s\mathcal{C}^{k-max}(\mathcal{S}) = \{{}^sC_n^{k-max}(\mathcal{S})\}_{n=1}^{\infty}$ *and define* $\mathrm{OMAX}_k^s(\mathcal{S}) = (\mathcal{S}, {}^s\mathcal{C}^{k-max}(\mathcal{S}), I_{\mathcal{S}})$.

Hence, ${}^s\mathcal{C}^{k-max}(\mathcal{S})$ is an operator system structure on $\mathcal{S}$, which we will call **the super k-maximal operator system structure**. We call $\mathrm{OMAX}_k^s(\mathcal{S})$ **the super k-maximal operator system**.

Proposition 2.3.7. *Let* $\mathcal{S}$ *be an operator system, fix* $k, h \in \mathbb{N}$ *with* $k \leq h$*. Then:*

(1) $\mathrm{id} : \mathrm{OMAX}_k^s(\mathcal{S}) \to \mathcal{S}$ *is completely positive.*

(2) $\mathrm{id} : \mathrm{OMAX}_k^s(\mathcal{S}) \to \mathrm{OMAX}_h^s(\mathcal{S})$ *is completely positive.*

(3) Given an operator system $\mathcal{T}$*, the map* $\phi : \mathcal{S} \to \mathcal{T}$ *is k-positive if and only if* $\phi : \mathrm{OMAX}_k^s(\mathcal{S}) \to \mathcal{T}$ *is completely positive.*

Proof. Refer to Section 2.2.2. □

Remark 2.3.8. *Let $\mathcal{S}$ be an operator system, fix $k \in \mathbb{N}$.*

(1) For $k = 1$, we have $S_1^s(\mathcal{S}) = S_1(\mathcal{S}) = S(\mathcal{S})$, the set of all states on S. It follows that the super 1-minimal operator system structure on $\mathcal{S}$ coincides with minimal operator system structure on $\mathcal{S}$, i.e $OMIN_1^s(\mathcal{S}) = OMIN(\mathcal{S})$. Moreover, for a fixed $1 < k \in \mathbb{N}$ we have

$$M_n(\mathcal{S})^+ \subseteq {}^sC_n^{k-min}(\mathcal{S}) \subseteq C_n^{min}(\mathcal{S}), \text{ for all } n \in \mathbb{N}.$$

And the equality holds for $n = 1$, i.e. $S^+ = {}^sC_1^{k-min}(\mathcal{S}) = C_1^{min}(\mathcal{S})$.
One can verify that $M_n(\mathcal{S})^+ = \bigcap_{k\in\mathbb{N}} {}^sC_n^{k-min}(\mathcal{S})$ for all $n \in \mathbb{N}$.

(2) For $k = 1$, one can easily check that the super 1-maximal operator system structure on $\mathcal{S}$ coincides with the maximal operator system structure on $\mathcal{S}$, i.e. $OMAX_1^s(\mathcal{S}) = OMAX(\mathcal{S})$. Moreover, for a fixed $1 < k \in \mathbb{N}$ we have ${}^sD_n^{k-max}(\mathcal{S}) \subseteq M_n^+(\mathcal{S})$ and since ${}^sC_n^{k-max}(\mathcal{S})$ is the closure of ${}^sD_n^{k-max}(\mathcal{S})$ in the order topology of $M_n(\mathcal{S})$, we get

$$C_n^{max}(\mathcal{S}) \subseteq {}^sC_n^{k-max}(\mathcal{S}) \subseteq M_n(\mathcal{S})^+, \text{ for all } n \in \mathbb{N}.$$

And the equality holds for $n = 1$, i.e. $C_1^{max}(\mathcal{S}) = {}^sC_1^{k-max}(\mathcal{S}) = S^+$.

To summarize, given an operator system $\mathcal{S}$ on a Hilbert space $\mathcal{H}$ and a fixed $k \in \mathbb{N}$, we can construct universal operator system structures with the following property:

$$C_n^{max}(\mathcal{S}) \subseteq {}^sC_n^{k-max}(\mathcal{S}) \subseteq M_n(\mathcal{S})^+ \subseteq {}^sC_n^{k-min}(\mathcal{S}) \subseteq C_n^{min}(\mathcal{S}), \text{ for all } n \in \mathbb{N}.$$

The natural operator system structure of $\mathcal{S}$ induces a natural matrix order on its dual space $\mathcal{S}'$, which makes $\mathcal{S}'$ an operator system too. The dual cones on $\mathcal{S}'$ can be described as follows:

$$M_n(\mathcal{S}')^+ = (M_n(\mathcal{S})^+)^d = \{f : M_n(\mathcal{S}) \to \mathbb{C} \mid f \text{ positive linear functional } \},$$

for all $n \in \mathbb{N}$. Moreover, one can verify that $M_n(\mathcal{S}')^+ \cong CP(\mathcal{S}, M_n)$ for all n.

Knowing the matricial state space of a given operator system $\mathcal{S}$, we would like to find the corresponding matricial state spaces of super k-minimal and super k-maximal operator systems.

Definition 2.3.9. *Let $\mathcal{S}$ be a given operator system. For a fixed $k \in \mathbb{N}$, set*

$$\begin{aligned} {}^sQ_n^{k-min}(\mathcal{S}') = \big\{ F_{X^*GX} : M_n(\mathcal{S}) \to \mathbb{C} \,\big|\, X \in M_{mk,n}, \\ G = diag(\phi_1, \dots, \phi_m) \text{ with } \phi_i \in CP(\mathcal{S}, M_k), m \in \mathbb{N} \Big\}, \end{aligned}$$

and

$$ {}^sQ_n^{k-max}(\mathcal{S}') = \big\{ (f_{ij}) \in M_n(\mathcal{S}') : \Big(f_{ij}^{(k)}(a) \Big) \in M_{nk}^+, \text{ for all } a \in M_k(\mathcal{S})^+ \, \big\}. $$

Theorem 2.3.10. *Let $\mathcal{S}$ be a given operator system. Then $\{{}^sQ_n^{k-min}(\mathcal{S}')\}_{n=1}^{\infty}$ and $\{{}^sQ_n^{k-max}(\mathcal{S}')\}_{n=1}^{\infty}$ are matrix orderings on $\mathcal{S}'$ with ${}^sQ_1^{k-min}(\mathcal{S}') = {}^sQ_1^{k-max}(\mathcal{S}') = (\mathcal{S}^+)^d$. Moreover, ${}^d({}^sQ_n^{k-min}(\mathcal{S}')) = {}^sC_n^{k-min}(\mathcal{S})$ and $({}^sC_n^{k-max}(\mathcal{S}))^d = {}^sQ_n^{k-max}(\mathcal{S}')$.*

Proof. We will leave it to the reader to check these claims. For more details, see Proposition 2.2.22 and Theorem 2.2.23. $\square$

Given (V, V^+, e) an AOU space, recall the minimal and maximal operator systems on V, $\mathrm{OMIN}(V)$ and $\mathrm{OMAX}(V)$. We will apply the super k-minimal and the super k-maximal operator system structures on these two operator systems.

Proposition 2.3.11. *Let (V, V^+, e) be an AOU space and let $\mathrm{OMIN}(V)$ and $\mathrm{OMAX}(V)$ be the minimal and maximal operator systems on V, respectively. Then, for each $n \in \mathbb{N}$*

$$ D_n^{k-max}(V) = {}^sD_n^{k-max}(\mathrm{OMIN}(V)) \text{ and } C_n^{k-min}(V) = {}^sC_n^{k-min}(\mathrm{OMAX}(V)). $$

Proof. Use the definitions of each operator system structure involved. $\square$

This proposition shows how k-minimal and k-maximal operator system structures on ordered spaces defined in Section 2.2, can be regarded as a special case of the super k-minimal and super k-maximal operator system structures on the given operator systems $\mathrm{OMAX}(V)$ and on $\mathrm{OMIN}(V)$, respectively.

2.4 k-Partially Entanglement Breaking Maps

In Quantum Information Theory, there is a great interest in quantum entanglement theory and the objects that support this theory like entangled states, separable states, and "entanglement breaking" maps. There is a well-known duality between the class of entanglement breaking maps and separable states defined on tensor composite systems. Based on this theory, a lot of work has been done to generalize the well-known class of entanglement breaking maps, and introducing the classes of "partially entanglement breaking" maps, which are related to "partially separable states". In this section, we will review these generalized concepts, and relate them to our construction of universal minimal and universal maximal operator system structures.

Let M_n be the full algebra of $n \times n$ matrices, $n \in \mathbb{N}$. It is clear that M_n is, in fact, an AOU space. Moreover, M_n is an operator system arising from the identification of M_n with $B(\mathbb{C}^n)$. For some $k \in \mathbb{N}$, let $\mathrm{OMIN}_k^s(M_n)$ be the super k-minimal operator system structure on M_n and $\mathrm{OMAX}_k^s(M_n)$ be the super k-maximal operator system structure on M_n. Then, we have

$$M_m(\mathrm{OMAX}_k^s(M_n))^+ \subseteq M_m(M_n)^+ \subseteq M_m(\mathrm{OMIN}_k^s(M_n))^+, \text{ for all } m \in \mathbb{N}.$$

Note that $\mathrm{OMIN}_k^s(M_n)$ is just the operator system $M_n \cong B(\mathbb{C}^n)$ for all $k \geq n$. The cone of positive elements of M_n for any of these operator system structures coincides with the set of all positive definite matrices in M_n.

Let $s : M_n \otimes M_m \to \mathbb{C}$ be a (quantum) state defined on the composite system $M_n \otimes M_m$, $n, m \in \mathbb{N}$. Then s is called **separable** if it is a convex combination of tensor states; i.e. if there exist $l \in \mathbb{N}$, states $s_i : M_n \to \mathbb{C}$, states $t_i : M_m \to \mathbb{C}$, and real numbers $r_i > 0$ with $\sum_{i=1}^{l} r_i = 1$ and such that $s = \sum_{i=1}^{l} r_i s_i \otimes t_i$. If the state $s : M_n \otimes M_m \to \mathbb{C}$ cannot be written as a convex combination of tensor states, then s is called **entangled**.

Recall that a state $s : M_n \otimes M_m \to \mathbb{C}$ can be represented by a positive semi-definite self-adjoint matrix operator of trace one, called a **density matrix**. One commonly denotes density matrices with lowercase Greek letters such as ρ, ξ, σ. The density matrix of a quantum state $s : M_n \otimes M_m \to \mathbb{C}$ is defined by $\rho_s = (s(E_{ij} \otimes E_{kl}))$, where $\{E_{ij}\}_{i,j=1}^{n}$ and $\{E_{kl}\}_{k,l=1}^{m}$ are the canonical matrix units for M_n and M_m, respectively, and it can be written as a sum of rank one positive semi-definite matrices $\rho_s = \sum_{l=1}^{p} U_l U_l^*$, where $U_l \in \mathbb{C}^n \otimes \mathbb{C}^m$. We will classify quantum states according to their level of entanglement or separability.

Recall that the Schmidt number of a density matrix tells us the "level of entanglement or separability" of the state. A state $s : M_n \otimes M_m \to \mathbb{C}$ is called **maximally entangled** if the Schmidt number of its density matrix is $k = \min(n, m)$. Also, note that separable states are represented by density matrices of the form $\rho = \sum_j \sigma_j \otimes \tau_j$, where each $\sigma_j = \sum_e u_e^j (u_e^j)^* \geq 0$, $\tau_j = \sum_f v_f^j (v_f^j)^* \geq 0$. These are exactly the density matrices, whose Schmidt numbers are equal to 1. A state $s : M_n \otimes M_m \to \mathbb{C}$ is called **k-separable** [29] if the Schmidt number of its density matrix ρ_s is at most k with $k \leq \min(n, m)$.

A nonzero positive linear functional $f : M_n \otimes M_m \to \mathbb{C}$ is called k-separable if and only if $\dfrac{f}{f(I_n \otimes I_m)}$ is a k-separable state. We recall that, given a completely positive map $\phi : M_p \to M_m$, we denote $\phi^{(n)} : M_n \otimes M_p \to M_n \otimes M_m$ the map given by $\phi^{(n)}((v_{ij})) = (\phi(v_{ij})) \in M_n \otimes M_m$. If $s : M_n \otimes M_m \to \mathbb{C}$ is a positive linear functional, then $s \circ \phi^{(n)} : M_n \otimes M_p \to \mathbb{C}$ is positive linear functional. If s is a state and ϕ is unital, then $s \circ \phi^{(n)}$ is a state.

A linear map $\phi : M_p \to M_m$ is called **k-partially entanglement breaking** [**5**] (k-PEB), if $s \circ \phi^{(n)} : M_n \otimes M_p \to \mathbb{C}$ is a k-separable state for every state $s : M_n \otimes M_m \to \mathbb{C}$, for all $n \in \mathbb{N}$.

In this section, we relate k-partially entanglement breaking maps to the super k-minimal and super k-maximal operator system structures studied in the previous section. We begin with a characterization of k-separable states.

Proposition 2.4.1. *Let $f : M_n \otimes M_m \to \mathbb{C}$ be a positive linear functional. Then f is k-separable if and only if $f : M_n(\mathrm{OMIN}_k^s(M_m)) \to \mathbb{C}$ is positive.*

Proof. Given a positive linear functional $f : M_n \otimes M_m \to \mathbb{C}$ with $f(I_n \otimes I_m) \neq 0$, then $\dfrac{f}{f(I_n \otimes I_m)} : M_n \otimes M_m \to \mathbb{C}$ becomes a state. Hence, we may assume f is a k-separable state, $k \leq \min(n, m)$. Assume that the density matrix of f is

$$\rho_f = \sum_{e,f=1}^{k} u_e u_f^* \otimes v_e v_f^*,$$

for some $\{u_1, u_2, \ldots, u_k\} \subseteq \mathbb{C}^n$ and $\{v_1, v_2, \ldots, v_k\} \subseteq \mathbb{C}^m$.

Define $\phi_{ef} : M_m \to \mathbb{C}$ by

$$\phi_{ef}(x) = \bar{v}_e^*(x)\bar{v}_f, \text{ for all } x \in M_m.$$

It is obvious that ϕ_{ef} is a well defined linear map on M_m. Note that the "density matrix" for each ϕ_{ef} is

$$\rho_{ef} = [\phi_{ef}(E_{kl})]_{k,l=1}^{m} = v_e v_f^*.$$

Now, look at $\phi = [\phi_{ef}] : M_m \to M_k$ given by

$$\phi(x) = [\phi_{ef}(x)] \quad = \quad [\bar{v}_e^*(x)\bar{v}_f]$$

$$
\begin{aligned}
&= \begin{bmatrix} \bar{v}_1^* \\ \vdots \\ \bar{v}_k^* \end{bmatrix} x \underbrace{\begin{bmatrix} \bar{v}_1 & \cdots & \bar{v}_k \end{bmatrix}}_{=A\in M_{m,k}} \\
&= A^* x A.
\end{aligned}
$$

Then, one can easily verify that ϕ is a completely positive map on M_m. Hence, we can write each function f as

$$
f = \sum_{e,f=1}^{k} u_e u_f^* \otimes \phi_{ef} = \begin{bmatrix} u_1 & \cdots & u_k \end{bmatrix} \phi \begin{bmatrix} u_1^* \\ \vdots \\ u_k^* \end{bmatrix}.
$$

This shows that $f \in {}^sQ_n^{k-min}(M_m)$, i.e. $f \in ({}^sC_n^{k-min}(M_m))^d$.

So, f is positive on $M_n(\mathrm{OMIN}_k^s(M_m))$.

Conversely, assume $f : M_n(\mathrm{OMIN}_k^s(M_m)) \to \mathbb{C}$ is positive , i.e. $f \in ({}^sC_n^{k-min}(M_m))^d = \overline{{}^sQ_n^{k-min}(M_m)}^{w^*}$. Without loss of generality, let $f = \Lambda\phi\Lambda^* \in {}^sQ_n^{k-min}(M_m)$, where $\Lambda = \begin{bmatrix} u_1 & u_2 & \cdots u_k \end{bmatrix} \in M_{n,k}$ and $\phi = [\phi_{ef}] : M_m \to M_k$ is completely positive. Then $f = \sum_{e,f=1}^k u_e u_f^* \otimes \phi_{ef}$. Since ϕ is completely positive, then $\phi(x) = \sum_{i=1}^l A_i^* x A_i$, for some Kraus operators $\{A_i\} \subseteq M_{m,k}$. Writing each $A_i = \begin{bmatrix} \bar{v}_1^i & \bar{v}_2^i & \cdots & \bar{v}_k^i \end{bmatrix}$, where each $\bar{v}_e^i \in \mathbb{C}^m$, then one can see that $\phi_{ef}(x) = \sum_{i=1}^l (\bar{v}_e^i)^* x (\bar{v}_f^i)$, and its density matrix $\rho_{\phi_{ef}} = \sum_{i=1}^l v_e^i (v_f^i)^*$. Hence, the density matrix for the function f will be

$$
\rho_f = \sum_{i=1}^{l} \sum_{e,f=1}^{k} u_e u_f^* \otimes v_e^i (v_f^i)^*.
$$

This shows that f is a k-separable map.

In general, any positive linear functional $f \in ({}^sC_n^{k-min}(M_m))^d$ (which becomes a state by dividing by its norm) is a weak*-limit of k-separable states. Such a limit exists, because k-separable states are the convex hull of a compact set, which is a compact set by

Caratheodory's theorem. $\square$

We now turn our attention to a duality result. Recall that the dual of a matrix ordered space is again a matrix ordered space. Let $\delta_{i,j} : M_n \to \mathbb{C}$ be the linear functional satisfying

$$\delta_{i,j}(E_{kl}) = \begin{cases} 1 & \text{when } (i,j) = (k,l) \\ 0 & \text{when } (i,j) \neq (k,l) \end{cases}$$

and let $\gamma_n : M_n \to M_n'$ be the linear isomorphism defined by $\gamma_n(E_{i,j}) = \delta_{i,j}$. The next result is certainly in some sense known, but the formal statement will be useful for us in the sequel.

Theorem 2.4.2 ([23], Theorem 6.2)**.** *The map $\gamma_n : M_n \to M_n'$ is a complete order isomorphism of matrix ordered spaces. Consequently, $(M_n', (M_n')^+, tr)$ is an AOU space that is order isomorphic to (M_n, M_n^+, I_n), where I_n denotes the identity matrix.*

Proposition 2.4.3. *The complete order isomorphism $\gamma_n : M_n \to M_n'$ gives rise to the identifications $\mathrm{OMIN}_k^s(M_n)' = \mathrm{OMAX}_k^s(M_n') = \mathrm{OMAX}_k^s(M_n)$ and $\mathrm{OMAX}_k^s(M_n)' = \mathrm{OMIN}_k^s(M_n')$ $= \mathrm{OMIN}_k^s(M_n)$.*

Proof. Let $\mathcal{S} = M_n'$, then one can observe that ${}^sQ_m^{k-min}(M_n) = {}^sD_m^{k-max}(\mathcal{S})$ by definitions of each cone. The unit ball of ${}^sD_m^{k-max}(\mathcal{S})$ is compact, therefore ${}^sD_m^{k-max}(\mathcal{S})$ is closed by the Krein-Shmulian Theorem. Hence, ${}^sD_m^{k-max}(\mathcal{S}) = {}^sC_m^{k-max}(\mathcal{S})$. Thus, we have that ${}^sQ_m^{k-min}(M_n) = {}^sC_m^{k-max}(\mathcal{S})$, i.e. $M_m(\mathrm{OMIN}_k^s(M_n)')^+ = M_m(\mathrm{OMAX}_k^s(M_n'))^+$, and so the identity map on M_n' yields a complete order isometry between the matrix ordered space $\mathrm{OMIN}_k^s(M_n)'$ and the operator system $\mathrm{OMAX}_k^s(M_n')$. Finally, the complete order isomorphism γ_n allows for the identification, $\mathrm{OMAX}_k^s(M_n') = \mathrm{OMAX}_k^s(M_n)$. The proof of the rest of the statement is similar. $\square$

Theorem 2.4.4. *Let $\phi : M_p \to M_m$ be a linear map. Then ϕ is a k-partially entanglement breaking map if and only if $\phi : \mathrm{OMIN}_k^s(M_p) \to M_m$ is completely positive.*

Proof. Assume $\phi : \mathrm{OMIN}_k^s(M_p) \to M_m$ is completely positive. Then $\phi' : M_m' \to \mathrm{OMIN}_k^s(M_p)'$ is completely positive too. If $f = (f_{ij}) \in M_n(M_m')^+$ is any state on $M_n \otimes M_m$, then $[\phi'(f_{ij})] \in M_n(\mathrm{OMIN}_k^s(M_p)')^+$. By Proposition 2.4.1, these are exactly the k-separable states on $M_n \otimes M_p$, i.e.

$$f \circ \phi^{(n)} = \left[\phi'(f_{ij})\right] = [f_{ij} \circ \phi] : M_n \otimes M_p \to \mathbb{C}$$

is k-separable, for all states $f : M_n \otimes M_m \to \mathbb{C}$. This implies ϕ is a k-PEB map.

Conversely, assume ϕ is k-PEB. Then, for any $f = (f_{ij}) \in M_n(M_m')^+$, we have $f \circ \phi^{(n)}$ is k-separable, i.e. $f \circ \phi^{(n)} = [\phi'(f_{ij})] \in M_n(\mathrm{OMIN}_k^s(M_p)')^+$, which implies that $\phi' : M_m' \to \mathrm{OMIN}_k^s(M_p)'$ is completely positive. As a result, we have $\phi : \mathrm{OMIN}_k^s(M_p) \to M_m$ is completely positive. $\square$

Let $U = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_k \end{bmatrix} = \sum_{j=1}^k e_j \otimes u_j \in \mathbb{C}^k \otimes \mathbb{C}^m$, where $u_j \in \mathbb{C}^m$, $1 \le j \le k$. Then U can be viewed as the $m \times k$ matrix $M_u = \begin{bmatrix} u_1 & u_2 & \cdots & u_k \end{bmatrix} \in M_{m,k}$. If $\lambda \in \mathbb{C}^k$, then we have

$$(\lambda^* \otimes I_m)(UU^*)(\lambda \otimes I_m) = M_u(\lambda\lambda^*)M_u^*.$$

Proposition 2.4.5. *Let $\phi : M_p \to M_m$ be a linear map. Then $\phi : M_p \to \mathrm{OMAX}_k^s(M_m)$ is completely positive if and only if there exist completely positive maps $\psi_l : M_p \to M_k$ and matrices $M_l \in M_{m,k}$, $l = 1, \dots, q$ such that $\phi(X) = \sum_{l=1}^q M_l \psi_l(X) M_l^*$.*

Proof. We have that $\phi : M_p \to \mathrm{OMAX}_k^s(M_m)$ is completely positive if and only if $(\phi(E_{ij})) \in M_p(\mathrm{OMAX}_k^s(M_m))^+ = {}^sC_p^{k-max}(M_m) = {}^sD_p^{k-max}(M_m)$, since the set ${}^sD_p^{k-max}(M_m)$ is closed. Thus, there exists an integer q,

$A_1, \ldots, A_q \in M_{k,p}$, positive matrices $D_1, \ldots, D_q \in M_k(M_m)^+$, such that

$$(\phi(E_{ij})) = \sum_{l=1}^{q} (A_l^* \otimes I_m) D_l (A_l \otimes I_m).$$

Write $A_l = \begin{bmatrix} \lambda_{1,l} & \lambda_{2,l} & \cdots & \lambda_{p,l} \end{bmatrix}$, where $\lambda_{i,l} \in \mathbb{C}^k$ for all $i = 1, \ldots, p$. Then, we have $\phi(E_{ij}) = \sum_{l=1}^{q} (\lambda_{i,l}^* \otimes I_m) D_l (\lambda_{j,l} \otimes I_m)$. Since $D_l \in M_k(M_m)^+$, then $D_l = \sum_{r=1}^{t} U_{r,l} U_{r,l}^*$, where $U_{r,l} \in \mathbb{C}^k \otimes \mathbb{C}^m$ for all $1 \leq r \leq t$. Without loss of generalization, assume $D_l = U_l U_l^*$, where $U_l = \begin{bmatrix} u_{1,l} \\ u_{2,l} \\ \vdots \\ u_{k,l} \end{bmatrix}$, each $u_{e,l} \in \mathbb{C}^m$, for all $1 \leq e \leq k$. This implies

$$\phi(E_{ij}) = \sum_{l=1}^{q} (\lambda_{i,l}^* \otimes I_m) D_l (\lambda_{j,l} \otimes I_m) = \sum_{l=1}^{q} M_l \left[(\bar{\lambda}_{i,l})(\bar{\lambda}_{j,l})^* \right] M_l^*,$$

where $M_l = \begin{bmatrix} u_{1,l} & u_{2,l} & \cdots & u_{k,l} \end{bmatrix} \in M_{m,k}$ is the corresponding matrix for U_l. If we define completely positive maps $\psi_l : M_p \to M_k$ by

$$\psi_l(X) = \sum_{i,j=1}^{p} (\bar{\lambda}_{i,l}) x_{ij} (\bar{\lambda}_{j,l})^* = \bar{A}_l X \bar{A}_l^*,$$

then we have that $\phi(E_{ij}) = \sum_{l=1}^{q} M_l \psi_l(E_{ij}) M_l^*$, for all $1 \leq i, j \leq p$, and hence $\phi(X) = \sum_{l=1}^{q} M_l \psi_l(X) M_l^*$ for every $X \in M_p$.

Conversely, given any completely positive map $\psi : M_p \to M_k$, then ψ can be written as $\psi(X) = (\bar{A}) X (\bar{A})^* = \sum_{i,j=1}^{p} (\bar{\lambda}_i) x_{ij} (\bar{\lambda}_j)^*$, where $A = \begin{bmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_p \end{bmatrix} \in M_{k,p}$ with $\lambda_i \in \mathbb{C}^k$. Thus, if $\phi(X) = \sum_{l=1}^{q} M_l \psi_l(X) M_l^*$, where $M_l = \begin{bmatrix} u_{1,l} & u_{2,l} & \cdots & u_{k,l} \end{bmatrix} \in M_{m,k}$ with $u_{e,l} \in \mathbb{C}^m$ for all $1 \leq e \leq k$, and $\psi_l : M_p \to M_k$ completely positive, then by increasing the number of terms in the sum we may assume that each ψ_l has the form

$\psi_l(X) = (\bar{A}_l)X(\bar{A}_l)^* = \sum_{i,j=1}^{p}(\bar{\lambda}_{i,l})x_{ij}(\bar{\lambda}_{j,l})^*$, and hence

$$\phi(E_{ij}) = \sum_{l=1}^{q} M_l\left[(\bar{\lambda}_{i,l})(\bar{\lambda}_{j,l})^*\right]M_l^* = \sum_{l=1}^{q}(\lambda_{i,l}^*\otimes I_m)D_l(\lambda_{j,l}\otimes I_m),$$

where $D_l = \begin{bmatrix} u_{1,l} \\ u_{2,l} \\ \vdots \\ u_{k,l} \end{bmatrix}\begin{bmatrix} u_{1,l}^* & u_{2,l}^* & \cdots & u_{k,l}^* \end{bmatrix} = U_lU_l^* \in M_k(M_m)^+$.

Thus $(\phi(E_{ij})) = \sum_{l=1}^{q}(A_l^*\otimes I_m)D_l(A_l\otimes I_m) \in {}^sD_p^{k-max}(M_m)$, and it follows that $\phi : M_p \to \mathrm{OMAX}_k^s(M_m)$ is completely positive.

□

Corollary 2.4.6. *If $\phi : M_p \to \mathrm{OMAX}_k^s(M_m)$ is completely positive, then ϕ is a k-partially entanglement breaking map.*

Proof. By Proposition 2.4.5, there exist completely positive maps $\psi_l : M_p \to M_k$ and matrices $M_l \in M_{m,k}$, $1 \le l \le q$, such that $\phi(X) = \sum_{l=1}^{q} M_l\psi_l(X)M_l^*$. Given any $n \in \mathbb{N}$ and any positive linear functional $f : M_n \otimes M_m \to \mathbb{C}$, we have $f\circ\phi^{(n)} : M_n \otimes M_p \to \mathbb{C}$ is k-separable if and only if $f\circ\phi^{(n)} : M_n(\mathrm{OMIN}_k^s(M_p)) \to \mathbb{C}$ is a positive linear functional by Proposition 2.4.1. Let $(X_{ij}) \in {}^sC_n^{k-min}(M_p)$, then we have

$$\begin{aligned} \phi^{(n)}((X_{ij})) = (\phi(X_{ij})) &= \sum_{l=1}^{q}(M_l\psi_l(X_{ij})M_l^*) \\ &= \sum_{l=1}^{q}(I_n\otimes M_l)\psi^{(n)}((X_{ij}))(I_n\otimes M_l^*) \ge 0, \end{aligned}$$

since $\psi^{(n)}((X_{ij})) \ge$ for all $(X_{ij}) \in {}^sC_n^{k-min}(M_p)$. Thus, $(f\circ\phi^{(n)})((X_{ij})) = f((\phi(X_{ij}))) \ge 0$ since f is a positive linear functional on $M_n \otimes M_m$ and $(\phi(X_{ij})) \in M_n(M_m)^+$. As a result, $f\circ\phi^{(n)}$ is k-separable, which implies that ϕ is k-PEB. □

Theorem 2.4.7. *Let $\phi : M_p \to M_m$ be a linear map, and $k \leq \min(p, m)$. Then the following are equivalent:*

(i) $\phi : \mathrm{OMIN}_k^s(M_p) \to M_m$ *is completely positive.*

(ii) ϕ *is k-partially entanglement breaking.*

(iii) $\phi : M_p \to \mathrm{OMAX}_k^s(M_m)$ *is completely positive.*

(iv) There exist completely positive maps $\psi_l : M_p \to M_k$ *and* $M_l \in M_{m,k}$*, for* $1 \leq l \leq q$ *such that* $\phi(X) = \sum_{l=1}^{q} M_l \psi_l(X) M_l^*$.

(v) There exist matrices $A_l \in M_{p,m}$*,* $1 \leq l \leq r$ *of rank at most* k*, such that* $\phi(X) = \sum_{l=1}^{s} A_l^* X A_l$.

(vi) $\phi : \mathrm{OMIN}_k^s(M_p) \to \mathrm{OMAX}_k^s(M_m)$ *is completely positive.*

Proof. The equivalence of (i) and (ii) is stated in Theorem 2.4.4, while the equivalence of (iii) and (iv) is stated in Proposition 2.4.5. By Corollary 2.4.6, (iii) implies (ii). Note that $\phi : M_p \to \mathrm{OMAX}_k^s(M_m)$ is completely positive if and only if $\phi' : \mathrm{OMAX}_k^s(M_m)' \to M_p'$ is completely positive. Using the identifications of Proposition 2.4.3, we have $\phi' : \mathrm{OMIN}_k^s(M_m') \to M_p'$ is completely positive if and only if $\phi^\flat = \gamma_p^{-1} \circ \phi' \circ \gamma_m : \mathrm{OMIN}_k^s(M_m) \to M_p$ is completely positive, i.e. $\phi^\flat : M_m \to M_p$ is k-PEB. Hence, if $\phi = (\phi^\flat)^\flat$ is k-PEB, then $\phi^\flat$ is k-PEB, which is equivalent to $\phi : M_p \to \mathrm{OMAX}_k^s(M_m)$ is completely positive. So (ii) implies (iii). Now we have the equivalence $(i) - (iv)$.

To show that (iv) implies (v), we may assume that each completely positive map $\psi_l : M_p \to M_k$ can be written as $\psi_l(X) = \sum_{j=1}^{r} B_{j,l}^* X B_{j,l}$, for some $B_{j,l} \in M_{p,k}$. Then,

$$\phi(X) = \sum_{l=1}^{q} \sum_{j=1}^{r} M_l B_{j,l}^* X B_{j.l} M_l^* = \sum_{l=1}^{s} A_l^* X A_l,$$

where each $A_l = B_{j,l} M_l^* \in M_{p,m}$ has rank at most k for all $1 \leq l \leq s$, since $\mathrm{rank}(A_l) \leq \min(\mathrm{rank}(B_{j,l}), \mathrm{rank}(M_l)) = k$.

To see that (v) implies (iv), each $A_l \in M_{p,m}$ of rank at most k, can be factorized as $A_l = B_l M_l$, where $M_l \in M_{k,m}$ is the reduced matrix of A_l containing only the k rows that span A_l, and $B_l \in M_{p,k}$ is the coefficient matrix of A_l. Set $\psi_l(X) = B_l^* X B_l$ which is completely positive, then $\phi(X) = \sum_{l=1}^{s} M_l^* \psi_l(X) M_l$.

Finally, clearly (vi) implies (i). One can easily check that (iv) implies (vi). $\square$

Chapter 3

The Spanning Index of a Finite Set of Matrices in $\mathbf{M_n}$

3.1 Introduction

Let M_n denote the space of $n \times n$ real or complex matrices, and let $\mathcal{P}_1 = \{A_1, A_2, \ldots, A_l\} \subseteq M_n$ be a finite set of matrices from this space. We define

$$\mathcal{P}_m = \underbrace{\mathcal{P}_1 \cdot \mathcal{P}_1 \cdot \cdots \cdot \mathcal{P}_1}_{m\, times} = \left\{X_1 \cdot X_2 \cdot \cdots \cdot X_m \,|\, X_i \in \mathcal{P}_1 \text{ for all } 1 \leq i \leq m\right\}$$

to be the set of matrices that are products of m matrices coming from $\mathcal{P}_1$ with repetitions allowed, and we call any element of $\mathcal{P}_m$ a **matrix of length m**. Furthermore, we denote $\mathcal{V}_m = \text{span}(\mathcal{P}_m)$, a subspace of M_n. If $\mathcal{V}_m = M_n$ for some $m \in \mathbb{N}$, then the set $\mathcal{P}_m$ contains n^2 linearly independent matrices. Moreover, for each (i, j), there exists at least one $A \in \mathcal{P}_m$ such that the $(i, j)^{th}$ entry is non-zero. It follows that the matrices in $\mathcal{P}_m$ can not be all lower-triangular, or all upper-triangular or all diagonal. Hence, the set $\mathcal{P}_1$ should contain at least one of each.

Given $\mathcal{P}_1 = \{A_1, A_2, \ldots, A_l\} \subseteq M_n$ and $\mathcal{P}_m$ the set of matrices of length m, then one can see that the cardinality of $\mathcal{P}_m$ is less than or equal to l^m. It follows that $\dim(\mathcal{V}_m) \leq l^m$. Thus, if $\mathcal{V}_m = M_n$, then $n^2 \leq l^m$. We want to find what is the least such $m \in \mathbb{N}$ for which we get $\mathcal{V}_m = M_n$. In [25], it was conjectured that this bound does not exceed n^2.

Conjecture 3.1.1 (Michael Wolf, [25])**.** *Let $\mathcal{P}_1 = \{A_1, A_2, \ldots, A_l\} \subseteq M_n$ be a finite set, let $\mathcal{P}_m$ be the set of matrices of length m of $\mathcal{P}_1$ for some $m \in \mathbb{N}$, and let $\mathcal{V}_m = \mathrm{span}(\mathcal{P}_m)$. If $\mathcal{V}_m = M_n$ for some $m \in \mathbb{N}$, then there exists $m_1 \leq n^2$ such that $\mathcal{V}_{m_1} = M_n$.*

One can easily see that such a bound depends on the beginning set $\mathcal{P}_1$ and its elements. Hence, some progress could be made only if we know some properties of the initial matrices of $\mathcal{P}_1$.

Proposition 3.1.2. *Let $\mathcal{P}_1 = \{A_1, A_2, \ldots, A_l\} \subseteq M_n$ be a finite set and let $\mathcal{P}_m$ be the set of matrices of length m of $\mathcal{P}_1$ for some $m \in \mathbb{N}$. Then*

$$\mathcal{V}_m = A_1 \cdot \mathcal{V}_{m-1} + A_2 \cdot \mathcal{V}_{m-1} + \cdots + A_l \cdot \mathcal{V}_{m-1},$$

where $\mathcal{V}_m = \mathrm{span}(\mathcal{P}_m)$ and $\mathcal{V}_{m-1} = \mathrm{span}(\mathcal{P}_{m-1})$.

Proof. We know that $\mathcal{P}_m = \mathcal{P}_1 \cdot \mathcal{P}_{m-1}$. It follows that

$$A_1 \cdot \mathcal{V}_{m-1} + A_2 \cdot \mathcal{V}_{m-1} + \cdots + A_l \cdot \mathcal{V}_{m-1} \subseteq \mathcal{V}_m.$$

On the other hand, using basic linear algebra techniques, one can easily show that

$$\mathcal{V}_m \subseteq A_1 \cdot \mathcal{V}_{m-1} + A_2 \cdot \mathcal{V}_{m-1} + \cdots + A_l \cdot \mathcal{V}_{m-1}.$$

Hence, the result follows. □

Proposition 3.1.3. *Let $\mathcal{P}_1 = \{A_1, A_2, \ldots, A_l\} \subseteq M_n$ be a finite set, let $\mathcal{P}_m$ be the set of matrices of length m of $\mathcal{P}_1$ for some $m \in \mathbb{N}$, and let $\mathcal{V}_m = \operatorname{span}(\mathcal{P}_m)$. If $\mathcal{V}_m = M_n$ for some $m \in \mathbb{N}$, then $\mathcal{V}_{m+1} = M_n$.*

Proof. Let $m \in \mathbb{N}$ such that $\mathcal{V}_m = M_n$. It is clear that $\mathcal{V}_{m-1} \subseteq \mathcal{V}_m = M_n$. Then, by using Proposition 3.1.2 and basic linear algebra knowledge, we have

$$\begin{aligned} M_n &= \mathcal{V}_m = A_1 \cdot \mathcal{V}_{m-1} + A_2 \cdot \mathcal{V}_{m-1} + \cdots + A_l \cdot \mathcal{V}_{m-1} \\ &\subseteq A_1 \cdot \mathcal{V}_m + A_2 \cdot \mathcal{V}_m + \cdots + A_l \cdot \mathcal{V}_m \\ &= \mathcal{V}_{m+1} \subseteq M_n. \end{aligned}$$

As a result, we conclude $\mathcal{V}_{m+1} = \operatorname{span}(P_{m+1}) = M_n$. $\square$

Definition 3.1.4. *Let $\mathcal{P}_1 = \{A_1, A_2, \ldots, A_l\} \subseteq \mathcal{M}_n$, $\mathcal{P}_m$ be the set of matrices of length m of $\mathcal{P}_1$ for some $m \in \mathbb{N}$, and $\mathcal{V}_m = \operatorname{span}(\mathcal{P}_m)$. If $\mathcal{V}_m = M_n$ for some $m \in \mathbb{N}$, then the least such m is called* **the spanning index of** $\mathcal{P}_1$ *and denoted by $m = \operatorname{index}(P_1)$. If $\mathcal{V}_m$ is never M_n, then we set $\operatorname{index}(\mathcal{P}_1) = +\infty$.*

Note that if $\mathcal{V}_m = M_n$ for some $m \in \mathbb{N}$, then $\dim(\mathcal{V}_m) = n^2$. Hence, we can alternatively define the spanning index of a given set $\mathcal{P}_1$ as

$$\operatorname{index}(\mathcal{P}_1) = \inf\{m \in \mathbb{N} : \dim(\mathcal{V}_m) = n^2\}.$$

Example 3.1.5. *We will consider two cases of finite sets of matrices in M_3:*

(i) *Let* $\mathcal{P}_1 = \left\{ I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\} \subseteq M_3.$

For the ease of notation, we will write the elements of $\mathcal{P}_1$ in terms of matrix units, i.e. $\mathcal{P}_1 = \{I_3 = E_{11} + E_{22} + E_{33},\ E_{21} + E_{32},\ E_{13}\}$. One can straightforwardly check *that*

$$\begin{aligned}\mathcal{P}_2 &= \{I_3,\ E_{21}+E_{32},\ E_{13},\ E_{31},\ E_{23},\ E_{12}\},\\ \mathcal{P}_3 &= \{E_{11},\ E_{12},\ E_{13},\ E_{22},\ E_{21}+E_{32},\ E_{23},\ E_{31},\ E_{33}\},\\ \mathcal{P}_4 &= \{E_{11},\ E_{12},\ E_{13},\ E_{21},\ E_{22},\ E_{23},\ E_{31},\ E_{32},\ E_{33}\}.\end{aligned}$$

It follows that $\mathcal{V}_4 = M_3$ *and therefore* $\operatorname{index}(\mathcal{P}_1) = 4$.

(ii) *Let* $\mathcal{Q}_1 = \{I_3 = E_{11}+E_{22}+E_{33},\ S = E_{21}+E_{32}+E_{13}\} \subseteq M_3$*,*
where I_3 *is the identity matrix and* S *is the cyclic forward shift matrix.*
Then one can easily check that $\mathcal{Q}_k = \{I_3, S,\ S^2\}$ *for all* $1 < k \in \mathbb{N}$*.*
It follows that $\operatorname{index}(\mathcal{Q}_1) = +\infty$.

Proposition 3.1.6. *Let* $\mathcal{P}_1 = \{A_1, A_2, \ldots, A_l\} \subseteq M_n$ *and* $\tilde{\mathcal{P}}_1 = \{A_1 + A_2, A_3, \ldots, A_l\}$*. Then* $\operatorname{index}(\mathcal{P}_1) \leq \operatorname{index}(\tilde{\mathcal{P}}_1)$*.*

Proof. If $\operatorname{index}(\tilde{\mathcal{P}}_1) = +\infty$, then the result is clear. Hence, assume $\operatorname{index}(\tilde{\mathcal{P}}_1) = m < +\infty$ and let $\tilde{\mathcal{V}}_m = \operatorname{span}(\tilde{\mathcal{P}}_m)$ and $\mathcal{V}_m = \operatorname{span}(\mathcal{P}_m)$. By Definition 3.1.4, we have $\tilde{\mathcal{V}}_m = M_n$. To prove that $\operatorname{index}(\mathcal{P}_1) \leq \operatorname{index}(\tilde{\mathcal{P}}_1)$, it is enough to show that $\tilde{\mathcal{V}}_m \subseteq \mathcal{V}_m$.

Claim: $\tilde{\mathcal{V}}_k \subseteq \mathcal{V}_k$ for all $k \in \mathbb{N}$: We will show this result by induction.

(i) Clearly, $\tilde{\mathcal{V}}_1 = \operatorname{span}(\tilde{\mathcal{P}}_1) \subseteq \operatorname{span}(\mathcal{P}_1) = \mathcal{V}_1$.

(ii) Next, we assume that $\tilde{\mathcal{V}}_k \subseteq \mathcal{V}_k$ holds for all $1, 2, \ldots, k \in \mathbb{N}$.

(iii) Then, we show that $\tilde{\mathcal{V}}_{k+1} \subseteq \mathcal{V}_{k+1}$. Using the definition of $\tilde{\mathcal{V}}_{k+1}$, Proposition 3.1.2 and induction hypothesis (ii), we have

$$\begin{aligned}\tilde{\mathcal{V}}_{k+1} &= \operatorname{span}(\tilde{\mathcal{P}}_{k+1}) = \operatorname{span}(\tilde{\mathcal{P}}_1 \cdot \tilde{\mathcal{P}}_k)\\ &= (A_1 + A_2) \cdot \tilde{\mathcal{V}}_k + A_3 \cdot \tilde{\mathcal{V}}_k + \cdots + A_l \cdot \tilde{\mathcal{V}}_k\\ &\subseteq (A_1 + A_2) \cdot \mathcal{V}_k + A_3 \cdot \mathcal{V}_k + \cdots + A_l \cdot \mathcal{V}_k\end{aligned}$$

$$\begin{aligned} &\subseteq A_1 \cdot \mathcal{V}_k + A_2 \cdot \mathcal{V}_k + A_3 \cdot \mathcal{V}_k + \cdots + A_l \cdot \mathcal{V}_k \\ &= \mathcal{V}_{k+1}. \end{aligned}$$

As a result, we conclude that $\tilde{\mathcal{V}}_k \subseteq \mathcal{V}_k$ for all $k \in \mathbb{N}$. Then, we have

$$M_n = \tilde{\mathcal{V}}_m \subseteq \mathcal{V}_m \subseteq M_n.$$

It follows that $\mathcal{V}_m = M_n$ and therefore $\text{index}(\mathcal{P}_1) \leq \text{index}(\tilde{\mathcal{P}}_1) < +\infty$. $\square$

Remark 3.1.7. *Note that the converse of Proposition 3.1.6 is not generally true. In other words, if we are given a finite set $\mathcal{P}_1 = \{A_1, A_2, \ldots, A_l\} \subseteq M_n$ with a finite spanning index and if we let $\tilde{\mathcal{P}}_1 = \{A_1 + A_2, A_3, \ldots, A_l\}$ where $A_1 + A_2$ can be any grouping of 2 elements of $\mathcal{P}_1$, then the spanning index of $\tilde{\mathcal{P}}_1$ might not be finite.*
Specifically, let $\mathcal{P}_1 = \{I_3 = E_{11} + E_{22} + E_{33},\ E_{21} + E_{32},\ E_{13}\} \subseteq M_3$ with $\text{index}(\mathcal{P}_1) = 4$, as shown in Example 3.1.5(i). Then we group two last elements of $\mathcal{P}_1$ and set $\tilde{\mathcal{P}}_1 = \{I_3,\ (E_{21} + E_{32}) + E_{13}\}$. It follows that $\tilde{\mathcal{P}}_1 = \{I_3,\ S\}$ as in Example 3.1.5(ii) and the spanning index of this new set does not exist. But, if we let $\hat{\mathcal{P}}_1 = \{I_3 + E_{13},\ E_{21} + E_{32}\}$, then $\text{index}(\hat{\mathcal{P}}_1) = 4$.
This shows that grouping elements of a given finite set whose spanning index exists, cannot be done randomly. We are interested in grouping elements since we can get a bigger bound for the spanning index.

Given a finite set of l matrices $\mathcal{P}_1 = \{A_1, A_2, \ldots, A_l\} \subseteq M_n$, the spanning index of $\mathcal{P}_1$ might get smaller when the number of initial matrices increases as in Proposition 3.1.6. In such a case, one might expect that the set $\mathcal{P}_1$ with two appropriate matrices has the biggest finite spanning index. We believe that to prove the conjecture, it is enough to consider the case of the set $\mathcal{P}_1$ containing 2 matrices (which are good enough to lead to a spanning of $\mathcal{M}_n$). Before considering such a case, it would be nice to first see what kind of matrices should be involved and the properties they should have, such that $\text{span}(\mathcal{P}_m) = \mathcal{M}_n$, for some $m \in \mathbb{N}$.

The set $\mathcal{P}_m$ depends on the choice of matrices for the initial set $\mathcal{P}_1$. Therefore, the set $\mathcal{P}_m$ spans M_n for some $m \in \mathbb{N}$ if and only if $\mathcal{P}_1$ contains matrices that have enough entries to make the spanning happen. When do these matrices have enough entries? To get an answer for this question, we consider the case which involves matrix units.

3.2 Matrix Units and Graphs

Let M_n be the space of $n \times n$ matrices and let $\{E_{ij}\}_{i,j=1}^{n}$ be the canonical matrix units of M_n. Let $\mathcal{E} = \{(i,j) : i,j \in \{1,2,\ldots,n\}\}$ be a subset of $\{1,2,\ldots,n\} \times \{1,2,\ldots,n\}$ and let $\mathcal{P}_1 = \left\{E_{ij} | (i,j) \in \mathcal{E}\right\}$ be a collection of matrix units in M_n. Note that a product of matrix units is again a matrix unit. Thus

$$\mathcal{P}_m = \left\{ \prod_{k=1}^{m} E_{(i_k,j_k)} \,|\, (i_k,j_k) \in \mathcal{E} \text{ for all } k = 1,2,\ldots,m \right\}$$

contains matrix units, each of which are product of m matrix units from $\mathcal{P}_1$ with repetitions allowed. In other words, we say $\mathcal{P}_m$ contains **matrix units of length m** and denote $\prod_{k=1}^{m} E_{(i_k,j_k)} = E_{ij}^{(m)}$. Hence, we have

$$\mathcal{P}_m = \left\{E_{ij}^{(m)} \mid \text{ matrix unit of length m, } (i,j) \in \mathcal{E} \right\}.$$

If $\text{span}(\mathcal{P}_m) = M_n$ for some $m \in \mathbb{N}$, then it's obvious that $\mathcal{P}_m$ contains all the n^2 matrix units of M_n, each of length m. We will find the necessary and sufficient conditions for the initial matrix units of length 1, given that $\mathcal{P}_m$ spans M_n at some level $m \in \mathbb{N}$. The matrix units of length 1 depend on the choice of the set $\mathcal{E} \subseteq \{1,2,\ldots,n\} \times \{1,2,\ldots,n\}$. In order to understand such sets, we will look at their graphs. Next, we will give a brief introduction to Directed Graph Theory and relate it to our problem.

3.2.1 Directed Graph Theory

A graph $\mathcal{G}$ is an ordered pair $\mathcal{G} = (V, \mathcal{E})$ consisting of a set V of **vertices** (also known as nodes) and a set $\mathcal{E} \subseteq V \times V$ of **edges** . An edge $e \in \mathcal{E}$ connects two vertices i and j. Graphs can be **directed** or **undirected**. A directed graph (also known as digraph), is a graph $\mathcal{G}$ whose each edge has a sense of direction from i to j and is written as an ordered pair $e = (i, j)$. Such directed edges are also known as **arcs**. In an undirected graph, an edge has no sense of direction and is written as an unordered pair $\{i, j\}$.

A **path** in a graph $\mathcal{G}$ is a sequence of vertices $(i_0, i_1, i_2, \ldots, i_m)$ such that each pair of consecutive vertices in sequence (i_j, i_{j+1}) (or $\{i_j, i_{j+1}\}$), is an edge in $\mathcal{G}$ for each $0 \leq j \leq m$. The path is **simple** if no two vertices are identical. The path is a **cycle** if the initial and final vertex are the same, $i_0 = i_m$.

The **length** of a path is the number of successive edges it contains. A **connected graph** is a graph such that there exists a path between all pairs of vertices. A **strongly connected graph** is a directed graph such that there exists a path of finite length from each vertex to every other vertex. A **primitive graph** is a directed graph such that there exists a path of length m from each vertex to every other vertex, for some $m \in \mathbb{N}$.

There is a strong relation between graphs and matrices. Let $V_n = \{1, 2, \ldots, n\}$ and let $\mathcal{G} = (V_n, \mathcal{E})$ be a directed graph with n vertices ($|V_n| = n$). We associate an $n \times n$ binary matrix $A = (a_{ij})$ with

$$a_{ij} = \begin{cases} 1, & (i, j) \in \mathcal{E} \\ 0, & (i, j) \notin \mathcal{E} \end{cases}$$

to this graph, which is called the **adjacency matrix** of the graph.

Each matrix unit $E_{ij} \in M_n$ can be thought as an adjacency matrix of directed graph $(V_n, (i,j))$ that consists of the set V_n with n vertices and the single edge (i,j), $i, j \in V_n$. Let $\mathcal{E} \subseteq V_n \times V_n$ and $\mathcal{P}_1 = \{E_{ij} \,|\, (i,j) \in \mathcal{E}\}$. Note that the pair $(V_n, \mathcal{E})$ is a directed graph that describes the matrix units of $\mathcal{P}_1$. We denote $\mathcal{G}(\mathcal{P}_1) = (V_n, \mathcal{E})$. Having this setting, any matrix unit $E_{ij} \in \mathcal{P}_1$ corresponds to the arc (i,j) of the directed graph $\mathcal{G}(\mathcal{P}_1)$. Moreover, the adjacency matrix for $\mathcal{G}(\mathcal{P}_1)$ is given by $A = \sum_{(i,j)\in\mathcal{E}} E_{ij}$.

If $(i,j), (j,k) \in \mathcal{E}$, then there exists a path of length 2 from vertex i to k in $\mathcal{G}(\mathcal{P}_1)$. In other words, given E_{ij} and $E_{jk} \in \mathcal{P}_1$ with $(i,j), (j,k) \in \mathcal{E}$, we obtain a matrix unit of length 2, $E_{ik} = E_{ij}E_{jk}$. Hence, $E_{ik} \in \mathcal{P}_2$ if and only if there exists a path of length 2 in $\mathcal{G}(\mathcal{P}_1)$ from i to k. As a result, given $\mathcal{P}_1 = \{E_{ij} : (i,j) \in \mathcal{E}\}$ and its graph $\mathcal{G}(\mathcal{P}_1)$, one can easily verify that the set $\mathcal{P}_m$ of matrix units of length m is equal to

$$\mathcal{P}_m = \{E_{ij} \,|\, \text{ there exists a path of length m from i to j in } \mathcal{G}(\mathcal{P}_1)\,\}.$$

A matrix $A = (a_{ij}) \in M_n$ is called **nonnegative** if $a_{ij} \geq 0$ for all $1 \leq i, j \leq n$, and **entrywise-positive** if $a_{ij} > 0$ for all i, j. Given an arbitrary matrix $A = (a_{ij}) \in M_n$, the absolute matrix $|A| = (|a_{ij}|)$ is clearly nonnegative. A nonnegative matrix A is called **primitive** if and only if there exists some $m \in \mathbb{N}$ for which A^m is entrywise-positive. The least exponent m that makes A^m an entrywise-positive matrix, is called **the primitivity index of matrix A**, denoted by $\mathrm{prim}(A)$.

Proposition 3.2.1. *Let $V_n = \{1, 2, \ldots, n\}$ and $\mathcal{E} \subseteq V_n \times V_n$ for $n \in \mathbb{N}$ be a directed graph. If $\mathcal{P}_1 = \{E_{ij} \,\big|\, (i,j) \in \mathcal{E}\,\}$, $\mathcal{G}(\mathcal{P}_1) = (V_n, \mathcal{E})$ is its graph and $A = \sum_{(i,j)\in\mathcal{E}} E_{ij}$ is the adjacency matrix, then the following are equivalent:*

(1) $\mathrm{index}(\mathcal{P}_1) < +\infty$.

(2) $\mathcal{G}(\mathcal{P}_1)$ is a primitive graph.

(3) A^m is entrywise-positive, for some $m \in \mathbb{N}$.

Proof. Let $\mathcal{P}_1 = \{E_{ij} \,|(i,j) \in \mathcal{E}\}$ with $\mathcal{E} \subset V_n \times V_n$. Let $A = \sum_{(i,j)\in\mathcal{E}} E_{ij}$ be the adjacency matrix of $\mathcal{G}(\mathcal{P}_1)$. We know that

$$\mathcal{P}_m = \{E_{ij} \,|\, \exists \text{ a path of length m from i to j in } \mathcal{G}(\mathcal{P}_1)\,\}.$$

($1 \iff 2$) We have $\text{span}(\mathcal{P}_m) = M_n$, i.e. $\mathcal{P}_m$ contains all n^2 matrix units $E_{ij} \in M_n$, each of length m, if and only if there exists a path of length m for each pair of vertices in the directed graph of $\mathcal{P}_1$, if and only if the graph $\mathcal{G}(\mathcal{P}_1)$ is primitive.

($1 \Rightarrow 3$) Assume that $\text{span}(\mathcal{P}_m) = M_n$ for some $m \in \mathbb{N}$. It follows that $\mathcal{G}(\mathcal{P}_1)$ is a primitive graph. Hence, there is a path of finite length m for each pair of vertices. Each such path corresponds to a matrix unit of length m, $E_{ij}^{(m)} = \prod_{k=1}^m E_{(i,j)_k}$ with $(i,j) \in \mathcal{E}$. Then, we have

$$\begin{aligned} A^m &= \Big(\sum_{(i,j)\in\mathcal{E}} E_{ij}\Big)^m = \sum E_{ij}^{(m)} \\ &= \sum \Big(\prod_{k=1}^m E_{(i_k,j_k)}\Big). \end{aligned}$$

It follows that A^m contains all possible matrix units of length m as summand, at least once (since some products might give the same matrix unit of length m). Therefore, $(A^m)_{ij} > 0$ for all $i,j \in V_n$.

($3 \Leftarrow 1$) Conversely, assume that there exists $m \in \mathbb{N}$ such that $(A^m)_{ij} > 0$ for all $i,j \in V_n$. We know that A^m is the sum of all possible matrix units of length m of $\mathcal{P}_1$. Since each entry of A^m is greater than zero, it follows that $\mathcal{P}_m$ contains all n^2 matrix units of M_n, at least once. As a result, $\mathcal{P}_m$ spans M_n and $\text{index}(\mathcal{P}_1) = m$. □

Lemma 3.2.2. *Let $A = (a_{ij}) \in M_n$ and let $\mathcal{P}_1 = \{a_{ij}E_{ij} : a_{ij} \neq 0\}$. Then* $\text{index}(\mathcal{P}_1) < +\infty$ *if and only if $|A|$ is a primitive matrix. Moreover,* $\text{index}(\mathcal{P}_1) = \text{prim}(A)$.

Proof. Note that A is the sum of all elements of $\mathcal{P}_1$. One can easily check that $\operatorname{index}(\mathcal{P}_1) = m$ for some $m \in \mathbb{N}$ if and only if $\mathcal{G}(\mathcal{P}_1)$ is primitive, if and only if there exists paths of length m for each pair of vertices, if and only if $|A|^m$ is entrywise-positive as $|A|^m$ is a sum of matrices of length m, if and only if $|A|$ is primitive (see [11], [18] for more details). It is clear that $\operatorname{index}(\mathcal{P}_1) = \operatorname{prim}(A)$. □

This lemma shows that the primitivity of a nonnegative matrix $A \in M_n$ depends on the location of zero entries and not on the magnitude of non-zero entries. As one can observe, a primitive matrix A should have at least $(n+1)$ entries. It follows that $\mathcal{P}_1$ should contain at least $(n+1)$ matrix units E_{ij} or their multiples, which altogether add up to a primitive matrix.

Corollary 3.2.3. *Let A be a primitive matrix, and let m be its primitivity index. Then A^k is entrywise-positive for all $k \geq m$.*

Proof. Use Proposition 3.1.3 and Lemma 3.2.2. □

Example 3.2.4. *Let $W \in M_n$ be* **Wielandt's matrix***. Then we have*

$$W = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ 1 & 1 & 0 & \cdots & 0 \end{bmatrix} = E_{12} + E_{23} + \cdots + E_{(n-1)n} + E_{n1} + E_{n2}.$$

Let $\mathcal{P}_1 = \{E_{12},\ E_{23},\ \ldots,\ E_{(n-1)n},\ E_{n1},\ E_{n2}\}$ be the collection of matrix units of W. In [11], it is shown that the primitivity index of W is $(n-1)^2+1$. By Lemma 3.2.2, one can verify that spanning index of $\mathcal{P}_1$ is equal to primitivity index of the adjacency matrix of the graph of $\mathcal{P}_1$. As a result, $\operatorname{index}(\mathcal{P}_1) = (n-1)^2+1$.

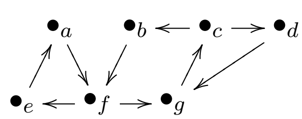
Remark 3.2.5. *It is already known that the primitivity index of the Wielandt matrix is the*

best bound for testing whether a matrix is primitive or not (see [11] for more details). Thus, for any collection $\mathcal{P}_1 \subseteq M_n$ of matrix units or their scalar multiples whose sum yields a primitive matrix, we have $\operatorname{index}(\mathcal{P}_1) \leq (n-1)^2 + 1$. *It follows that the Conjecture 3.1.1 is true for such sets $\mathcal{P}_1$.*
In the next section, we will prove this result in a different way using properties of their graphs.

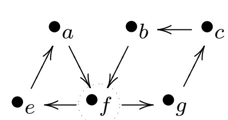
3.3 Primitive Graphs

Let $\mathcal{G} = (V_n, \mathcal{E})$ be a strongly connected graph on n vertices. Then there exists a path of finite length from each vertex $i \in V_n$ to every other vertex $j \in V_n$. When $i = j$, we get a cycle of finite length. Hence, $\mathcal{G}$ consists of cycles joined to each other by a vertex or a path of length l, $l \geq 1$, in common.

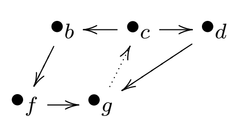
Example 3.3.1. *Let $\mathcal{G}$ be a directed graph on 7 vertices, as it is given below:*



One can easily check that there is a path of finite length from each vertex to every other one. Therefore, $\mathcal{G}$ is strongly connected.
This graph consists of three cycles: the first 2 cycles have only the vertex f in common and the last two cycles have the edge (g, c) in common (one edge is a path of length one).



and



Let $A \in M_n$ be a primitive matrix. By Proposition 3.2.1 and Lemma 3.2.2, the graph of A is primitive. Primitive graphs are very interesting to be studied. Firstly, a primitive graph on n vertices cannot be just a simple cycle. Graphs consisting of simple cycles are strongly connected, but not primitive. One can easily verify that the adjacency matrix of a simple cycle on n vertices is just a permutation matrix in M_n. Therefore, a primitive graph should contain at least two joined cycles.

Given a primitive graph $\mathcal{G}$ on n vertices and its adjacency matrix A, we let $\mathcal{P}_1 = \{E_{ij} : (i, j) \text{ is an edge in } \mathcal{G}\}$. By Proposition 3.2.1, we know that the spanning index of $\mathcal{P}_1$ exists. In Corollary 3.3.9, we will show that index$(\mathcal{P}_1)$ is bounded above by the Wielandt primitivity index $(n-1)^2 + 1$.

Lemma 3.3.2. *Let* $\gcd(p, q) = 1$. *If* $J = a_1 p + b_1 q = a_2 p + b_2 q$, *then there exists* $l \in \mathbb{Z}$ *such that* $a_2 = a_1 + lq$, $b_2 = b_1 - lp$.

Proof. We have $0 = (a_2 - a_1)p + (b_2 - b_1)q$. It follows that $q|(a_1 - a_2)p$. Since $\gcd(p, q) = 1$, then $q|(a_1 - a_2)$. Therefore, there exists $l \in \mathbb{Z}$ such that $a_2 - a_1 = lq$, i.e. $a_2 = a_1 + lq$ and $b_2 = b_1 - lp$. □

Theorem 3.3.3. *Let* $p, q > 0$ *be two integers such that* $\gcd(p, q) = 1$. *Then there exist nonnegative integers* a, b *such that* $ap + bq = J$, *for all* $J \geq (p-1)(q-1)$.

Proof. By algebra theory, we know that if $\gcd(p, q) = 1$, then there exists $\alpha, \beta \in \mathbb{Z}$ such that $\alpha p + \beta q = 1$. It is clear that one of these coefficients is positive and the other negative. Therefore, without loss of generality, assume $\alpha < 0$, $\beta > 0$. (There is a similar argument for the other option). Note that $(\alpha - nq)p + (\beta + np)q = 1$ for all $n \in \mathbb{Z}$, which shows that there are infinitely many integer pairs (α, β) satisfying $\alpha p + \beta q = 1$. Therefore, let α be the biggest possible negative integer, and let β be the smallest possible positive integer satisfying $\alpha p + \beta q = 1$. Clearly, $|\alpha| < q$, $\beta < p$. Denote $J_l = (p-1)(q-1) + l$ where $l \geq 0$.

One can easily verify that

$$\underbrace{[(l+1)\alpha - 1]}_{<0} p + \underbrace{[(l+1)\beta + p - 1]}_{>0} q = J_l, \text{ for all } l \in \mathbb{N}.$$

Claim: There exist integers $a, b \geq 0$ such that $ap + bq = J_l$ for all $l \geq 0$.

(i) We will show this claim by induction. For $l = 0$, let $a = \alpha - 1 + q$ and $b = \beta + p - 1 - p = \beta - 1$. One can check that $a > 0$, $b \geq 0$ and $ap + bq = J_0$. Hence, the result follows.

(ii) Assume that there exist $a, b \geq 0$ such that $ap + bq = J_l$, for $l \in \mathbb{N}$.

(iii) We will show that this is true for $l + 1$, too. We know $ap + bq = J_l$ for some $a, b \geq 0$ by induction hypothesis. Then we have

$$(a + \alpha)p + (b + \beta)q = (ap + bq) + (\alpha p + \beta q) = J_l + 1 = J_{l+1}.$$

If $a + \alpha \geq 0$, then we let $\tilde{a} = a + \alpha \geq 0$ and $\tilde{b} = b + \beta > 0$ and the result follows. If $a + \alpha < 0$, then $a + \alpha + q > 0$ since $|\alpha| < q$. One can easily check that $0 < a + \alpha + q < q$. Let $\tilde{a} = a + \alpha + q > 0$ and $\tilde{b} = b + \beta - p$. Then we have $\tilde{a}p + \tilde{b}q = J_{l+1}$. If $\tilde{b} \geq 0$, then the result follows:

$$\begin{aligned}
b + \beta - p &= \frac{J_l - ap}{q} + \beta - p \\
&= \frac{(p-1)(q-1) + l - ap + \beta q - pq}{q} \\
&= \frac{pq - p - q + 1 + l - ap + 1 - \alpha p - pq}{q} \\
&= \frac{l + 2 - p - q - p(a + \alpha + q)}{q} \\
&> \frac{l + 2 - p - q + pq}{q} = \frac{l + 1 + (p-1)(q-1)}{q} \\
&= \frac{J_{l+1}}{q} > 0.
\end{aligned}$$

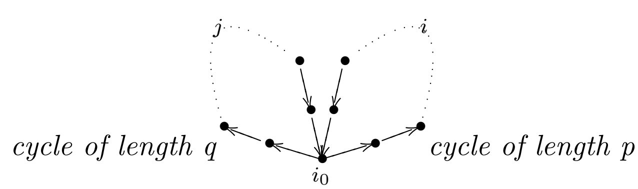
Hence, there exists integers $a, b \geq 0$ such that $ap + bq = J_l$ for all $l \in \mathbb{N}$. $\square$

Remark 3.3.4. *The equation* $ap + bq = J - 1 = (p-1)(q-1) - 1$ *has no solution in positive integers! To see this, on the contrary, suppose there exist nonnegative integers* a, b *such that* $ap + bq = pq - p - q$*. It follows that* $pq = (a+1)p + (b+1)q$*, which implies* $q|(a+1)$ *and* $p|(b+1)$ *since* $\gcd(p,q) = 1$*. Hence,* $a + 1 > q$ *and* $b + 1 > p$*. Then* $pq = (a+1)p + (b+1)q > pq + pq = 2pq$ *a contradiction.*

Proposition 3.3.5. *Let* $V_n = \{1, 2, \ldots, n\}$ *be a set of* n *vertices and let* $\mathcal{E} \subseteq V_n \times V_n$ *be a collection of edges such that the directed graph* $\mathcal{G} = (V_n, \mathcal{E})$ *consists of two cycles, each of length* p *and* q*, that are joined by a common path of length* $L \geq 0$*. Then* $\mathcal{G}$ *is primitive if and only if* $\gcd(p,q) = 1$*.*

Proof. Let $\mathcal{G}$ be a directed graph on n vertices, with 2 cycles, each of length p and q, that are joined by a common path of length $L \geq 0$. Note that $n = p + q - L - 1$. Assume that $\mathcal{G}$ is primitive. Then, for each pair of vertices $(i,j) \in V_n \times V_n$, there exists a path of length m, for some $m \in \mathbb{N}$.

Case 1: Assume that the cycles of the primitive graph $\mathcal{G}$ are joined by a common path of length $L = 0$, i.e. by a vertex. Then such a graph will look like the following graph:



For every pair of vertices $(i,j) \in V_n \times V_n$, there exists a path of length $m = \text{index}(\mathcal{P}_1)$. To get a path of a given length from i to j, one has to go through the cycles as many times as needed.

(i) If $i = j = i_0$ (vertex in common), then to get a path of length m from i to itself, there exist $\alpha \geq 0, \beta \geq 0$ such that

$$\alpha p + \beta q = m \text{ with } 1 \leq p, q \leq n.$$

One can check that even when $i = j \neq i_0$, then again there exist nonnegative integers α, β such that $\alpha p + \beta q = m$.

(ii) If $i = i_0$ and $j \neq i_0$ is in one of the cycles, then to get a path of length m from i to j there exists $\alpha \geq 0$, $\beta \geq 0$ such that

$$\alpha p + \beta q + j = m \text{ with } 1 \leq j \leq \max\{p-1, q-1\}.$$

(iii) If $i \neq i_0$ is in the cycle of length p and $j = i_0$, then to get a path of length m from i to j there exists $\alpha \geq 0$, $\beta \geq 0$ such that

$$(p-i) + \alpha p + \beta q = m \text{ with } 1 \leq p - i \leq p - 1.$$

Similarly, if $i \neq i_0$ is in the cycle of length q, one would have

$$(q-i) + \alpha p + \beta q = m \text{ with } 1 \leq q - i \leq q - 1.$$

(iv) If $i \neq i_0$, $j \neq i_0$ and $i \neq j$, then there are two possible cases:

(1) If i and j are in different cycles, say i is in the cycle of length p and j in the other one, then to get a path of length m from i to j there exist $\alpha \geq 0, \beta \geq 0$ such that

$$(p-i) + \alpha p + \beta q + j = m \text{ with } 2 \leq p - i + j \leq p - 1 + q - 1.$$

One can argue similarly when i and j reverse the roles.

(2) If both i and j are in the same cycle, say in the cycle of length p, and are positioned as $i_0 \to i \to j \to i_0$, then to get a path of length m from i to j there exist $\alpha \geq 0, \beta \geq 0$ such that

$$(j-i) + \alpha p + \beta q = m \text{ with } 1 \leq j-i \leq p-1.$$

If they are positioned as $i_0 \to j \to i \to i_0$, then one would have

$$(p-i+j) + \alpha p + \beta q = m \text{ with } 1 \leq p-i+j \leq p-1.$$

One can argue similarly when both i and j are in the other cycle.

To generalize, for a path of length m from i to j, there exists $\alpha \geq 0, \beta \geq 0$ such that

$$\alpha p + \beta q + k = m \text{ with } 0 \leq k \leq n.$$

Rewrite this result as $a_l p + b_l q + k = m$, for some nonnegative integers a_l, b_l and $0 \leq k \leq n$. Then we have

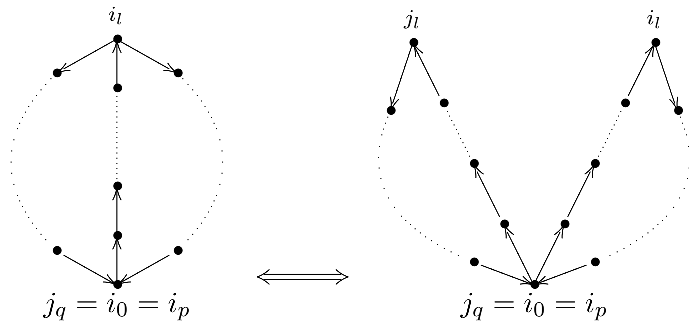
$$(*) \quad a_l p + b_l q = m - k = K \text{ with } m - n \leq K \leq m.$$

Since $\mathcal{G}$ is primitive, then there exist a path of length $m+1, m+2, \ldots$ from i to j, for all $i, j \in V_n$. Hence, we have

$$\begin{aligned} a_l p + b_l q &= K \\ \underline{a_{l+1} p + b_{l+1} q} &\underline{= K + 1} \\ (a_{l+1} - a_l)p + (b_{l+1} - b_l)q &= 1. \end{aligned}$$

It follows that $\gcd(p, q) = 1$.

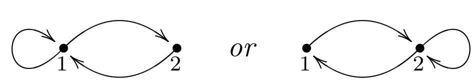
Case 2: Assume that the cycles of the primitive graph $\mathcal{G}$ are joined by a common path of length $L > 0$. Then such a graph can be considered as a graph of two cycles with a common vertex as the figure shows:



Hence, one can straightforwardly verify that $\gcd(p, q) = 1$ by using similar arguments as in Case 1. One can easily verify that the converse is true by reversing the arguments of the proof. $\square$

Remark 3.3.6. (i) *Let $\mathcal{G}$ be a directed graph on n vertices consisting of 2 cycles, each of length p and q, that are joint by a common path of length $L \geq 0$. Note that such a graph $\mathcal{G}$ contains $p + q - L = n + 1$ edges.*
Conversely, if $\mathcal{G}$ is a primitive graph that contains $(n+1)$ edges, then $\mathcal{G}$ contains exactly two cycles described as above: Being a primitive graph, $\mathcal{G}$ should have at least two cycles, and being a graph with only $(n+1)$ edges, $\mathcal{G}$ cannot have more than two cycles.
This result shows that primitive graphs on n vertices that contain only $(n+1)$ edges are the simplest form of primitive graphs. The more complicated the primitive graphs are, the more edges they contain, the shorter the length of paths becomes for every pair of vertices.

(ii) *A primitive graph on 2 vertices is just the trivial one, i.e.*



there exists a path of length 2 for every pair of vertices. Hence, when speaking of primitive *graphs on n vertices, we will always assume that $n \geq 3$.*

Theorem 3.3.7. *Let $V_n = \{1, 2, \ldots, n\}$, $\mathcal{E} \subseteq V_n \times V_n$ for $n \in \mathbb{N}$, and let $\mathcal{P}_1 = \{E_{ij} \,\big|\, (i,j) \in \mathcal{E}\,\}$ be a collection of some matrix units E_{ij}. If $\mathcal{G}(\mathcal{P}_1) = (V_n, \mathcal{E})$ is a primitive graph consisting of 2 joined cycles, each of length p and q, then*

$$\operatorname{index}(\mathcal{P}_1) = n + (p-1)(q-1).$$

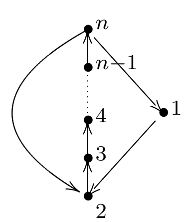
Proof. Let $\mathcal{P}_1 = \{E_{ij} \,\big|\, (i,j) \in \mathcal{E}\,\}$ be a collection of matrix units E_{ij} such that $\mathcal{G}(\mathcal{P}_1)$ is a primitive graph consisting of 2 cycles, each of length p and q. By Proposition 3.2.1, we know that $\operatorname{index}(\mathcal{P}_1)$ exists, since its graph is primitive. Assume $\operatorname{index}(\mathcal{P}_1) = m$ for some $m \in \mathbb{N}$, i.e. there exists a path of (least) length m for every pair of vertices $(i,j) \in V_n \times V_n$. By Proposition 3.3.5, we have $\gcd(p,q) = 1$. Recall the result $(*)$ from the proof of Proposition 3.3.5: There exist nonnegative integers a, b such that

$$ap + bq = K \text{ with } m - n \leq K \leq m,$$

where $m = \operatorname{index}(\mathcal{P}_1)$. Since $\gcd(p,q) = 1$, then by Theorem 3.3.3 $ap + bq = K$ has solution in nonnegative integers if and only if $K \geq (p-1)(q-1)$. It follows that $\operatorname{index}(\mathcal{P}_1) = n + (p-1)(q-1)$. $\square$

Corollary 3.3.8. *Let $\mathcal{P}_1 = \{E_{12}, E_{23}, \ldots, E_{(n-1)n}, E_{n1}, E_{n2}\}$ be the collection of matrix units of the Wielandt matrix. Then $\operatorname{index}(\mathcal{P}_1) = (n-1)^2 + 1$.*

Proof. One can easily realize that the graph of $\mathcal{P}_1$ consists of two cycles of length n and $(n-1)$, that are joined by a path of length $(n-2)$.



Since $\gcd(n, n-1) = 1$, then the graph of $\mathcal{P}_1$ is primitive. It follows that the spanning index of $\mathcal{P}_1$ exists. By Theorem 3.3.7 we have

$$\text{index}(\mathcal{P}_1) = n + (n-1)(n-2) = (n-1)^2 + 1.$$

□

Corollary 3.3.9. *Let $\mathcal{P}_1$ be a collection of $(n+1)$ matrix units such that its directed graph $\mathcal{G}(\mathcal{P}_1)$ is primitive. Then* $\text{index}(\mathcal{P}_1) \leq (n-1)^2 + 1$.

Proof. Since $\mathcal{G}(\mathcal{P}_1)$ is a primitive graph with $(n+1)$ edges, then by Proposition 3.3.5 and Remark 3.3.6, we know that $\mathcal{G}(\mathcal{P}_1)$ contains two cycles of length p and q joined by a path of length $l \geq 0$, with $\gcd(p, q) = 1$. Hence, $\text{index}(\mathcal{P}_1) = n + (p-1)(q-1)$. Using calculus and $n = p + q - L - 1$, one can easily show that $n + (p-1)(q-1) \leq (n-1)^2 + 1$ for all $n > 2$. □

This corollary shows that the Conjecture 3.1.1 is true for these kinds of sets P_1, whose graphs are primitive. The more entries $\mathcal{P}_1$ contains, the smaller the spanning index we obtain. For such $\mathcal{P}_1$, the real problem stands when we divide these matrix units into 2 groups, add them up into 2 matrices, and form $\tilde{\mathcal{P}}_1$, we get $\text{index}(\mathcal{P}_1) \leq \text{index}(\tilde{\mathcal{P}}_1)$ by Proposition 3.1.6. We don't know yet for sure, whether this spanning index of $\tilde{\mathcal{P}}_1$ is less than n^2. We conjecture this is true. In the following section, we will illustrate this idea with some examples.

3.4 Wielandt Matrices' Decomposition and Jordan Canonical Form of Matrices

Let $W = E_{12} + E_{23} + \cdots + E_{(n-1)n} + E_{n1} + E_{n2} \in M_n$ be the Wielandt matrix. We know that W is a primitive matrix and W^m is entrywise-positive for all $m \geq (n-1)^2 + 1$.

In Corollary 3.3.8, we have shown that if $\mathcal{P}_1$ is the collection of Wielandt matrix units separately, then the spanning index of $\mathcal{P}_1$ is $(n-1)^2 + 1$.

We will rewrite $\mathcal{P}_1$ as a collection of two matrices, say $\mathcal{Q}_1 = \{A, B\}$, with $A = E_{12} + E_{23} + \cdots + E_{(n-1)n}$ and $B = E_{n1} + E_{n2}$ such that $W = A + B$. Note that $A^n = 0$ and $B^2 = 0$. The computations shown in the following table, will be used to construct the matrices of length m of $\mathcal{Q}_1$, $m \in \mathbb{N}$.

Matrices of length $(\cdot)$ using A and B	
$A = E_{12} + E_{23} + \cdots + E_{(n-1)n}$	$B = E_{n1} + E_{n2}$
$A^2 = E_{13} + E_{24} + \cdots + E_{(n-2)n}$	$BA = E_{n2} + E_{n3}$
$A^3 = E_{14} + E_{25} + \cdots + E_{(n-3)n}$	$BA^2 = E_{n3} + E_{n4}$
$\vdots$	$\vdots$
$A^{n-2} = E_{1(n-1)} + E_{2n}$	$BA^{n-2} = E_{n(n-1)} + E_{nn}$
$A^{n-1} = E_{1n}$	$BA^{n-1} = E_{nn}$
$A^n = 0$	$BA^n = 0$
$A = E_{12} + E_{23} + \cdots + E_{(n-1)n}$	$BA^iB = 0,\ 0 \leq i \leq n-3$
$AB = E_{(n-1)1} + E_{(n-1)2}$	$BA^{n-2}B = BA^{n-1}B = B$
$A^2B = E_{(n-2)1} + E_{(n-2)2}$	$BA^nB = 0$
$\vdots$	$(BA^{n-2})^i = BA^{n-2},\ i \in \mathbb{N}$
$A^{n-2}B = E_{21} + E_{22}$	$(BA^{n-2})(BA^j) = BA^j,\ j \in \mathbb{N}$
$A^{n-1}B = E_{11} + E_{12}$	$(BA^{n-1})^k = BA^{n-1},\ k \in \mathbb{N}$
$A^nB = 0.$	$(BA^{n-1})(BA^l) = BA^l,\ l \in \mathbb{N}.$

Note that the matrices BA^i, $0 \leq i \leq n-1$, are matrix units or sum of matrix units that span the n^{th} row of any matrix in M_n. One can easily verify that

$$\mathrm{span}\{BA^i : 0 \leq i \leq n-1\} = \mathrm{span}\{E_{nj} : 1 \leq j \leq n\}.$$

Both A and B are sums of matrix units. For a fixed $m \in \mathbb{N}$, the product of m such matrices can be a matrix unit or a sum of matrix units as the table shows. Given $\mathcal{Q}_1 = \{A, B\}$, we compute $\mathcal{Q}_2 = \{A^2, AB, BA\}$, $\mathcal{Q}_3 = \{A^3, A^2B, ABA, BA^2\}$ and in general

$$\mathcal{Q}_m = \left\{ X^{(m)} = \prod_{i=1}^{m} X_i : X_i = A,\, B \right\}.$$

One can notice that the elements of $\mathcal{Q}_m$ are sums of matrix units. There exists $m \in \mathbb{N}$ such that at least one element $X^{(m)} \in \mathcal{Q}_m$ is a matrix unit. One can inductively show that $\mathcal{Q}_m$ contains the first matrix unit at $m = n-1$, i.e. $A^{n-1} = E_{1n} \in \mathcal{Q}_{n-1}$.

In fact, $\mathcal{Q}_{n-1}$ is the first set that contains n matrices with just one nonzero row. For each $1 \leq i \leq n$, there is just one matrix in $\mathcal{Q}_{n-1}$ whose i^{th} row is nonzero. Thus, by using the table information, we have:

$$\mathcal{Q}_{n-1} = \{A^{n-1}, A^{n-2}B, A^{n-3}BA, \ldots, ABA^{n-3}, BA^{n-2}\}.$$

It is easy to check that $\mathcal{Q}_n = A \cdot \mathcal{Q}_{n-1} \cup B \cdot \mathcal{Q}_{n-1}$ will contain $n+1$ matrices with just one nonzero row. For each $1 \leq i \leq n-1$, there is just one matrix in $\mathcal{Q}_n$ whose i^{th} row is nonzero, and there are 2 matrices whose n^{th} row is nonzero, specifically $E_{nn} = BA^{n-1}$ and $B = BA^{n-2}B = E_{n1} + E_{n2}$.

Observe that, at the initial step $\mathcal{Q}_1 = \{A, B\}$ contains only one element which consists of matrix units that lie the n^{th} row, i.e.

$$B = E_{n1} + E_{n2} \in \mathcal{Q}_1.$$

Then, at the n^{th} place after $(n-1)$ steps, besides the other elements, $\mathcal{Q}_{(n-1)+1}$ contains 2 matrices which consist of matrix units that lie in the n^{th} row, i.e. we have

$$BA^{n-1} = E_{nn} \text{ and } B = (BA^{n-2})B = E_{n1} + E_{n2} \in \mathcal{Q}_{(n-1)+1}.$$

Using the table information, one can obviously see that $BA^{n-1} = E_{nn}$ and $B = (BA^{n-2})B = E_{n1} + E_{n2}$ will be repeated again after $(n-1)$ more steps, since

$$BA^{n-1} = \underbrace{(BA^{n-2})}_{(n-1)}(BA^{n-1}) = E_{nn} \text{ and } B = \underbrace{(BA^{n-2})}_{(n-1)}(BA^{n-2})B = E_{n1} + E_{n2}.$$

Moreover, at this step we obtain a third matrix which consists of matrix units that lie in the n^{th} row,

$$BA^{n-2} = (BA^{n-1})\underbrace{(BA^{n-2})}_{(n-1)} = E_{n(n-1)} + E_{nn} \in \mathcal{Q}_{2(n-1)+1}.$$

Thus, one can check that $\mathcal{Q}_{2(n-1)+1}$ contains $2n+1$ matrices with just one nonzero row. For each $1 \leq i \leq n-1$ there are two matrices in $\mathcal{Q}_{2(n-1)+1}$ whose i^{th} row is nonzero, and there are three matrices which consist of matrix units that lie in the n^{th} row.

As a result, after each group of $(n-1)$ steps, we obtain one more matrix with just one nonzero row, for each row. It follows that after $(n-1)^2$ steps, $\mathcal{Q}_{(n-1)^2+1}$ will contain $(n-1)$ matrices whose i^{th} row is nonzero for all $1 \leq i \leq n-1$ and $\mathcal{Q}_{(n-1)^2+1}$ will contain n matrices which consist of matrix units that lie in the n^{th} row. More specifically, $\mathcal{Q}_{(n-1)^2+1}$ will contain all BA^i, $0 \leq i \leq n-1$, since each matrix BA^i can be written as a matrix of length $(n-1)^2+1$ using initial matrices A and B. We have

$$BA^i = \begin{cases} (BA^{n-1})^{n-1-i}(BA^{n-2})^{i-1}BA^i, & 1 \leq i \leq n-1 \\ (BA^{n-2})^{n-1}B, & i = 0 \end{cases}$$

where each matrix BA^i is of length

$$n(n-1-i)+(n-1)(i-1)+(i+1) = (n-1)^2+1 \text{ for all } 1 \leq i \leq n-1.$$

The case $i=0$ is clear. We know that $\mathrm{span}\{BA^i : 0 \leq i \leq n-1\} = \mathrm{span}\{E_{nj} : 1 \leq j \leq n\}$, therefore we can equivalently say that $\mathcal{Q}_{(n-1)^2+1}$ contains all matrix units of n^{th} row. This is the first place, where all these matrix units happen to occur at the same time.

Theorem 3.4.1. *Let* $A = E_{12}+E_{23}+\cdots+E_{(n-1)n}$, $B = E_{n1}+E_{n2}$ *in* M_n *and let* $\mathcal{Q}_1 = \{A, B\}$. *Then* $\mathrm{index}(\mathcal{Q}_1) = (n-1)^2+n$.

Proof. Firstly, we know that besides the other elements, $\mathcal{Q}_{(n-1)^2+1}$ contains all n matrix units of the n^{th} row, i.e. $E_{n1}, E_{n2}, \ldots, E_{n(n-1)}, E_{nn} \in \mathcal{Q}_{(n-1)^2+1}$. Next, we compute

$$\mathcal{Q}_{(n-1)^2+2} = \mathcal{Q}_1 \cdot \mathcal{Q}_{(n-1)^2+1} = \mathcal{Q}_{(n-1)^2+1} \cdot \mathcal{Q}_1.$$

Besides the other elements, $\mathcal{Q}_{(n-1)^2+2}$ contains all matrix units that span the $(n-1)^{th}$ row,

$$A \cdot \{E_{n1}, E_{n2}, \ldots, E_{n(n-1)}, E_{nn}\} = \{E_{(n-1)1}, E_{(n-1)2}, \ldots, E_{(n-1)(n-1)}, E_{(n-1)n}\},$$

and all matrix units that span the n^{th} row,

$$\{E_{n1}, E_{n2}, \ldots, E_{n(n-1)}, E_{nn}\} \cdot \mathcal{Q}_1 = \{E_{n1}, E_{n2}, \ldots, E_{n(n-1)}, E_{nn}\}.$$

Hence, $\mathcal{Q}_{(n-1)^2+2}$ contains all the matrix units that span the last 2 rows of any matrix in

M_n. Note that there might be other matrices of length $(n-1)^2+2$ that yield these results. In a similar way, we have

$$\mathcal{Q}_{(n-1)^2+3} = \mathcal{Q}_1 \cdot \mathcal{Q}_{(n-1)^2+2} \cup \mathcal{Q}_{(n-1)^2+2} \cdot \mathcal{Q}_1.$$

Besides other elements, $\mathcal{Q}_{(n-1)^2+3}$ contains all matrix units that span the $(n-2)^{th}$ row,

$$\begin{aligned} A \cdot \{&E_{(n-1)1}, E_{(n-1)2}, \ldots, E_{(n-1)(n-1)}, E_{(n-1)n}\} \\ &= \{E_{(n-2)1}, E_{(n-2)2}, \ldots, E_{(n-2)(n-1)}, E_{(n-2)n}\}, \end{aligned}$$

all matrix units that span the $(n-1)^{th}$ row,

$$\begin{aligned} \{&E_{(n-1)1}, E_{(n-1)2}, \ldots, E_{(n-1)(n-1)}, E_{(n-1)n}\} \cdot \mathcal{Q}_1 \\ &= \{E_{(n-1)1}, E_{(n-1)2}, \ldots, E_{(n-1)(n-1)}, E_{(n-1)n}\}, \end{aligned}$$

and all matrix units that span the n^{th} row,

$$\{E_{n1}, E_{n2}, \ldots, E_{n(n-1)}, E_{nn}\} \cdot \mathcal{Q}_1 = \{E_{n1}, E_{n2}, \ldots, E_{n(n-1)}, E_{nn}\}.$$

Hence, $\mathcal{Q}_{(n-1)^2+3}$ contains all matrix units that span the last 3 rows of any matrix in M_n. Note that we obtain all the matrix units of a new row in every step. Therefore, after $(n-1)$ steps, $\mathcal{Q}_{(n-1)^2+1+(n-1)} = \mathcal{Q}_{(n-1)^2+n}$ will contain all the matrix units E_{ij}, $1 \le i, j \le n$. As a result, we conclude that $\text{index}(\mathcal{Q}_1) = (n-1)^2 + n$. □

Remark 3.4.2. (1) *If $\mathcal{P}_1$ is the collection of Wielandt matrix units, then* $\text{index}(\mathcal{P}_1) = (n-1)^2+1$ *as we have shown in Corollary 3.3.8. If we divide these matrix units into two groups and form $\mathcal{Q}_1 = \{A, B\}$ with $A = E_{12} + E_{23} + \cdots + E_{(n-1)n}$ and $B = E_{n1} + E_{n2}$, then* $\text{index}(\mathcal{Q}_1) = (n-1)^2 + n = \text{index}(\mathcal{P}_1) + (n-1)$ *as shown in the theorem above. In fact,* *we claim that any set* $\mathcal{Q}_1 = \{A, B\}$ *of two elements such that* $A + B = W$ *is the Wielandt matrix, has a finite spanning index which is bounded by*

$$(n-1)^2 + 1 \leq \text{index}(\mathcal{Q}_1) \leq (n-1)^2 + n.$$

The first inequality $(n-1)^2 + 1 (= \text{index}(\mathcal{P}_1)) \leq \text{index}(\mathcal{Q}_1)$ *is obvious by Proposition 3.1.6 and Corollary 3.3.8. The second inequality* $\text{index}(\mathcal{Q}_1) \leq \text{index}(\mathcal{P}_1) + (n-1) = (n-1)^2 + n$ *depends on the grouping of Wielandt matrix units. The grouping we discussed in Theorem 3.4.1 is the one that appears to give the biggest possible spanning index among all sets of two elements whose sum yields the Wielandt matrix, because the matrix* A *vanishes at the* n^{th} *step which makes the process of obtaining matrix units slower, and* B *associated with any* A *gives mostly the sum of matrix units. On the other hand, any other choice of matrices we can have will give matrix units at a lower level most of the time, which will likely lead to a smaller spanning index.*

(2) *If* $\mathcal{P}_1$ *is a collection of* $(n+1)$ *matrix units whose graph is primitive and* $A_{\mathcal{P}_1}$ *is its adjacency matrix, then* $\text{index}(\mathcal{P}_1) \leq (n-1)^2 + 1$ *by Corollary 3.3.9. By following the same arguments as done in the case of the Wielandt matrix units, we conclude that the spanning index of any set* $\mathcal{Q}_1 = \{A, B\}$ *of any two elements whose sum gives the primitive matrix* $A_{\mathcal{P}_1}$ *is bounded by* $\text{index}(\mathcal{P}_1) \leq \text{index}(\mathcal{Q}_1) \leq \text{index}(\mathcal{P}_1) + (n-1)$.

(3) *Note that when a set* $\mathcal{P}_1$ *of a finite spanning index, contains only* $(n+1)$ *matrix units, then any grouping of the given matrix units will yield a finite spanning index of the new formed set, as shown in the above facts* (1) *and* (2).
But this is not true for sets $\mathcal{P}_1$ *of finite spanning index that contain more than* $(n+1)$ *matrix units* (*look at Remark 3.1.7 for a counterexample*).

To summarize, a set $\mathcal{P}_1$ should contain at least $(n+1)$ matrix units and their sum should add up to a primitive matrix $A_{\mathcal{P}_1}$. The more matrix units are involved, the smaller is the primitivity index of their sum matrix. Depending on the grouping of matrix units, the spanning index of $\mathcal{Q}_1 = \{A, B\}$ with $A + B = A_{\mathcal{P}_1}$, if exists, is bounded as

$$\mathrm{index}(\mathcal{P}_1) \leq \mathrm{index}(\mathcal{Q}_1) \leq \mathrm{index}(\mathcal{P}_1) + n.$$

Another example we want to consider, is Jordan Canonical Form of Matrices. Let $\mathcal{P}_1$ be a set of 2 matrices (worst case), where for simplicity we'll use the Jordan canonical form of one of them, and the other being transformed appropriately. Still, to make easier, let consider the trivial Jordan form of matrix, i.e. a diagonal matrix, and the cyclic forward shift matrix. So, let $\mathcal{P}_1 = \{D_1, S\}$, where $D_1 = \begin{bmatrix} d_1 & & & 0 \\ & d_2 & & \\ & & \ddots & \\ 0 & & & d_n \end{bmatrix} \neq 0$, and $S = \begin{bmatrix} 0 & 0 & \cdots & 1 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 1 & 0 \end{bmatrix}$.

Note that $S^n = I_n$, the identity matrix in M_n. One can easily verify the following facts:

(i) The matrices of length m in $\mathcal{P}_m$ containing n times the cyclic forward shift matrix S (or multiple of n times), will be always diagonal matrices.

(ii) In a similar way, the matrices of length m in $\mathcal{P}_m$ containing i times the shift matrix S, where $1 \leq i < n$, (or the number of the repeated shift matrix equivalent to i in modulo n), will be matrices of the form S^i.

We begin with one diagonal matrix D_1 and one shift matrix S in $\mathcal{P}_1$. Note that $\mathcal{P}_n$ is the first set to contain 2 diagonal matrices, D_1^n and $S^n = I_n$. Therefore, the first diagonal matrices, different from the powers of D_1 and the identity I_n, happen at $(n+1)^{th}$ place. It goes as follows:

$D_1 S^n = S^n D_1 = \mathrm{diag}(d_1, d_2, \ldots, d_n) = D_1$

$SD_1S^{n-1} = \text{diag}(d_n, d_1, d_2, \ldots, d_{n-1}) = D_2$

$S^2D_1S^{n-2} = \text{diag}(d_{n-1}, d_n, d_1, d_2, \ldots, d_{n-3}) = D_3$

$\vdots$

$S^{n-2}D_1S^2 = \text{diag}(d_3, d_4, \ldots, d_n, d_1, d_2) = D_n$

$S^{n-1}D_1S = \text{diag}(d_2, d_3, d_4, \ldots, d_n, d_1) = D_n.$

Case I:

If the initial diagonal matrix D_1 is given by

$$D_1 = \text{diag}(1, \omega, \omega^2, \ldots, \omega^{n-1}) = \begin{bmatrix} 1 & & & 0 \\ & \omega & & \\ & & \ddots & \\ 0 & & & \omega^{n-1} \end{bmatrix},$$

where ω is the primitive n^{th} root of unity with $\omega^n = 1$, then all the above matrices of length $(n+1)$ would be equivalent in the following sense:

$$D_1 = \omega D_2 = \cdots = \omega^{n-1} D_n.$$

Hence, $\mathcal{P}_{n+1}$ contains only one diagonal matrix $D_1 = D_1^{n+1}$. As a result, we conclude that $\mathcal{P}_m$ with $m \in \mathbb{N}$ will always contain only one diagonal matrix, specifically

$$D_1^m = \begin{cases} D_1^i, & m \equiv i \mod n \\ I_n, & m \equiv 0 \mod n \end{cases}.$$

This shows that such a choice of matrices can not lead to a spanning of all diagonal matrices. There is a similar situation for the matrices of form S^i. It follows that the spanning index of $\mathcal{P}_1$ does not exist in this case.

Even if we begin with a nonzero scalar multiple of such a diagonal matrix, say $\tilde{D}_1 = \alpha D_1$ and $\alpha \neq \omega$, then one can calculate that $\tilde{D}_1^m = \begin{cases} \alpha^m D_1^i, & m \equiv i \mod n \\ \alpha^m I_n, & m \equiv 0 \mod n \end{cases}$.

It follows that $\mathcal{P}_m$ will always contain only one diagonal matrix, specifically a power of $\tilde{D}_1$ or the identity matrix. As a result, no spanning can occur.

Case II:

If all diagonal entries of $\mathcal{D}_1$ are the same, i.e. $D_1 = aI_n$ is a multiple of the identity matrix, then $\mathcal{P}_m$ will always contain only one diagonal matrix, basically a multiple of identity. Similarly, one can argument for matrices of the form S^i, $1 \leq i \leq n$. It follows that no spanning can occur under these conditions.

Case III:

We will assume that the initial diagonal matrix D_1 contains at least 2 different diagonal entries. Then $\mathcal{P}_{n+1}$ will contain n different diagonal matrices, which span the set of diagonal matrices. Similarly, one can show that $\mathcal{P}_{n+1}$ contains n different matrices of the form S^i, which span the set of the matrices of the form S^i for each i.

Given $A = (a_{ij}) \in M_n$, one can we write A as $A = D_A + S_A^1 + S_A^2 + \cdots + S_A^{n-1}$, where

$$D_A = \begin{bmatrix} a_{11} & & & 0 \\ & a_{22} & & \\ & & \ddots & \\ 0 & & & a_{nn} \end{bmatrix}, \quad S_A^1 = \begin{bmatrix} 0 & 0 & \cdots & a_{1n} \\ a_{21} & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & a_{n(n-1)} & 0 \end{bmatrix},$$

$$S_A^2 = \begin{bmatrix} 0 & \cdots & \cdots & a_{1(n-1)} & 0 \\ 0 & \ddots & & & a_{2n} \\ a_{31} & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & a_{n(n-2)} & 0 & 0 \end{bmatrix}, \ \ldots, \ S_A^{n-1} = \begin{bmatrix} 0 & a_{12} & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & a_{(n-1)n} \\ a_{n1} & \cdots & 0 & 0 \end{bmatrix}.$$

It follows that $\text{span}(\mathcal{P}_{n+1}) = M_n$.

3.5 Some Results and Applications

Let $\mathcal{P}_1 = \{A_1, A_2, \ldots, A_l\}$ be a finite set of matrices in M_n. Recall the matrix-vector correspondence shown in Section 2.2.3: if $X \in M_n$ is a scalar matrix written in terms of its columns as $X = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}$ with $x_i \in \mathbb{C}^n$, then the vector $\text{vec}(X) = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{C}^n \otimes \mathbb{C}^n$ is called the vectorization of X. Hence, every matrix $A_i \in \mathcal{P}_1$ corresponds to its vectorization $\text{vec}(A_i) \in \mathbb{C}^{n^2}$. We set

$$V_{\mathcal{P}_1} = \begin{bmatrix} \text{vec}(A_1) & \text{vec}(A_2) & \cdots & \text{vec}(A_l) \end{bmatrix} \in M_{n^2,l}$$

and call $V_{\mathcal{P}_1}$ **the vectorization matrix** of $\mathcal{P}_1$.

Proposition 3.5.1. *Let $\mathcal{P}_1 = \{A_1, A_2, \ldots, A_l\} \subseteq M_n$ be a finite set, $\mathcal{P}_m$ be the set of matrices of length m of $\mathcal{P}_1$, $\mathcal{V}_m = \text{span}(\mathcal{P}_m)$ and $V_{\mathcal{P}_m}$ be the vectorization matrix of $\mathcal{P}_m$ for any $m \in \mathbb{N}$. Then* $\dim(\mathcal{V}_m) = \text{rank}(V_{\mathcal{P}_m})$.

Proof. Assume $\dim(\mathcal{V}_m) = k$ with $k \leq n^2$. It follows that there exists k linearly independent matrices of length m of $\mathcal{P}_1$, say $A_e^{(m)} \in \mathcal{P}_m$ with $1 \leq e \leq k$, whose linear spanning yields

$\mathcal{V}_m$. One can easily verify that the linearly independence of these matrices $A_e^{(m)}$'s with $1 \leq e \leq k$ implies the linearly independence of their vectorizations $\mathrm{vec}(A_e^{(m)})$'s and vice-versa. Therefore the rank of the vectorization matrix $V_{\mathcal{P}_m}$ is equal to the number of linearly independent columns of itself. As a result, we conclude $\dim(\mathcal{V}_m) = \mathrm{rank}(V_{\mathcal{P}_m})$. $\square$

Corollary 3.5.2. *Let $\mathcal{P}_1 = \{A_1, A_2, \ldots, A_l\} \subseteq M_n$ be a finite set, $\mathcal{P}_m$ be the set of matrices of length m of $\mathcal{P}_1$, $\mathcal{V}_m = \mathrm{span}(\mathcal{P}_m)$ and $V_{\mathcal{P}_m}$ be the vectorization matrix of $\mathcal{P}_m$ for some $m \in \mathbb{N}$. Then $\mathcal{V}_m = M_n$ if and only if $\mathrm{rank}(V_{\mathcal{P}_m}) = n^2$.*

Let $\phi : M_n \to M_n$ be a linear map. This map can be represented by its Choi matrix $C_\phi = (\phi(E_{ij}))$, where $\{E_{ij}\}$ are the canonical matrix units of M_n (see Subsection 1.4.2). Recall that the map ϕ is called completely positive provided that the natural extension $\phi^{(m)} : M_m \otimes M_n \to M_m \otimes M_n$ given by $\phi^{(m)}((A_{ij})) = (\phi(A_{ij}))$ is positive for all $m \in \mathbb{N}$ (see Section 1.3).

Given a finite set $\mathcal{P}_1 = \{A_1, A_2, \ldots, A_l\} \subseteq M_n$, we define a map $\phi : M_n \to M_n$ by

$$\phi(X) = \sum_{i=1}^{l} A_i X A_i^*.$$

One can easily verify that ϕ is a completely positive linear map. By Theorem 1.3.1, $\mathcal{P}_1$ is referred as the family of Kraus operators for the map ϕ.

Proposition 3.5.3. *Let $\mathcal{P}_1 = \{A_1, A_2, \ldots, A_l\}$ be a finite set of matrices in M_n and let ϕ be its corresponding completely positive map given by $\phi(X) = \sum_{i=1}^{l} A_i X A_i^*$. Then $\mathrm{rank}(C_\phi) = \dim(\mathcal{V}_1)$, where $\mathcal{V}_1 = \mathrm{span}(\mathcal{P}_1)$.*

Proof. Let's assume that the dimension of $\mathcal{V}_1$ is equal to k for some $1 \leq k \leq l$. By Proposition 3.5.1, we have that $\dim(\mathcal{V}_1) = \mathrm{rank}(V_{\mathcal{P}_1})$, where

$$V_{\mathcal{P}_1} = \begin{bmatrix} \mathrm{vec}(A_1) & \mathrm{vec}(A_2) & \cdots & \mathrm{vec}(A_l) \end{bmatrix}.$$

Now consider the completely positive map $\phi : M_n \to M_n$ given by $\phi(X) = \sum_{i=1}^{l} A_i X A_i^*$. One can easily verify that

$$(A_r E_{ij} A_r^*) = \mathrm{vec}(A_r)\mathrm{vec}(A_r)^* \text{ for all } 1 \le r \le l.$$

Then the Choi matrix of ϕ is equal to

$$\begin{aligned}
(\phi(E_{ij})) &= \left(\sum_{r=1}^{l} A_r E_{ij} A_r^*\right) = \sum_{r=1}^{l} \mathrm{vec}(A_r)\mathrm{vec}(A_r)^* \\
&= \begin{bmatrix} \mathrm{vec}(A_1) & \mathrm{vec}(A_2) & \cdots & \mathrm{vec}(A_l) \end{bmatrix} \begin{bmatrix} \mathrm{vec}(A_1)^* \\ \mathrm{vec}(A_2)^* \\ \vdots \\ \mathrm{vec}(A_l)^* \end{bmatrix} \\
&= V_{\mathcal{P}_1} V_{\mathcal{P}_1}^*.
\end{aligned}$$

It follows that $\mathrm{rank}(C_\phi) = \mathrm{rank}(V_{\mathcal{P}_1})$. As a result, we have $\mathrm{rank}(C_\phi) = \dim(\mathcal{V}_1)$. $\square$

For a given finite set $\mathcal{P}_1 = \{A_1, A_2, \ldots, A_l\} \subseteq M_n$ and its corresponding completely positive map $\phi(X) = \sum_{i=1}^{l} A_i X A_i^*$, consider the composition of the map ϕ with itself. Then we have $\phi^2 = \phi \circ \phi : M_n \to M_n$ given by

$$\begin{aligned}
\phi^2(X) &= \phi(\phi(X)) \\
&= \sum_{i=1}^{l} A_i \phi(X) A_i^* \\
&= \sum_{i=1}^{l} A_i (\sum_{j=1}^{l} A_j X A_j^*) A_i^* \\
&= \sum_{i,j=1}^{l} (A_i A_j) X (A_i A_j)^*.
\end{aligned}$$

Note that $A_i A_j \in \mathcal{P}_2$ for all $1 \le i, j \le l$. Thus, the composition of the completely positive map ϕ with itself, $\phi^2 : M_n \to M_n$, is defined by

$$\phi^2(X) = \sum_{A^{(2)} \in \mathcal{P}_2} A^{(2)} X A^{(2)^*},$$

with repetitions allowed and $A^{(2)}$ is a matrix of length 2.

Inductively, given a finite set $P_1 \subseteq M_n$ and its corresponding completely positive map $\phi(X) = \sum_{A \in P_1} AXA^*$, the composition of this map with itself m times

$$\phi^m = \underbrace{\phi \circ \phi \circ \cdots \circ \phi}_{\text{m times}} : M_n \to M_n$$

is defined by

$$\phi^m(X) = \sum_{A^{(m)} \in \mathcal{P}_m} A^{(m)} X A^{(m)^*},$$

with repetitions allowed and $A^{(m)} \in \mathcal{P}_m$ is a matrix of length m.

Corollary 3.5.4. *Let $\mathcal{P}_1 = \{A_1, A_2, \ldots, A_l\}$ be a finite set of matrices in M_n and let ϕ be its corresponding completely positive map given by $\phi(X) = \sum_{i=1}^{l} A_i X A_i^*$. If $\mathcal{P}_m$ is the set of matrices of length m of $\mathcal{P}_1$, $\mathcal{V}_m = \operatorname{span}(\mathcal{P}_m)$, ϕ^m the composition of the map ϕ m times and C_{ϕ^m} is its Choi matrix, then* $\operatorname{rank}(C_{\phi^m}) = \dim(\mathcal{V}_m)$.

Proposition 3.5.5. *Let $\mathcal{P}_1 = \{A_1, A_2, \ldots, A_l\}$ be a finite set of matrices in M_n and let ϕ be its corresponding completely positive map. If $\mathcal{P}_m$ is the set of matrices of length m of $\mathcal{P}_1$, ϕ^m the composition of the map ϕ m times and C_{ϕ^m} is its Choi matrix, then* $\operatorname{index}(\mathcal{P}_1) = m$ *if and only if* $\operatorname{rank}(C_{\phi^m}) = n^2$.

Proof. Use Corollaries 3.5.2 and 3.5.4. ∎

Remark 3.5.6. *For a given finite set $\mathcal{P}_1 \subseteq M_n$ with finite spanning index, say index$(\mathcal{P}_1) = m < +\infty$, we know that $\mathcal{V}_m = span(\mathcal{P}_m) = M_n$ implies that $\mathcal{V}_{m+1} = M_n$ by Proposi-*

tion 3.1.3. By using the results of Proposition 3.5.5, we deduce that $\operatorname{rank}(C_{\phi^m}) = n^2$ *implies that* $\operatorname{rank}(C_{\phi^{m+1}}) = n^2$.

Lemma 3.5.7. *Let* $\mathcal{P}_1 = \{A_1, A_2, \ldots, A_l\}$ *and* S *be an invertible matrix in* M_n. *Set*

$$\tilde{\mathcal{P}}_1 = S^{-1}\mathcal{P}_1 S = \{S^{-1}A_1S, S^{-1}A_2S, \ldots, S^{-1}A_lS\} \text{ and } \mathcal{P}_1^* = \{A_1^*, A_2^*, \ldots, A_l^*\}.$$

Then, if it exists, $\operatorname{index}(\mathcal{P}_1) = \operatorname{index}(\tilde{\mathcal{P}}_1) = \operatorname{index}(\mathcal{P}_1^*)$.

Proof. We will show $\operatorname{index}(\mathcal{P}_1) = \operatorname{index}(\tilde{\mathcal{P}}_1)$. Note that $\tilde{\mathcal{P}}_m = S^{-1}\mathcal{P}_m S$, $m \in \mathbb{N}$. The linear independence of matrices of length m in $\mathcal{P}_m$ will not be changed if we left-multiply these matrices by S^{-1} and right-multiply by S. It follows that the number of linearly independent matrices in $\tilde{\mathcal{P}}_m$ will be the same as in $\mathcal{P}_m$, for each m. Hence, the result follows.
We will leave to the reader to check the other identity $\operatorname{index}(\mathcal{P}_1) = \operatorname{index}(\mathcal{P}_1^*)$. □

Let $\mathcal{P}_1 = \{A_1, A_2, \ldots, A_l\}$ be a set of matrices in M_n such that $\operatorname{index}(\mathcal{P}_1) < +\infty$. Then one can easily verify that $P = \sum_{i=1}^{l} A_iA_i^*$ is a positive definite matrix with $\operatorname{rank}(P) = n$, i.e. P is invertible. Note that if $\sum_{i=1}^{l} A_iA_i^* = I_n$, then the map $\phi : M_n \to M_n$ given by $\phi(X) = \sum_{i=1}^{l} A_iXA_i^*$ becomes a quantum channel, i.e. a trace-preserving completely positive map.(see Section 1.3).

Proposition 3.5.8. *Let* $\mathcal{P}_1 = \{A_1, A_2, \ldots, A_l\}$ *be a finite set of matrices in* M_n *such that* $\sum_{i=1}^{l} A_iA_i^*$ *is invertible, and let* $\phi : M_n \to M_n$ *be its corresponding completely positive map. Then there exists an invertible matrix* $S \in M_n$ *such that for* $\tilde{A}_i = S^{-1}A_iS$ *with* $A_i \in \mathcal{P}_1$, *the map* $\tilde{\phi}(X) = \sum_{i=1}^{l} \tilde{A}_iX\tilde{A}_i^*$ *is a quantum channel if and only if* ϕ *has a fixed point of full rank.*

Proof. Assume that there exists an invertible matrix $S \in M_n$ such that $\tilde{A}_i = S^{-1}A_iS$ with $A_i \in \mathcal{P}_1$ and $\tilde{\phi}(X) = \sum_{i=1}^{l} \tilde{A}_iX\tilde{A}_i^*$ is a quantum channel. Then we have

$$I_n = \tilde{\phi}(I_n) = \sum_{i=1}^{l} \tilde{A}_i\tilde{A}_i^*$$

$$\begin{aligned} &= \sum_{i=1}^{l}(S^{-1}A_iS)(S^{-1}A_iS)^* \\ &= \sum_{i=1}^{l} S^{-1}A_iSS^*A_i^*(S^{-1})^* \\ &= S^{-1}\left(\sum_{i=1}^{l} A_i(SS^*)A_i^*\right)(S^{-1})^*. \end{aligned}$$

It follows that $SS^* = \sum_{i=1}^{l} A_i(SS^*)A_i^* = \phi(SS^*)$. Hence, ϕ has a fixed point of full rank, since SS^* is invertible. The converse is similar. $\square$

Recall the brief description of Matrix Product States (MPS) given in Section 1.5 and their representations [25]. It is already known in Quantum Information Theory that any translationally invariant MPS ξ of N spins on some n-dimensional virtual Hilbert spaces $\mathcal{H}$ that are connected to the real physical l-dimensional spaces through a map, is defined by a family of Kraus operators $\mathcal{P}_1 = \{A_1, A_2, \ldots, A_l\}$ such

$$\xi = \sum_{i_1,\ldots,i_N} \operatorname{tr}(A_{i_1}\cdots A_{i_N})e_{i_1}\otimes\cdots\otimes e_{i_N} \in \mathbb{C}^N \otimes \mathbb{C}^n,$$

where e_{i_j} are the orthonormal basis for each n-dimensional Hilbert space $\mathcal{H} \cong \mathbb{C}^n$.

Let $\phi : M_n \to M_n$ be the completely positive map given by $\phi(X) = \sum_{A\in\mathcal{P}_1} AXA^*$. Since the Kraus operators of the completely positive map ϕ are uniquely determined up to unitaries, it follows that ϕ determines the MPS up to local unitaries in the physical system(for more details see [25]). This property associated to Kraus operators gives nice classifications of the corresponding completely positive maps.

Let $\Gamma_L : M_n \to \mathbb{C}^m \otimes \mathbb{C}^n$ be the map given by

$$\Gamma_L(X) = \sum_{i_1,\dots,i_L} \mathrm{tr}(XA_{i_1} \dots A_{i_L}) e_{i_1} \otimes \dots \otimes e_{i_L}.$$

Recall the inner product of two matrices $A, B \in M_n$ from matrix theory

$$\langle A, B \rangle = \sum_{i,j=1}^{n} a_{ij}\bar{b}_{ij} = \mathrm{tr}(AB^*).$$

Assume that index$(\mathcal{P}_1) = m < +\infty$. We claim that Γ_m is one-to-one:

Let $X \in M_n$ such that $\Gamma_m(X) = 0$. This implies that each entry of the vector $\Gamma_m(X)$ is 0, i.e. $\mathrm{tr}(XA_{i_1}A_{i_2} \dots A_{i_m}) = 0$ for all $A_{i_1}A_{i_2} \dots A_{i_m} \in \mathcal{P}_m$. For the sake of notations we write $A^{(m)} = A_{i_1}A_{i_2} \dots A_{i_m} \in \mathcal{P}_m$. Thus, we have

$$\begin{aligned} \mathrm{tr}(XA^{(m)}) = 0 \text{ for all } A^{(m)} \in \mathcal{P}_m \quad & \text{if and only if} \quad \langle X, (A^{(m)})^* \rangle = 0 \text{ for all } (A^{(m)})^* \in \mathcal{P}_m^*, \\ & \text{if and only if} \quad X \perp (A^{(m)})^* \text{ for all } (A^{(m)})^* \in \mathcal{P}_m^*, \\ & \text{if and only if} \quad X \perp \mathcal{V}_m^* = \mathrm{span}(\mathcal{P}_m^*). \end{aligned}$$

Since $\mathcal{V}_m = \mathrm{span}(\mathcal{P}_m) = M_n$, then by Lemma 3.5.7 we have $\mathcal{V}_m^* = M_n$. It follows that $X = 0$ and therefore, Γ_m is one-to-one. We know that if $(\mathcal{V}_m) = M_n$, then $\mathcal{V}_{m+1} = M_n$ too. This implies that the map Γ_{m+1} is one-to-one.

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